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A Novel Numerical Method of $\mathcal{O}(h^4)$ for Three-Dimensional Non-Linear Triharmonic Equations

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Abstract. In this article, we present two new novel finite difference approximations of order two and four, respectively, for the three dimensional non-linear triharmonic partial differential equations on a compact stencil where the values of u, $\partial^2 u/\partial n^2$ and $\partial^4 u/\partial n^4$ are prescribed on the boundary. We introduce new ideas to handle the boundary conditions and there is no need to discretize the derivative boundary conditions. We require only 7- and 19-grid points on the compact cell for the second and fourth order approximation, respectively. The Laplacian and the biharmonic of the solution are obtained as by-product of the methods. We require only system of three equations to obtain the solution. Numerical results are provided to illustrate the usefulness of the proposed methods.

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1 Introduction

We are concerned with the numerical solution of three dimensional non-linear triharmonic partial differential equation of the form:

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$$
\begin{aligned}
&\nabla^6 u(x,y,z)\\
\equiv&\frac{\partial^6 u}{\partial x^6}+\frac{\partial^6 u}{\partial y^6}+\frac{\partial^6 u}{\partial z^6}+6\frac{\partial^6 u}{\partial x^2\partial y^2\partial z^2}+3\Big(\frac{\partial^6 u}{\partial x^4\partial y^2}+\frac{\partial^6 u}{\partial x^2\partial y^4}\\
&\qquad+\frac{\partial^6 u}{\partial z^4\partial y^2}+\frac{\partial^6 u}{\partial z^2\partial y^4}+\frac{\partial^6 u}{\partial x^4\partial z^2}+\frac{\partial^6 u}{\partial x^2\partial z^4}\Big)\\
=&f(x,y,z,u,u_x,u_y,u_z,\nabla^2 u,\nabla^2 u_x,\nabla^2 u_y,\nabla^2 u_z,\nabla^4 u,\nabla^4 u_x,\nabla^4 u_y,\nabla^4 u_z),\quad (x,y,z)\in\Omega, \quad (1.1)
\end{aligned}
$$

where $\Omega=\{(x,y,z)|0<x,y,z<1\}$ is the solution region with boundary $\partial\Omega$ and

$$
\nabla^2 u\equiv\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\quad\text{and}\quad \nabla^4 u\equiv\frac{\partial^4 u}{\partial x^4}+\frac{\partial^4 u}{\partial y^4}+\frac{\partial^4 u}{\partial z^4}+2\Big(\frac{\partial^4 u}{\partial x^2\partial y^2}+\frac{\partial^4 u}{\partial z^2\partial y^2}+\frac{\partial^4 u}{\partial x^2\partial z^2}\Big)
$$

represent the three dimensional Laplacian and biharmonic of the function $u(x,y,z)$. We assume that the solution $u(x,y,z)$ is smooth enough to maintain the order of accuracy as high as possible of the finite difference schemes under consideration.

The values of

$$
u,\quad \frac{\partial^2 u}{\partial n^2}\quad\text{and}\quad\frac{\partial^4 u}{\partial n^4}\ \text{are prescribed on the boundary } \partial\Omega. \tag{1.2}
$$

The boundary conditions prescribed by (1.2) are called second kind boundary conditions. Since the grid lines are parallel to coordinate axes, we assume that the boundary values prescribed by (1.2) must satisfy the consistency conditions at all twelve edges and eight corner points of the boundary $\partial\Omega$, i.e., $u_{xx}=u_{yy}=u_{zz}$ and $u_{xxxx}=u_{yyyy}=u_{zzzz}$ at all corner points, $u_{xx}=u_{yy}$ and $u_{xxxx}=u_{yyyy}$ at all points on the lines parallel to z-axis, $\cdots$, etc. The triharmonic equation is a sixth order elliptic partial differential equation which is encountered in viscous flow problems. Not many researchers have tried to solve the triharmonic equations, because it is difficult to discretize the differential equations and boundary conditions on a compact cell and moreover triharmonic problems require large computing power and place huge amount of memory requirements on the computational systems. Such computing power has only recently begun to become available for academic research. Different techniques for the numerical solution of the 2D non-linear biharmonic and 3D non-linear biharmonic equations have been considered in the literature, but not for the 3D non-linear triharmonic equations. A popular technique is to split the biharmonic equation into two coupled Poisson equations each of which may be discretized using the standard approximations and solving using any of the Poisson solvers. Difficulty with this approach is that the boundary conditions for the new variable are undefined and need to be approximated at the boundary. Smith [1] and Ehrlich [2,3] have solved 2D biharmonic equations using coupled second order accurate finite difference approximations. Bauer and Riess [4] have used block iterative method to solve the equation. Later, Kwon et al. [5], Stephenson [6], Evans and Mohanty [7], Mohanty et al. [8–11] have developed certain second- and fourth-order finite difference approximations for the biharmonic problems using 9-point compact cell. The compact approach involves discretizing the biharmonic equations using not just the grid values of the unknown solution u but also the values of the derivatives u_{xx}, u_{yy} and u_{zz} at the selected grid points. For 2D and 3D problems, the aforesaid researchers have solved three and four system of equations to obtain the values of u, u_{xx}, u_{yy} and u,u_{xx}, u_{yy}, u_{zz}, respectively. Fourth order compact finite difference schemes have become quite popular as against the other lower order accurate schemes which require high mesh refinement and hence are computationally inefficient. On the other hand, the higher order accuracy of the fourth order compact methods combined with the compactness of the difference stencil yields highly accurate numerical solutions on relatively coarser grids with greater computational efficiency. Using 19-point compact cell Jain et al. [12] have derived compact fourth order method for the solution of three dimensional nonlinear elliptic boundary value problems and obtained convergent solution. Later, using off-step discretization, Mohanty and Singh [13] discussed compact fourth order method for three dimensional singularly perturbed non-linear elliptic equations. In the recent past many well known researchers like Spotz and Carey [14], Li et al. [15], Tian et al. [16–18], Erturk and Gokcol [19] have proposed compact fourth order schemes for the solution of nonlinear fluid dynamics problems. A conventional approach for solving the 3D triharmonic equation is to discretize the differential equation (1.1) on a uniform grid using 343-point approximations with truncation error of order h^2. This approximation connects the values of central point in terms of 342 neighbouring values of u in $7\times7\times7$ grid. We note that the central value of u is connected to grid points three grids away in each direction from the central point and the difference approximations needs to be modified at grid points near the boundaries. There are serious computational difficulties with solution of the linear and non-linear systems obtained through 343-point discretization of the 3D triharmonic equation. Approximations using compact cells avoid these difficulties. The compact approach involves discretizing the biharmonic equations using not just the grid values of the unknown solution u but also the values of the derivatives u_{xx} and u_{yy} at selected grid points (see Mohanty et al. [8]). Recently, Singh et al. [20], Khattar et al. [21] and Mohanty et al. [22–24] have developed single-cell compact finite-difference discretization of order two and four for multi dimensional biharmonic and triharmonic problems.

In this article, we split the differential equation into system of three Poisson equations and introduce new ideas to handle boundary conditions without discretizing them in the system of equations. The present work is the extension of the work of 2D biharmonic and triharmonic problems described in [23, 24]. Further, using 19-point and 7-point compact cell (see Fig. 1), Mohanty [22] has discussed fourth and second order approximations for the triharmonic equation $\nabla^6 u = f(x,y,z)$, which is a particular case of (1.1) and linear in nature; whereas in this paper, using the same 19-point and 7-point compact cell we discuss fourth and second order approximations for the nonlinear triharmonic equation (1.1). The given second kind boundary conditions are exactly satisfied and no approximations for derivatives need to be carried out at the boundaries. The proposed new

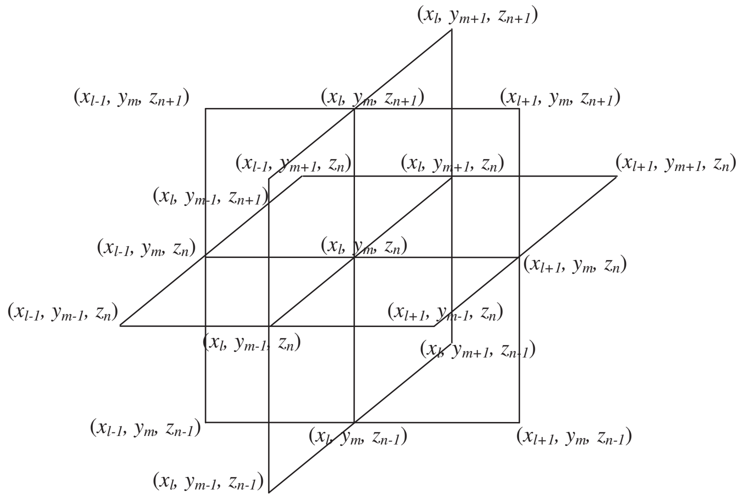

Figure 1: 19-point 3D single computational cell.

technique is not applicable to the triharmonic problem of first kind. Using this approach, we cannot obtain 19-point compact cell (see Fig. 1) fourth order approximations for the triharmonic problem of first kind. In next section, we discuss the finite difference approximation for the nonlinear triharmonic equations (1.1). In Section 3, we give the complete derivation of the method. In Section 4, we have discussed the stability analysis and illustrated the methods and their convergence by solving two problems. Concluding remarks are given in Section 5.

2 Three-dimensional discretization

Consider a three-dimensional uniform grid centred at the point (x_l,y_m,z_n), where $h>0$ is the constant mesh length in x-, y- and z- directions and $x_l=lh$, $y_m=mh$, $z_n=nh$, $l,m,n=0,1,2,\cdots,N$ with $(N+1)h=1$. Let $U_{l,m,n}$and $u_{l,m,n}$ be the exact and approximate solution values of $u(x,y,z)$ at the grid point (x_l,y_m,z_n), respectively.

The second kind boundary conditions are given by (1.2). Since the grid lines are parallel to coordinate axes and the values of u are exactly known on the boundary, this implies, the successive tangential partial derivatives of u are known exactly on the boundary. For example, on the plane $y=0$, the values of $u(x,0,z)$ and $u_{yy}(x,0,z)$ are known, i.e., the values of $u_x(x,0,z)$, $u_z(x,0,z)$, $u_{xx}(x,0,z)$, $u_{zz}(x,0,z)$, $\cdots$, etc. are known on the plane $y=0$. This implies, the values of u, $\nabla^2 u\equiv u_{xx}+u_{yy}+u_{zz}$ and $\nabla^4 u\equiv u_{xxxx}+u_{yyyy}+u_{zzzz}+2(u_{xxyy}+u_{yyzz}+u_{zzxx})$ are known on the plane $y=0$. Similarly the values of u, $\nabla^2 u$ and $\nabla^4 u$ are known on all plane sides of the cubic region Ω.

Thus, the second kind boundary conditions (1.2) may be replaced by

$$u=g_1(x,y,z),\qquad \nabla^2 u=g_2(x,y,z),\qquad \nabla^4 u=g_3(x,y,z),\qquad (x,y,z)\in\partial\Omega. \tag{2.1}$$

Let us denote $\nabla^2 u = v$, $\nabla^2 v = w$. Then we can re-write the differential equation (1.1) in a system of three Poisson equations of the form:

$$\nabla^2 u(x,y,z) \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = v(x,y,z), \qquad (x,y,z) \in \Omega, \tag{2.2a}$$

$$\nabla^2 v(x,y,z) \equiv \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = w(x,y,z), \qquad (x,y,z) \in \Omega, \tag{2.2b}$$

$$\begin{aligned}\nabla^2 w(x,y,z) &\equiv \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \\ &= f(x,y,z,u,v,w,u_x,v_x,w_x,u_y,v_y,w_y,u_z,v_z,w_z), \qquad (x,y,z) \in \Omega,\end{aligned} \tag{2.2c}$$

and the Dirichlet type boundary conditions (2.1) may be re-written as

$$u = g_1(x,y,z), \quad v = g_2(x,y,z), \quad w = g_3(x,y,z), \quad (x,y,z) \in \partial\Omega. \tag{2.3}$$

We assume that

$$f(x,y,z,u,v,w,u_x,v_x,w_x,u_y,v_y,w_y,u_z,v_z,w_z) \text{ is continuous}, \tag{2.4a}$$

$$\frac{\partial f}{\partial s}, \ \frac{\partial f}{\partial s_x}, \ \frac{\partial f}{\partial s_y}, \ \frac{\partial f}{\partial s_z} \ \text{exist and are continuous}; \quad s = u,v,w, \tag{2.4b}$$

$$\frac{\partial f}{\partial s} \geq 0, \quad s = u,v,w, \tag{2.4c}$$

$$\left|\frac{\partial f}{\partial s_x}\right| \leq K, \quad \left|\frac{\partial f}{\partial s_y}\right| \leq K, \quad \left|\frac{\partial f}{\partial s_z}\right| \leq K, \quad s = u,v,w, \tag{2.4d}$$

where K is a positive constants. These conditions are must for the convergent solution (see [12, 13]). In addition for a meaningful fourth order approximation, we assume that $u(x,y,z)$ is sufficiently differentiable (at least $u \in C^{10}$) as high order as possible.

Let at the grid points (x_l, y_m, z_n), the exact and approximate solution values of $v(x,y,z)$ and $w(x,y,z)$ be denoted as $V_{l,m,n}$, $W_{l,m,n}$ and $v_{l,m,n}$, $w_{l,m,n}$, respectively.

In order to obtain fourth order approximations on the 19-point compact cell for the system of non-linear differential equations (2.2a)-(2.2c), for $S = U,V,W$, let

$$\overline{S}_{xl,m,n} = \frac{1}{2h}(S_{l+1,m,n} - S_{l-1,m,n}), \qquad \overline{S}_{yl,m,n} = \frac{1}{2h}(S_{l,m+1,n} - S_{l,m-1,n}), \tag{2.5a}$$

$$\overline{S}_{zl,m,n} = \frac{1}{2h}(S_{l,m,n+1} - S_{l,m,n-1}), \qquad \overline{S}_{xl\pm1,m,n} = \frac{1}{2h}(\pm 3S_{l\pm1,m,n} \mp 4S_{l,m,n} \pm S_{l\mp1,m,n}), \tag{2.5b}$$

$$\overline{S}_{xl,m\pm1,n} = \frac{1}{2h}(S_{l+1,m\pm1,n} - S_{l-1,m\pm1,n}), \qquad \overline{S}_{xl,m,n\pm1} = \frac{1}{2h}(S_{l+1,m,n\pm1} - S_{l-1,m,n\pm1}), \tag{2.5c}$$

$$\overline{S}_{yl\pm1,m,n} = \frac{1}{2h}(S_{l\pm1,m+1,n} - S_{l\pm1,m-1,n}), \qquad \overline{S}_{yl,m\pm1,n} = \frac{1}{2h}(\pm 3S_{l,m\pm1,n} \mp 4S_{l,m,n} \pm S_{l,m\mp1,n}), \tag{2.5d}$$

$$\overline{S}_{yl,m,n\pm1} = \frac{1}{2h}(S_{l,m+1,n\pm1} - S_{l,m-1,n\pm1}), \qquad \overline{S}_{zl\pm1,m,n} = \frac{1}{2h}(S_{l\pm1,m,n+1} - S_{l\pm1,m,n-1}), \tag{2.5e}$$

$$\overline{S}_{zl,m\pm1,n} = \frac{1}{2h}(S_{l,m\pm1,n+1} - S_{l,m\pm1,n-1}), \qquad \overline{S}_{zl,m,n\pm1} = \frac{1}{2h}(\pm 3S_{l,m,n\pm1} \mp 4S_{l,m,n} \pm S_{l,m,n\mp1}), \tag{2.5f}$$

$$\overline{S}_{xxl,m\pm1,n}=\frac{1}{h^2}(S_{l+1,m\pm1,n}-2S_{l,m\pm1,n}+S_{l-1,m\pm1,n}), \tag{2.5g}$$

$$\overline{S}_{xxl,m,n\pm1}=\frac{1}{h^2}(S_{l+1,m,n\pm1}-2S_{l,m,n\pm1}+S_{l-1,m,n\pm1}), \tag{2.5h}$$

$$\overline{S}_{yyl\pm1,m,n}=\frac{1}{h^2}(S_{l\pm1,m+1,n}-2S_{l\pm1,m,n}+S_{l\pm1,m-1,n}), \tag{2.5i}$$

$$\overline{S}_{yyl,m,n\pm1}=\frac{1}{h^2}(S_{l,m+1,n\pm1}-2S_{l,m,n\pm1}+S_{l,m-1,n\pm1}), \tag{2.5j}$$

$$\overline{S}_{zzl\pm1,m,n}=\frac{1}{h^2}(S_{l\pm1,m,n+1}-2S_{l\pm1,m,n}+S_{l\pm1,m,n-1}), \tag{2.5k}$$

$$\overline{S}_{zzl,m\pm1,n}=\frac{1}{h^2}(S_{l,m\pm1,n+1}-2S_{l,m\pm1,n}+S_{l,m\pm1,n-1}). \tag{2.5l}$$

Now we define the approximation:

$$\begin{aligned}\overline{F}_{l\pm1,m,n}=f(&x_{l\pm1},y_m,z_n,U_{l\pm1,m,n},V_{l\pm1,m,n},W_{l\pm1,m,n},\overline{U}_{xl\pm1,m,n},\overline{V}_{xl\pm1,m,n},\overline{W}_{xl\pm1,m,n},\\ &\overline{U}_{y_{l\pm1,m,n}},\overline{V}_{y_{l\pm1,m,n}},\overline{W}_{y_{l\pm1,m,n}},\overline{U}_{zl\pm1,m,n},\overline{V}_{zl\pm1,m,n},\overline{W}_{zl\pm1,m,n}).\end{aligned} \tag{2.6}$$

Similarly, we can define the approximations $\overline{F}_{l,m\pm1,n}$ and $\overline{F}_{l,m,n\pm1}$, respectively.

Further, we define

$$\begin{aligned}\overline{\overline{U}}_{xl,m,n}=&\overline{U}_{xl,m,n}-\frac{h}{12}(V_{l+1,m,n}-V_{l-1,m,n})+\frac{h}{12}(\overline{U}_{yy_{l+1,m,n}}-\overline{U}_{yy_{l-1,m,n}})\\ &+\frac{h}{12}(\overline{U}_{zzl+1,m,n}-\overline{U}_{zzl-1,m,n}),\end{aligned} \tag{2.7a}$$

$$\begin{aligned}\overline{\overline{V}}_{xl,m,n}=&\overline{V}_{xl,m,n}-\frac{h}{12}(W_{l+1,m,n}-W_{l-1,m,n})+\frac{h}{12}(\overline{V}_{yy_{l+1,m,n}}-\overline{V}_{yy_{l-1,m,n}})\\ &+\frac{h}{12}(\overline{V}_{zzl+1,m,n}-\overline{V}_{zzl-1,m,n}),\end{aligned} \tag{2.7b}$$

$$\begin{aligned}\overline{\overline{W}}_{xl,m,n}=&\overline{W}_{xl,m,n}-\frac{h}{12}(\overline{F}_{l+1,m,n}-\overline{F}_{l-1,m,n})+\frac{h}{12}(\overline{W}_{yy_{l+1,m,n}}-\overline{W}_{yy_{l-1,m,n}})\\ &+\frac{h}{12}(\overline{W}_{zzl+1,m,n}-\overline{W}_{zzl-1,m,n}).\end{aligned} \tag{2.7c}$$

Similarly replacing the variable x by y and z and interchanging the subscripts $(l\pm1,m,n)$ by $(l,m\pm1,n)$ and $(l,m,n\pm1)$, respectively, in (2.7a)-(2.7c), we can define the approximations for $\overline{\overline{U}}_{yl,m,n}$, $\overline{\overline{V}}_{yl,m,n}$, $\overline{\overline{W}}_{yl,m,n}$ and $\overline{\overline{U}}_{zl,m,n}$, $\overline{\overline{V}}_{zl,m,n}$, $\overline{\overline{W}}_{zl,m,n}$, respectively.

Finally, let

$$\begin{aligned}\overline{\overline{F}}_{l,m,n}=f(&x_l,y_m,z_n,U_{l,m,n},V_{l,m,n},W_{l,m,n},\overline{\overline{U}}_{xl,m,n},\overline{\overline{V}}_{xl,m,n},\overline{\overline{W}}_{xl,m,n},\\ &\overline{\overline{U}}_{yl,m,n},\overline{\overline{V}}_{yl,m,n},\overline{\overline{W}}_{yl,m,n},\overline{\overline{U}}_{zl,m,n},\overline{\overline{V}}_{zl,m,n},\overline{\overline{W}}_{zl,m,n}).\end{aligned} \tag{2.8}$$

Then at each internal grid point (x_l,y_m,z_n); $l,m,n=1(1)N$, the given system of differential equations (2.2) are discretized by

$$\begin{aligned}
L[U] \equiv\, & U_{l,m-1,n-1}+U_{l-1,m,n-1}+2U_{l,m,n-1}+U_{l+1,m,n-1}+U_{l,m+1,n-1} \\
& +U_{l-1,m-1,n}+2U_{l,m-1,n}+U_{l+1,m-1,n}+2U_{l-1,m,n}-24U_{l,m,n} \\
& +2U_{l+1,m,n}+U_{l-1,m+1,n}+2U_{l,m+1,n}+U_{l+1,m+1,n}+U_{l,m-1,n+1} \\
& +U_{l-1,m,n+1}+2U_{l,m,n+1}+U_{l+1,m,n+1}+U_{l,m+1,n+1} \\
=\, & \frac{h^2}{2}[V_{l+1,m,n}+V_{l-1,m,n}+V_{l,m+1,n}+V_{l,m-1,n}+V_{l,m,n+1} \\
& +V_{l,m,n-1}+6V_{l,m,n}]+\mathcal{O}(h^6),
\end{aligned} \tag{2.9a}$$

$$\begin{aligned}
L[V] = & \frac{h^2}{2}[W_{l+1,m,n}+W_{l-1,m,n}+W_{l,m+1,n}+W_{l,m-1,n}+W_{l,m,n+1} \\
& +W_{l,m,n-1}+6W_{l,m,n}]+\mathcal{O}(h^6),
\end{aligned} \tag{2.9b}$$

$$\begin{aligned}
L[W] = & \frac{h^2}{2}[\overline{F}_{l+1,m,n}+\overline{F}_{l-1,m,n}+\overline{F}_{l,m+1,n}+\overline{F}_{l,m-1,n}+\overline{F}_{l,m,n+1} \\
& +\overline{F}_{l,m,n-1}+6\overline{\overline{F}}_{l,m,n}]+\overline{T}_{l,m,n},
\end{aligned} \tag{2.9c}$$

where $L[V]$ and $L[W]$ can be obtained from $L[U]$ by replacing U by V and W, respectively and $\overline{T}_{l,m,n}=\mathcal{O}(h^6)$. Note that, the approximations (2.9a)-(2.9c) require only 19- grid points with a single computational cell. Incorporating the Dirichlet boundary conditions given by (2.3) into the difference methods (2.9a)-(2.9c), we obtain the three sparse system of tri-block-block diagonal matrix equations, which can be solved by appropriate iterative methods (see [25–30]).

3 Derivation of the fourth order discretization

For the derivation of the fourth order method (2.9a)-(2.9c), we follow the technique given by Mohanty [23, 24].

At the grid point (x_l,y_m,z_n), let us denote

$$S_{ijk}=\frac{\partial^{i+j+k}S}{\partial x_l{}^i \partial y_m{}^j \partial z_n{}^k}, \quad \alpha^{(S)}_{l,m,n}=\frac{\partial f}{\partial S_{xl,m,n}}, \quad \beta^{(S)}_{l,m,n}=\frac{\partial f}{\partial S_{yl,m,n}}, \quad \gamma^{(S)}_{l,m,n}=\frac{\partial f}{\partial S_{zl,m,n}}, \\ S=U,V,W. \tag{3.1}$$

Further, at the grid point (x_l,y_m,z_n), we define

$$\begin{aligned}
F_{l,m,n}=f(&x_l,y_m,z_n,U_{l,m,n},V_{l,m,n},W_{l,m,n},U_{xl,m,n},V_{xl,m,n},W_{xl,m,n}, \\
&U_{yl,m,n},V_{yl,m,n},W_{yl,m,n},U_{zl,m,n},V_{zl,m,n},W_{zl,m,n}).
\end{aligned} \tag{3.2}$$

Using Taylor expansion about the grid point (x_l,y_m,z_n), we first obtain

$$\begin{aligned}
L[W] = & \frac{h^2}{2}[F_{l+1,m,n}+F_{l-1,m,n}+F_{l,m+1,n}+F_{l,m-1,n}+F_{l,m,n+1} \\
& +F_{l,m,n-1}+6F_{l,m,n}]+\mathcal{O}(h^6).
\end{aligned} \tag{3.3}$$

Now by the help of the approximations (2.5a)-(2.5l), from (2.6), we obtain

$$\overline{F}_{l\pm1,m,n} = F_{l\pm1,m,n} + \frac{h^2}{6} T_1 \pm \mathcal{O}(h^3). \tag{3.4}$$

Similarly,

$$\overline{F}_{l,m\pm1,n} = F_{l,m\pm1,n} + \frac{h^2}{6} T_2 \pm \mathcal{O}(h^3), \tag{3.5a}$$

$$\overline{F}_{l,m,n\pm1} = F_{l,m,n\pm1} + \frac{h^2}{6} T_3 \pm \mathcal{O}(h^3), \tag{3.5b}$$

where

$$\begin{aligned} T_1 = & -2U_{300}\alpha^{(U)}_{l,m,n} - 2V_{300}\alpha^{(V)}_{l,m,n} - 2W_{300}\alpha^{(W)}_{l,m,n} + U_{030}\beta^{(U)}_{l,m,n} + V_{030}\beta^{(V)}_{l,m,n} + W_{030}\beta^{(W)}_{l,m,n} \\ & + U_{003}\gamma^{(U)}_{l,m,n} + V_{003}\gamma^{(V)}_{l,m,n} + W_{003}\gamma^{(W)}_{l,m,n}, \\ T_2 = & \, U_{300}\alpha^{(U)}_{l,m,n} + V_{300}\alpha^{(V)}_{l,m,n} + W_{300}\alpha^{(W)}_{l,m,n} - 2U_{030}\beta^{(U)}_{l,m,n} - 2V_{030}\beta^{(V)}_{l,m,n} - 2W_{030}\beta^{(W)}_{l,m,n} \\ & + U_{003}\gamma^{(U)}_{l,m,n} + V_{003}\gamma^{(V)}_{l,m,n} + W_{003}\gamma^{(W)}_{l,m,n}, \\ T_3 = & \, U_{300}\alpha^{(U)}_{l,m,n} + V_{300}\alpha^{(V)}_{l,m,n} + W_{300}\alpha^{(W)}_{l,m,n} + U_{030}\beta^{(U)}_{l,m,n} + V_{030}\beta^{(V)}_{l,m,n} + W_{030}\beta^{(W)}_{l,m,n} \\ & - 2U_{003}\gamma^{(U)}_{l,m,n} - 2V_{003}\gamma^{(V)}_{l,m,n} - 2W_{003}\gamma^{(W)}_{l,m,n}. \end{aligned}$$

Now we consider the linear combination:

$$\begin{aligned} \overline{\overline{\overline{U}}}_{xl,m,n} = & \, \overline{U}_{xl,m,n} + ha_{11}(V_{l+1,m,n} - V_{l-1,m,n}) + ha_{12}(\overline{U}_{yyl+1,m,n} - \overline{U}_{yyl-1,m,n}) \\ & + ha_{13}(\overline{U}_{zzl+1,m,n} - \overline{U}_{zzl-1,m,n}), \end{aligned} \tag{3.6a}$$

$$\begin{aligned} \overline{\overline{\overline{V}}}_{xl,m,n} = & \, \overline{V}_{xl,m,n} + hb_{11}(W_{l+1,m,n} - W_{l-1,m,n}) + hb_{12}(\overline{V}_{yy\,l+1,m,n} - \overline{V}_{yy\,l-1,m,n}) \\ & + hb_{13}(\overline{V}_{zzl+1,m,n} - \overline{V}_{zzl-1,m,n}), \end{aligned} \tag{3.6b}$$

$$\begin{aligned} \overline{\overline{\overline{W}}}_{xl,m,n} = & \, \overline{W}_{xl,m,n} + hc_{11}(\overline{F}_{l+1,m,n} - \overline{F}_{l-1,m,n}) + hc_{12}(\overline{W}_{yy\,l+1,m,n} - \overline{W}_{yy\,l-1,m,n}) \\ & + hc_{13}(\overline{W}_{zzl+1,m,n} - \overline{W}_{zzl-1,m,n}), \end{aligned} \tag{3.6c}$$

$$\begin{aligned} \overline{\overline{\overline{U}}}_{yl,m,n} = & \, \overline{U}_{yl,m,n} + ha_{21}(V_{l,m+1,n} - V_{l,m-1,n}) + ha_{22}(\overline{U}_{xxl,m+1,n} - \overline{U}_{xxl,m-1,n}) \\ & + ha_{23}(\overline{U}_{zzl,m+1,n} - \overline{U}_{zzl,m-1,n}), \end{aligned} \tag{3.6d}$$

$$\begin{aligned} \overline{\overline{\overline{V}}}_{yl,m,n} = & \, \overline{V}_{yl,m,n} + hb_{21}(W_{l,m+1,n} - W_{l,m-1,n}) + hb_{22}(\overline{V}_{xxl,m+1,n} - \overline{V}_{xxl,m-1,n}) \\ & + hb_{23}(\overline{V}_{zzl,m+1,n} - \overline{V}_{zzl,m-1,n}), \end{aligned} \tag{3.6e}$$

$$\begin{aligned} \overline{\overline{\overline{W}}}_{yl,m,n} = & \, \overline{W}_{yl,m,n} + hc_{21}(\overline{F}_{l,m+1,n} - \overline{F}_{l,m-1,n}) + hc_{22}(\overline{W}_{xxl,m+1,n} - \overline{W}_{xxl,m-1,n}) \\ & + hc_{23}(\overline{W}_{zzl,m+1,n} - \overline{W}_{zzl,m-1,n}), \end{aligned} \tag{3.6f}$$

$$\begin{aligned} \overline{\overline{\overline{U}}}_{zl,m,n} = & \, \overline{U}_{zl,m,n} + ha_{31}(V_{l,m,n+1} - V_{l,m,n-1}) + ha_{32}(\overline{U}_{xxl,m,n+1} - \overline{U}_{xxl,m,n-1}) \\ & + ha_{33}(\overline{U}_{yy\,l,m,n+1} - \overline{U}_{yy\,l,m,n-1}), \end{aligned} \tag{3.6g}$$

$$\begin{aligned}\overline{\overline{V}}_{zl,m,n}=&\overline{V}_{zl,m,n}+hb_{31}(W_{l,m,n+1}-W_{l,m,n-1})+hb_{32}(\overline{V}_{xxl,m,n+1}-\overline{V}_{xxl,m,n-1})\\&+hb_{33}(\overline{V}_{yyl,m,n+1}-\overline{V}_{yyl,m,n-1}),\end{aligned}\tag{3.6h}$$

$$\begin{aligned}\overline{\overline{W}}_{zl,m,n}=&\overline{W}_{zl,m,n}+hc_{31}(\overline{F}_{l,m,n+1}-\overline{F}_{l,m,n-1})+hc_{32}(\overline{W}_{xxl,m,n+1}-\overline{W}_{xxl,m,n-1})\\&+hc_{33}(\overline{W}_{yyl,m,n+1}-\overline{W}_{yyl,m,n-1}),\end{aligned}\tag{3.6i}$$

where a_{ij}, b_{ij} and c_{ij}, $i,j=1,2,3$ are parameters to be determined.

Now using the approximations defined in previous section and (3.4)-(3.5b), from (3.6a)-(3.6i), we obtain

$$\overline{\overline{U}}_{xl,m,n}=U_{xl,m,n}+\frac{h^2}{6}T_4+\mathcal{O}(h^4),\qquad \overline{\overline{V}}_{xl,m,n}=V_{xl,m,n}+\frac{h^2}{6}T_5+\mathcal{O}(h^4),\tag{3.7a}$$

$$\overline{\overline{W}}_{xl,m,n}=W_{xl,m,n}+\frac{h^2}{6}T_6+\mathcal{O}(h^4),\qquad \overline{\overline{U}}_{yl,m,n}=U_{yl,m,n}+\frac{h^2}{6}T_7+\mathcal{O}(h^4),\tag{3.7b}$$

$$\overline{\overline{V}}_{yl,m,n}=V_{yl,m,n}+\frac{h^2}{6}T_8+\mathcal{O}(h^4),\qquad \overline{\overline{W}}_{yl,m,n}=W_{yl,m,n}+\frac{h^2}{6}T_9+\mathcal{O}(h^4),\tag{3.7c}$$

$$\overline{\overline{U}}_{zl,m,n}=U_{zl,m,n}+\frac{h^2}{6}T_{10}+\mathcal{O}(h^4),\qquad \overline{\overline{V}}_{zl,m,n}=V_{zl,m,n}+\frac{h^2}{6}T_{11}+\mathcal{O}(h^4),\tag{3.7d}$$

$$\overline{\overline{W}}_{zl,m,n}=W_{zl,m,n}+\frac{h^2}{6}T_{12}+\mathcal{O}(h^4),\tag{3.7e}$$

where

$$\begin{aligned}
T_4&=(1+12a_{11})U_{300}+12(a_{11}+a_{12})U_{120}+12(a_{11}+a_{13})U_{102},\\
T_5&=(1+12b_{11})V_{300}+12(b_{11}+b_{12})V_{120}+12(b_{11}+b_{13})V_{102},\\
T_6&=(1+12c_{11})W_{300}+12(c_{11}+c_{12})W_{120}+12(c_{11}+c_{13})W_{102},\\
T_7&=(1+12a_{21})U_{030}+12(a_{21}+a_{22})U_{210}+12(a_{21}+a_{23})U_{012},\\
T_8&=(1+12b_{21})V_{030}+12(b_{21}+b_{22})V_{210}+12(b_{21}+b_{23})V_{012},\\
T_9&=(1+12c_{21})W_{030}+12(c_{21}+c_{22})W_{210}+12(c_{21}+c_{23})W_{012},\\
T_{10}&=(1+12a_{31})U_{003}+12(a_{31}+a_{32})U_{201}+12(a_{31}+a_{33})U_{021},\\
T_{11}&=(1+12b_{31})V_{003}+12(b_{31}+b_{32})V_{201}+12(b_{31}+b_{33})V_{021},\\
T_{12}&=(1+12c_{31})W_{003}+12(c_{31}+c_{32})W_{201}+12(c_{31}+c_{33})W_{021}.
\end{aligned}$$

By the help of the approximations (3.7a)-(3.7e), from (2.8), we get

$$\overline{\overline{F}}_{l,m,n}=F_{l,m,n}+\frac{h^2}{6}T_{13}+\mathcal{O}(h^4),\tag{3.8}$$

where

$$\begin{aligned}T_{13}=&T_4\alpha^{(U)}_{l,m,n}+T_5\alpha^{(V)}_{l,m,n}+T_6\alpha^{(W)}_{l,m,n}+T_7\beta^{(U)}_{l,m,n}+T_8\beta^{(V)}_{l,m,n}\\&+T_9\beta^{(W)}_{l,m,n}+T_{10}\gamma^{(U)}_{l,m,n}+T_{11}\gamma^{(V)}_{l,m,n}+T_{12}\gamma^{(W)}_{l,m,n}.\end{aligned}$$

Finally, by the help of the relations (3.4)-(3.5b) and (3.8), from (2.9c) and (3.3), we obtain the local truncation error

$$\overline{T}_{l,m,n}=\frac{-h^4}{6}[T_1+T_2+T_3+3T_{13}]+\mathcal{O}(h^6). \tag{3.9}$$

The proposed difference methods (2.9a)-(2.9c) are to be of $\mathcal{O}(h^4)$, the coefficient of h^4 in (3.9) must be zero, and we obtain a relation

$$T_1+T_2+T_3+3T_{13}=0. \tag{3.10}$$

Substituting the values of T_1, T_2, T_3 and T_{13} in (3.10), we obtain the values of parameters $a_{i1}=b_{i1}=c_{i1}=-1/12$ for $i=1,2,3$ and $a_{ij}=b_{ij}=c_{ij}=1/12$ for $i=1,2,3$ and $j=2,3$, and the local truncation error (3.9) reduces to $\overline{T}_{l,m,n}=\mathcal{O}(h^6)$.

The details of the convergence analysis of the difference scheme for a scalar 3D nonlinear elliptic equation has been discussed in [12], in which it has been shown that $u^h\rightarrow u$ as $h\rightarrow 0$. The same error analysis can be carried out for the system of three elliptic equations. We have omitted the convergence analysis for the proposed fourth order approximations (2.9a)-(2.9c) because of the crude mathematical calculation involved in error analysis with three unknown variables u, v and w.

4 Stability analysis and numerical results

Consider the test equation

$$\nabla^6 u(x,y,z)=G(x,y,z), \qquad 0<x,y,z<1. \tag{4.1}$$

Applying the proposed method (2.9a)-(2.9c) to the above equation, we obtain

$$L[U]=\frac{h^2}{2}[V_{l+1,m,n}+V_{l-1,m,n}+V_{l,m+1,n}+V_{l,m-1,n}+V_{l,m,n+1}+V_{l,m,n-1}+6V_{l,m,n}], \tag{4.2a}$$

$$L[V]=\frac{h^2}{2}[W_{l+1,m,n}+W_{l-1,m,n}+W_{l,m+1,n}+W_{l,m-1,n}+W_{l,m,n+1}+W_{l,m,n-1}+6W_{l,m,n}], \tag{4.2b}$$

$$L[W]=\frac{h^2}{2}[G_{l+1,m,n}+G_{l-1,m,n}+G_{l,m+1,n}+G_{l,m-1,n}+G_{l,m,n+1}+G_{l,m,n-1}+6G_{l,m,n}], \tag{4.2c}$$

where $G_{l,m,n}=G(x_l,y_m,z_n),\cdots$, etc.

An iterative method for (4.2a)-(4.2c) can be written as:

$$24\mathbf{I}\mathbf{u}^{(k+1)}=\mathbf{A}\mathbf{u}^{(k)}-\frac{h^2}{2}\mathbf{B}\mathbf{v}^{(k)}+\mathbf{0}\mathbf{w}^{(k)}+\mathbf{RHU}, \tag{4.3a}$$

$$24\mathbf{I}\mathbf{v}^{(k+1)}=\mathbf{0}\mathbf{u}^{(k)}+\mathbf{A}\mathbf{v}^{(k)}-\frac{h^2}{2}\mathbf{B}\mathbf{w}^{(k)}+\mathbf{RHV}, \tag{4.3b}$$

$$24\mathbf{I}\mathbf{w}^{(k+1)}=\mathbf{0}\mathbf{u}^{(k)}+\mathbf{0}\mathbf{v}^{(k)}+\mathbf{A}\mathbf{w}^{(k)}+\mathbf{RHW}, \tag{4.3c}$$

where $\mathbf{u}^{(k)}$, $\mathbf{v}^{(k)}$, $\mathbf{w}^{(k)}$ are solution vectors and **RHU**, **RHV**, **RHW** are right hand side vectors consists of boundary and homogenous function values.

Above iterative method in matrix form can be written as:

$$\begin{bmatrix} \mathbf{U}^{(k+1)} \\ \mathbf{V}^{(k+1)} \\ \mathbf{W}^{(k+1)} \end{bmatrix} = \mathbf{G} \begin{bmatrix} \mathbf{U}^{(k)} \\ \mathbf{V}^{(k)} \\ \mathbf{W}^{(k)} \end{bmatrix} + \mathbf{RH}, \tag{4.4}$$

where

$$\mathbf{G} = \frac{1}{24} \begin{bmatrix} \mathbf{A} & \dfrac{-h^2}{2}\mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} & \dfrac{-h^2}{2}\mathbf{B} \\ \mathbf{0} & \mathbf{0} & \mathbf{A} \end{bmatrix}, \qquad \mathbf{RH} = \begin{bmatrix} \mathbf{RHU} \\ \mathbf{RHV} \\ \mathbf{RHW} \end{bmatrix}.$$

We denote $\mathbf{0}$ as the N-th order null matrix, $\mathbf{I}=[0,1,0]$ as the N-th order identity matrix and $\mathbf{H}=[1,2,1]$, $\mathbf{Q}=[2,0,2]$, $\mathbf{J}=[1,6,1]$ as N-th order tridiagonal matrices and $\mathbf{C}=[I,H,I]$, $\mathbf{D}=[H,Q,H]$, $\mathbf{E}=[0,I,0]$, $\mathbf{F}=[I,J,I]$, as the N-th order block-tridiagonal matrices, where, in general, we denote

$$[a,b,c] = \begin{bmatrix} b & c & & & \mathbf{0} \\ a & b & c & & \\ & & \ddots & & \\ & & a & b & c \\ \mathbf{0} & & & a & b \end{bmatrix}_{N\times N}$$

as N-th order tridiagonal matrix whose eigen values are given by

$$b+2\sqrt{ac}\cos\Big(\frac{\pi j}{N+1}\Big), \qquad j=1,2,\cdots,N. \tag{4.5}$$

The N-th order block-block-tridiagonal matrices $\mathbf{A}$ and $\mathbf{B}$ associated with (4.3a)-(4.3c) are defined by

$$\mathbf{A} = \begin{bmatrix} \mathbf{D} & \mathbf{C} & & & \mathbf{0} \\ \mathbf{C} & \mathbf{D} & \mathbf{C} & & \\ & & \ddots & & \\ & & \mathbf{C} & \mathbf{D} & \mathbf{C} \\ \mathbf{0} & & & \mathbf{C} & \mathbf{D} \end{bmatrix}_{N\times N} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} \mathbf{F} & \mathbf{E} & & & \mathbf{0} \\ \mathbf{E} & \mathbf{F} & \mathbf{E} & & \\ & & \ddots & & \\ & & \mathbf{E} & \mathbf{F} & \mathbf{E} \\ \mathbf{0} & & & \mathbf{E} & \mathbf{F} \end{bmatrix}_{N\times N}.$$

The eigenvalues of $\mathbf{I}$, $\mathbf{H}$ and $\mathbf{Q}$ are 1 (N-times), $2+2\cos(k\pi h)$ and $4\cos(k\pi h)$, $k=1,2,\cdots,N$, respectively, where $(N+1)h=1$.

The eigenvalues of **C** and **D** are given by

$$\xi_{jk}=2+2[\cos(j\pi h)+\cos(k\pi h)], \qquad j,k=1(1)N, \tag{4.6}$$
$$\eta_{jk}=4[\cos(j\pi h)+\cos(k\pi h)+\cos(j\pi h)\cos(k\pi h)], \qquad j,k=1(1)N. \tag{4.7}$$

The matrix **A** associated with the iteration matrix **G** whose eigenvalues are given by

$$\gamma_{ijk}=\eta_{jk}+2\xi_{jk}\cos(i\pi h), \qquad i,j,k=1,2,\cdots,N$$

or

$$\begin{aligned}\gamma_{ijk}=&4[\cos(i\pi h)+\cos(j\pi h)+\cos(k\pi h)+\cos(i\pi h)\cos(j\pi h)\\&+\cos(j\pi h)\cos(k\pi h)+\cos(k\pi h)\cos(i\pi h)], \qquad i,j,k=1,2,\cdots,N.\end{aligned}$$

Similarly, the eigenvalues of the matrix **B** associated with the iteration matrix **G** are given by

$$\sigma_{ijk}=6+2[\cos(i\pi h)+\cos(j\pi h)+\cos(k\pi h)], \qquad i,j,k=1,2,\cdots,N. \tag{4.8}$$

The iterative method (4.4) is stable as long as $\rho(\mathbf{G})\le 1$, where $\rho(\mathbf{G})$ is the spectral radius of **G**.

The characteristic equation of the matrix **G** is given by

$$det\begin{bmatrix}\frac{1}{24}\gamma_{ijk}-\lambda & \frac{-h^2}{48}\sigma_{ijk} & \mathbf{0}\\ \mathbf{0} & \frac{1}{24}\gamma_{ijk}-\lambda & \frac{-h^2}{48}\sigma_{ijk}\\ \mathbf{0} & \mathbf{0} & \frac{1}{24}\gamma_{ijk}-\lambda\end{bmatrix}=0, \qquad i,j,k=1,2,\cdots,N. \tag{4.9}$$

Thus the eigenvalues of **G** are given by

$$\begin{aligned}\lambda_{ijk}=\lambda=\frac{1}{24}\gamma_{ijk}=\frac{1}{6}&[\cos(i\pi h)+\cos(j\pi h)+\cos(k\pi h)+\cos(i\pi h)\cos(j\pi h)\\&+\cos(j\pi h)\cos(k\pi h)+\cos(k\pi h)\cos(i\pi h)], \quad i,j,k=1,2,\cdots,N.\end{aligned} \tag{4.10}$$

The maximum values of all eigenvalues of **G** occur at $i=j=k=1$, hence

$$\rho(\mathbf{G})=\max|\lambda_{ijk}|=\frac{\cos(\pi h)}{2}[1+\cos(\pi h)]\le 1, \tag{4.11}$$

which is satisfied for all variable angles πh. Hence the iterative method (4.3a)-(4.3c) is stable.

The second order approximations for the system of differential equations (2.2a)-(2.2c) are straightforward and can be written as:

$$\begin{aligned}
&U_{l,m,n-1}+U_{l,m-1,n}+U_{l-1,m,n}-6U_{l,m,n}+U_{l+1,m,n}+U_{l,m+1,n}+U_{l,m,n+1}\\
=&h^2V_{l,m,n}+\mathcal{O}(h^4), \qquad (4.12a)\\
&V_{l,m,n-1}+V_{l,m-1,n}+V_{l-1,m,n}-6V_{l,m,n}+V_{l+1,m,n}+V_{l,m+1,n}+V_{l,m,n+1}\\
=&h^2W_{l,m,n}+\mathcal{O}(h^4), \qquad (4.12b)\\
&W_{l,m,n-1}+W_{l,m-1,n}+W_{l-1,m,n}-6W_{l,m,n}+W_{l+1,m,n}+W_{l,m+1,n}+W_{l,m,n+1}\\
=&h^2f\big(x_l,y_m,z_n,U_{l,m,n},V_{l,m,n},W_{l,m,n},\overline{U}_{xl,m,n},\overline{V}_{xl,m,n},\overline{W}_{xl,m,n},\\
&\quad\overline{U}_{yl,m,n},\overline{V}_{yl,m,n},\overline{W}_{yl,m,n},\overline{U}_{zl,m,n},\overline{V}_{zl,m,n},\overline{W}_{zl,m,n}\big)+\mathcal{O}(h^4). \qquad (4.12c)
\end{aligned}$$

Note that, the second order approximations (4.12a)-(4.12c) require only 7-grid points on a single computational cell (see Fig. 1).

By combining the difference equations at each internal grid points, we obtain a large sparse system of matrix to solve. At each interior mesh point, we have three unknowns u, $\nabla^2 u\equiv v$ and $\nabla^4 u\equiv w$, that is, the number of bands with non-zero entries is increased, and so is the size of the final matrix for the same mesh size. However, by this new method, the values of the Laplacian and biharmonic which are often of interest are also computed.

Whenever $f(x,y,z,u,v,w,u_x,v_x,w_x,u_y,v_y,w_y,u_z,v_z,w_z)$ is linear (or, non-linear) in u, v, w, u_x, v_x, w_x, u_y, v_y, w_y, u_z, v_z, w_z, the difference equations (2.9a)-(2.9c) or (4.12a)-(4.12c) form a linear (or non-linear) system. To solve such a system or indeed to demonstrate the existence of a solution, we use iterative methods. In this section, we solve the following three test problems in the region $0<x,y,z<1$, whose exact solutions are known. The boundary conditions and right hand side homogeneous functions are obtained by using the exact solutions. In all cases, we have considered $u^{(0)}=0$ as the initial approximation and the iterations were stopped when the absolute error tolerance $|u^{(k+1)}-u^{(k)}|\leq 10^{-12}$ was achieved. In all cases, we have calculated maximum absolute errors (l_∞-norm) for different grid sizes. All computations were performed using double precision arithmetic.

Example 4.1. (Variable coefficient problems)

(a) The equation

$$\begin{aligned}\nabla^6 u=&(1+x^2)(\nabla^4 u)_x+(1+z^2)(\nabla^2 u)_z+(1+x^2)u_x\\ &+(1+z^2)u_z+G(x,y,z),\qquad 0<x,y,z<1. \qquad (4.13)\end{aligned}$$

The exact solution is $u(x,y,z)=\sin(\pi x)\sin(\pi y)\sin(\pi z)$.

(b) The equation

$$\begin{aligned}\nabla^6 u=&(1+\cos^2 x)(\nabla^4 u)_x+(1+\cos^2 z)(\nabla^2 u)_z+(1+\cos^2 x)u_x\\ &+(1+\cos^2 z)u_z+G(x,y,z),\qquad 0<x,y,z<1. \qquad (4.14)\end{aligned}$$

The exact solution is $u(x,y,z)=e^{x+y+z}$.

Table 1: The maximum absolute errors.

h		Example 4.1(a)		Example 4.1(b)	
		$\mathcal{O}(h^4)$-Method	$\mathcal{O}(h^2)$-Method	$\mathcal{O}(h^4)$-Method	$\mathcal{O}(h^2)$-Method
$\frac{1}{4}$	u	0.3056(-02)	0.1672(+00)	0.4072(-05)	0.3254(-02)
	$\nabla^2 u$	0.8884(-01)	0.3213(+01)	0.6836(-04)	0.1233(-01)
	$\nabla^4 u$	0.7455(+00)	0.4642(+02)	0.8206(-04)	0.2698(-01)
$\frac{1}{8}$	u	0.1664(-03)	0.3924(-01)	0.2610(-06)	0.9828(-03)
	$\nabla^2 u$	0.4622(-02)	0.7692(+00)	0.3427(-05)	0.3566(-02)
	$\nabla^4 u$	0.3477(-01)	0.1133(+02)	0.4673(-05)	0.7401(-02)
$\frac{1}{16}$	u	0.7889(-05)	0.9678(-02)	0.1247(-07)	0.2614(-03)
	$\nabla^2 u$	0.2845(-03)	0.1800(+00)	0.1611(-06)	0.1024(-02)
	$\nabla^4 u$	0.1224(-02)	0.2820(+01)	0.3054(-06)	0.2918(-02)
$\frac{1}{32}$	u	0.4416(-06)	0.2408(-02)	0.8742(-09)	0.8256(-04)
	$\nabla^2 u$	0.1054(-04)	0.4756(-01)	0.8055(-08)	0.7766(-03)
	$\nabla^4 u$	0.7772(-04)	0.7040(+00)	0.1293(-07)	0.8751(-03)
$\frac{1}{64}$	u	0.2612(-07)	0.5182(-03)	0.5419(-10)	0.1821(-04)
	$\nabla^2 u$	0.6518(-06)	0.1118(-01)	0.4984(-09)	0.1442(-03)
	$\nabla^4 u$	0.4518(-05)	0.8288(-01)	0.8055(-08)	0.1919(-03)

The maximum absolute errors are tabulated in Table 1.

Example 4.2. (Navier-Stokes model equation in terms of stream function ψ) The steady-state two dimensional incompressible Navier-Stokes equations in the traditional velocity-pressure formulation is given by

$$\frac{1}{R_e}(u_{xx}+u_{yy})=uu_x+vu_y+p_x, \tag{4.15a}$$

$$\frac{1}{R_e}(v_{xx}+v_{yy})=uv_x+vv_y+p_y, \tag{4.15b}$$

$$u_x+v_y=0, \tag{4.15c}$$

where u, v are velocities along the x- and y-directions respectively, p represents the pressure and $Re>0$ is the Reynolds number. Though this formulation represents the fluid flow phenomena, its direct solution traditionally has been difficult to obtain due to the presence of pressure term in Eqs. (4.15a) and (4.15b). Partly in order to avoid handling the pressure variable, an alternative formulation using stream-function and vorticity has been used for several decades. The relation between stream function ψ and velocity components are given by

$$u(x,y)=\psi_y, \qquad v(x,y)=-\psi_x. \tag{4.16}$$

The relation (4.16) satisfies automatically the Eq. (4.15c). Substituting (4.16) into the Eqs. (4.15a) and (4.15b) and eliminating the pressure terms, we obtain the Navier-Stokes equation in terms of stream function ψ as:

$$\frac{1}{R_e}(\psi_{xxxx}+2\psi_{xxyy}+\psi_{yyyy})=\psi_y(\psi_{xxx}+\psi_{xyy})-\psi_x(\psi_{xxy}+\psi_{yyy}). \tag{4.17}$$

We consider a more general form of this equation to incorporate a forcing term $G(x,y)$:

$$\frac{1}{R_e}(\psi_{xxxx}+2\psi_{xxyy}+\psi_{yyyy})=\psi_y\nabla^2\psi_x-\psi_x\nabla^2\psi_y+G(x,y). \tag{4.18}$$

This formulation has been very successful and has been used by a large number of researchers (see [14–19]) to test new methods for the numerical solutions of a variety of fluid flow problems.

Similarly, we choose three-dimensional model Navier-Stokes equations in terms of stream function ψ:

$$\begin{aligned}\frac{1}{R_e}\nabla^6\psi=&\psi_y(\nabla^2\psi)_x-\psi_x(\nabla^2\psi)_y+(\nabla^2\psi)_y(\nabla^4\psi)_x-(\nabla^2\psi)_x(\nabla^4\psi)_y\\&+\psi_x(\nabla^4\psi)_y-\psi_y(\nabla^4\psi)_x+G(x,y,z),\qquad 0<x,y,z<1.\end{aligned} \tag{4.19}$$

The exact solution is given by $\psi(x,y,z)=e^x\sin(\pi y)\sin(\pi z)$.

The maximum absolute errors are tabulated in Table 2 for various values of R_e.

Table 2: The maximum absolute errors.

h		$\mathcal{O}(h^4)$-Method		$\mathcal{O}(h^2)$-Method
		$Re=10^2$	$Re=10^4, 10^6, 10^8$	$Re=10^2, 10^4, 10^6, 10^8$
$\frac{1}{4}$	ψ	0.2791(-02)	0.2798(-02)	Over Flow
	$\nabla^2\psi$	0.5231(-01)	0.5311(-01)	
	$\nabla^4\psi$	0.6734(+00)	0.6884(+00)	
$\frac{1}{8}$	ψ	0.1277(-03)	0.1287(-03)	Over Flow
	$\nabla^2\psi$	0.2416(-02)	0.2524(-02)	
	$\nabla^4\psi$	0.3235(-01)	0.3363(-01)	
$\frac{1}{16}$	ψ	0.6540(-05)	0.6688(-05)	Over Flow
	$\nabla^2\psi$	0.1118(-03)	0.1228(-03)	
	$\nabla^4\psi$	0.1628(-02)	0.1741(-02)	
$\frac{1}{32}$	ψ	0.3318(-06)	0.3424(-06)	Over Flow
	$\nabla^2\psi$	0.7659(-05)	0.7772(-05)	
	$\nabla^4\psi$	0.8452(-04)	0.8460(-04)	
$\frac{1}{64}$	ψ	0.2068(-07)	0.2112(-07)	Over Flow
	$\nabla^2\psi$	0.4522(-06)	0.4604(-06)	
	$\nabla^4\psi$	0.5202(-05)	0.5204(-05)	

5 Final remarks

In this article, we derived two new novel compact finite difference discretizations of order two and four, respectively, for the 3D non-linear triharmonic partial differential equations. The methods are derived on 7- and 19-point compact stencils, respectively, using the values of u, its Laplacian and biharmonic as the unknowns. We have obtained the numerical solutions of Laplacian and biharmonic of u as by-products of the methods, which are quite often of interest in many applied mathematics problems. Our methods are used to solve two problems including Navier Stokes model equation in terms of stream function ψ. While solving Navier Stokes equations of motion, numerical results confirm that the proposed fourth order discretization method produces oscillation free solution for high Reynolds number, whereas the corresponding second order method becomes unstable and produces overflow solution. We are currently working to extend this technique to solve time dependent parabolic partial differential equations.

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