

## A Variational Binary Level Set Method for Structural Topology Optimization

Xiaoxia Dai<sup>1</sup>, Peipei Tang<sup>1,\*</sup>, Xiaoliang Cheng<sup>2</sup> and Minghui Wu<sup>1</sup>

<sup>1</sup> School of Computing Science, Zhejiang University City College, Hangzhou, P.R. China.

<sup>2</sup> Department of Mathematics, Zhejiang University, Hangzhou, P.R. China.

Received 16 September 2011; Accepted (in revised version) 11 May 2012

Available online 8 October 2012

---

**Abstract.** This paper proposes a variational binary level set method for shape and topology optimization of structural. First, a topology optimization problem is presented based on the level set method and an algorithm based on binary level set method is proposed to solve such problem. Considering the difficulties of coordination between the various parameters and efficient implementation of the proposed method, we present a fast algorithm by reducing several parameters to only one parameter, which would substantially reduce the complexity of computation and make it easily and quickly to get the optimal solution. The algorithm we constructed does not need to re-initialize and can produce many new holes automatically. Furthermore, the fast algorithm allows us to avoid the update of Lagrange multiplier and easily deal with constraints, such as piecewise constant, volume and length of the interfaces. Finally, we show several optimum design examples to confirm the validity and efficiency of our method.

**AMS subject classifications:** 65K10, 74B05, 74S05

**Key words:** Structural optimization, binary level set method, finite element method.

---

### 1 Introduction

The topology optimization of continuum structures, as one of the most challenging tasks, has been widely investigated in the relevant literatures [1, 2, 24, 26, 29, 32, 35]. The homogenization method, which was first developed by Bendsoe and Kikuchi (see [3]), is a popular and important approach for structural topology optimization. By changing a

---

\*Corresponding author. *Email addresses:* daixiaoxia@zucc.edu.cn (X. Dai), wjktpp@yahoo.com.cn (P. Tang), xiaoliangcheng@zju.edu.cn (X. Cheng), mhwu@zucc.edu.cn (M. Wu)

difficult topology design problem into a relatively simpler 'sizing' problem, the homogenization technique provides a method for simultaneous shape and topology optimization. As a simplification and variation of the homogenization method, the solid isotropic material with penalization (SIMP) method has many advantages and has been widely considered [2, 17, 31]. The basic idea of SIMP method is to use of a fictitious isotropic material. The elasticity tensor of this material is assumed to be a penalized material density function which is represented by an exponent parameter. However, both of the homogenization method and the SIMP method have a problem of numerical instability such as checkerboards, mesh-dependencies and local minimal solutions [7, 20]. Furthermore, there is an approach called the evolutionary structural optimization (ESO) method for structural topology optimization which was proposed in [30, 33]. In this method, the design domain is discretized by a finite element mesh, and shape and topology of the structure are obtained by progressively removing the material region with low stress.

Recently, the level set method, which was first developed for tracking the propagation of fluid interface [14], has been successfully used to solve a wide range of shape and topology optimization problems [1, 27]. Such method implicitly represents the geometric boundary by a higher dimensional function. Generally, a signed distance function is used to represent the level set function and trace the evolution of the boundaries with respect to time by solving Hamilton-Jacobi partial differential equation [8, 15], with a suitable normal velocity which is the moving boundary velocity normal to the interface. During the evolution, the so-called re-initialization [19, 21] process should be performed periodically to ensure that the level set function is always a signed distance function. This is a sufficient strategy to maintain numerical accuracy, but it has strong computational difficulties and is quite inefficient [28].

In order to avoid the unnecessary signed distance function and the re-initialization of level set function, an alternative piecewise constant level set (PCLS) method was proposed by Lie-Lysaker-Tai [4, 9, 10]. The PCLS method was originally used to handle image segmentation [11], see also [25]. In [5, 12], the PLCS method was successfully applied to solve inverse problems and interface problems. Wei and Wang [29] also proposed a PCLS method to solve topology optimization problems. Compared with the standard level set method, the PLCS method can easily create small holes without topological derivatives during the evolution. Thus PLCS method reduces dependence on the initial design. However, the PLCS method needs to add a piecewise constant constraint to the level set function.

Binary level set (BLS) method, as a special PCLS method, was developed in [9, 10] for problems involving the motion of curves and surfaces. Later it was applied to solve elliptic inverse problems with discontinuous coefficients [13] and elliptic shape optimization problems [36], both of them achieved good results. Compared with general PCLS method, the biggest advantage of BLS method is that it only requires  $N$  level set functions to represent a structure of  $m = 2^N$  different material phases. For example, four material phases can be represented using only two level set functions. The level set function in BLS method has only two values 1 and -1.

In this paper, we employ the binary level set method for structural optimization. The PCLS method uses one function with several different values to represent the different material phases, one value represent one material phase. And the BLS method uses combinations of several functions with two values to represent different material phases. According to BLS scheme, we propose an acceleration of algorithm by gradually converting material from solid to void. This technique allows us to avoid the update of Lagrange multiplier and easily deal with constraints, such as piecewise constant, volume and length of the interfaces, which is different from PCLS method in [29]. We reduce several parameters to only one parameter, which would substantially reduce the complexity of computation and make it easily and quickly to get the optimal solution. The new algorithm does not need to initialize and can produce many new holes automatically.

This paper is organized as follows. In Section 2, we give a brief introduction of BLS method. And then, a topology optimization problem is formulated in Section 3 based on the BLS method. We use the ersatz material approach to compute the equilibrium equations of the structure. In Section 4, we propose an acceleration of algorithm to solve topology optimization problem. In Section 5, we present some numerical examples to validate our method for structural optimization. Finally, we give some conclusions at the last section.

## 2 Binary level set method

In order to introduce the binary level set method, we first divide an open bounded set  $\Omega \subseteq D \subset R^d$  ( $d=2$  or  $3$ ) into a set of  $2^N$  subregions  $\Omega_1, \Omega_2, \dots, \Omega_{2^N}$ , such that

$$\Omega = \bigcup_{i=1}^{2^N} (\Omega_i \cup \Gamma_i), \quad (2.1)$$

where  $\Gamma_i$  is the boundary of the subregion  $\Omega_i$ . Then we define piecewise constant level set functions  $\phi_j$  ( $j=1, 2, \dots, N$ ) satisfying  $\phi_j^2 = 1$ , e.g.,  $\phi_j$  has value 1 or -1 at every point in  $\Omega$ . Now  $2^N$  subregions can be represented by  $N$  level set functions. To illustrate this, we set  $N=1$ , then the level set function  $\phi$  can be defined as

$$\phi(x) = \begin{cases} 1, & \text{if } x \in \Omega_1, \\ -1, & \text{if } x \in \Omega_2. \end{cases} \quad (2.2)$$

The characteristic functions associated with  $\Omega_1$  and  $\Omega_2$  can be represented by

$$\psi_1 = \frac{1}{2}(\phi + 1) \quad \text{and} \quad \psi_2 = \frac{1}{2}(1 - \phi), \quad (2.3)$$

respectively. With the characteristic functions, we can describe a predefined property in  $\Omega$  with an associated value distinctive in each subregion  $\Omega_i$ , for example, material

density. Assume the material density  $\rho(x) = c_1$  in  $\Omega_1$  and  $\rho(x) = c_2$  in  $\Omega_2$ . Then  $\rho(x)$  can be written as the following sum

$$\rho(x) = \frac{c_1}{2}(\phi+1) + \frac{c_2}{2}(1-\phi). \tag{2.4}$$

If we have  $N=4$  subregions, the material density  $\rho(x)$  can be defined as follows

$$\rho(x) = \begin{cases} c_1, & \text{if } \phi_1(x) = 1, \phi_2(x) = 1, \\ c_2, & \text{if } \phi_1(x) = 1, \phi_2(x) = -1, \\ c_3, & \text{if } \phi_1(x) = -1, \phi_2(x) = 1, \\ c_4, & \text{if } \phi_1(x) = -1, \phi_2(x) = -1, \end{cases} \tag{2.5}$$

where we set two level set functions  $\phi_1$  and  $\phi_2$ . More precisely, the function  $\rho(x)$  is given as

$$\begin{aligned} \rho(x) = & \frac{c_1}{4}(\phi_1+1)(\phi_2+1) + \frac{c_2}{4}(\phi_1+1)(1-\phi_2) \\ & + \frac{c_3}{4}(1-\phi_1)(\phi_2+1) + \frac{c_4}{4}(1-\phi_1)(1-\phi_2), \end{aligned} \tag{2.6}$$

where  $\psi_1 = \frac{1}{4}(\phi_1+1)(\phi_2+1)$ ,  $\psi_2 = \frac{1}{4}(\phi_1+1)(1-\phi_2)$ ,  $\psi_3 = \frac{1}{4}(1-\phi_1)(\phi_2+1)$  and  $\psi_4 = \frac{1}{4}(1-\phi_1)(1-\phi_2)$  are the characteristic functions of corresponding subregions.

More generally, we use  $N$  level set functions to represent  $2^N$  phases. For  $i=1,2,\dots,2^N$ , let  $(b_i^{i-1}, b_2^{i-1}, \dots, b_N^{i-1})$  be the binary representation of  $i-1$ , where  $b_j^{i-1} = 0 \vee 1$ . Furthermore, we set

$$s(i) = \sum_{j=1}^N b_j^{i-1} \tag{2.7}$$

and write characteristic function  $\psi_i$  as the product

$$\psi_i = \frac{(-1)^{s(i)}}{2^N} \prod_{j=1}^N (\phi_j + 1 - 2b_j^{i-1}), \tag{2.8}$$

then  $\psi_i = 1$  in  $\Omega_i$ ,  $\psi_i = 0$  in  $\Omega \setminus \Omega_i$ . For a given set of scalars  $c_i$ , the piecewise constant function  $\rho(x)$  is defined as

$$\rho(x) = \sum_{i=1}^{2^N} c_i \psi_i, \tag{2.9}$$

which represents the different material density  $\rho(x) = c_i$  in each subregion  $\Omega_i$ .

With the simple structure of the characteristic functions, we can calculate the length of the boundary of  $\Omega_i$  and the area inside  $\Omega_i$  by

$$|\Omega_i| = \int_{\Omega} \psi_i dx \quad \text{and} \quad |\partial\Omega_i| = \int_{\Omega} |\nabla \psi_i| dx. \tag{2.10}$$

### 3 Structure optimization with the BLS Method

We now describe the problem of structural optimization with the proposed binary level set method. The boundary of  $\Omega$  is denoted as  $\Gamma = \partial\Omega = \Gamma_D \cup \Gamma_N$  with Dirichlet boundary condition on  $\Gamma_D$  and Neumann boundary condition on  $\Gamma_N$ . The displacement field  $u$  in  $\Omega$  is the unique solution of the following linear elastic system

$$\begin{cases} -\operatorname{div}\sigma(u) = f, & \text{in } \Omega, \\ u = u_0, & \text{on } \Gamma_D, \\ \sigma(u) \cdot n = g, & \text{on } \Gamma_N, \end{cases} \quad (3.1)$$

where the strain tensor  $\varepsilon$  and the stress tensor  $\sigma$  are given in the usual form as

$$\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T), \quad \sigma(u) = E\varepsilon(u), \quad (3.2)$$

with  $E$  to be the elasticity tensor,  $f$  the body forces,  $g$  the boundary traction force applied on the part  $\Gamma_N$ ,  $u_0$  the prescribed displacement on the admissible Dirichlet boundary  $\Gamma_D$  and  $n$  the outward normal to the boundary.

The general problem of structure optimization is specified as follows:

$$\begin{aligned} \min_{\Omega} \quad & \tilde{J}(u, \Omega) = \int_{\Omega} F(u) dx \\ \text{subject to} \quad & G(u, \Omega) = \int_{\Omega} dx - V_{\max} \leq 0, \end{aligned} \quad (3.3)$$

where  $F(u) = \frac{1}{2}E\varepsilon(u) : \varepsilon(u)$  is the strain energy density and  $V_{\max}$  is the upper limit of the volume constraint. The linear elastic system is expressed in a general variational form as

$$\int_{\Omega} E\varepsilon(u) : \varepsilon(v) dx = \int_{\Omega} f \cdot v dx + \int_{\Gamma_N} g \cdot v ds, \quad \text{for all } v \in U, \quad (3.4)$$

where  $U = \{u : u \in H^1(\Omega), u = u_0 \text{ on } \Gamma_D\}$  denoting the space of the kinematically admissible displacement fields. The goal of optimization is to find a minimizer  $\Omega$  for the optimization criterion  $\tilde{J}(u, \Omega)$  which yields an optimized structure. This is a standard notion of structural optimization [2, 16]. For simplicity, the reference design domain  $D$  will be partitioned into two subregions of solid  $\Omega$  and void  $D \setminus \Omega$  (Multiple subregions can be found in Section 5.4). Thus, the material density ratio  $\rho$  is defined as

$$\rho(\phi) = \frac{1}{2}(\phi + 1) + \frac{\epsilon}{2}(1 - \phi), \quad (3.5)$$

as described by Eq. (2.4) in the BLS method. Here  $\phi$  is the level set function defined by Eq. (2.2) and  $c_1 = 1$  represent the solid material,  $c_2 = \epsilon$  ( $\epsilon > 0$  is a small constant) represent the void material. Therefore, the material density ratio  $\rho$  is also defined as

$$\rho(\phi) = \begin{cases} 1, & \text{if } \phi = 1, \\ \epsilon, & \text{if } \phi = -1. \end{cases} \quad (3.6)$$

With the BLS function  $\phi$ , the structure optimization problem (3.3) is now reformulated as:

$$\begin{aligned} \min \quad & J(u, \phi) = \int_D F(u) \rho(\phi) dx + \beta \int_D |\nabla \phi| dx \\ \text{s. t.} \quad & G(\phi) = \frac{1}{2} \int_D (\phi + 1) dx - V_{\max} \leq 0, \\ & H(\phi) = (\phi + 1)(\phi - 1) = 0. \end{aligned} \quad (3.7)$$

The constraint  $H(\phi) = 0$  is used to guarantee that there is no vacuum and overlap between different subregions. The second term in the objective function is the regularization term required to restrict the solution space because of the ill-posedness of the original problem. The variational equation (3.4) is written in the energy bilinear and the load linear form as

$$a(u, v, \phi) = l(v, \phi), \quad (3.8)$$

where

$$a(u, v, \phi) = \int_D E \varepsilon(u) : \varepsilon(v) \rho(\phi) dx, \quad (3.9a)$$

$$l(v, \phi) = \int_D f \cdot v \rho(\phi) dx + \int_{\Gamma_N} g \cdot v ds. \quad (3.9b)$$

We use the Lagrange multiplier method to solve the constrained problem (3.7). The Lagrange functional of (3.7) is defined by

$$L(\phi, \lambda) = J(\phi) - a(u, v, \phi) + l(v, \phi) + \lambda_1 G(\phi) + \lambda_2 \int_D H(\phi) dx, \quad (3.10)$$

where the Lagrange multipliers  $\lambda_1 \in \mathbb{R}$  and  $\lambda_2 \in L^2(D)$ . To solve this problem, we actually minimize  $L$  with respect to  $\phi$  and maximize  $L$  with respect to  $\lambda$ . This is a saddle point problem, its optimal solution is at a stationary point of the Lagrangian, which yield the necessary optimality conditions

$$\frac{\partial L}{\partial \phi} = 0, \quad (3.11a)$$

$$\frac{\partial L}{\partial \lambda_1} = G(\phi) = 0, \quad (3.11b)$$

$$\frac{\partial L}{\partial \lambda_2} = H(\phi) = 0. \quad (3.11c)$$

In what follows, we calculate the concrete formula of Eq. (3.11). Using the adjoint variable

method, the first variation of  $L$  in direction  $h$  is

$$\begin{aligned} \frac{\partial L}{\partial \phi} \cdot h = & \int_D F(u)\rho'(\phi) \cdot h dx + \int_D E\varepsilon(u) : \varepsilon\left(\frac{\partial u}{\partial \phi} \cdot h\right) \rho(\phi) dx + \beta \int_D \nabla \cdot \left(\frac{\nabla \phi}{|\nabla \phi|}\right) \cdot h dx \\ & - \int_D E\varepsilon(u) : \varepsilon(u)\rho'(\phi) \cdot h dx - \int_D E\varepsilon(u) : \varepsilon\left(\frac{\partial u}{\partial \phi} \cdot h\right) \rho(\phi) dx + \lambda_1 \int_D \frac{1}{2} \cdot h dx \\ & + \lambda_2 \int_D H'(\phi) \cdot h dx, \end{aligned} \tag{3.12}$$

where the adjoint field is defined as follows:

$$a(v, u, \phi) = l(u, \phi), \quad \text{for } \forall u \in U, v \in U,$$

and

$$\begin{aligned} \rho'(\phi) &= \frac{1}{2}(1 - \epsilon), \\ H'(\phi) &= 2\phi. \end{aligned}$$

Thus, the concrete formula of Eq. (3.11) is as follows:

$$\frac{\partial L}{\partial \phi} = -\frac{1}{4}(1 - \epsilon)E\varepsilon(u) : \varepsilon(u) + \beta \nabla \cdot \left(\frac{\nabla \phi}{|\nabla \phi|}\right) + \frac{1}{2}\lambda_1 + 2\phi\lambda_2. \tag{3.13}$$

We introduce an artificial time term and solve the following ordinary differential equation to the steady state (see [12, 23, 34])

$$\begin{cases} \phi_t + \frac{\partial L}{\partial \phi} = 0, \\ \phi(0) = \phi_0. \end{cases} \tag{3.14}$$

In our approach, we use the explicit update scheme

$$\phi^{n+1} = \phi^n - \Delta t^n \frac{\partial L}{\partial \phi}(\phi^n, \lambda^n), \tag{3.15}$$

where  $\Delta t$  is time step.

Finally, the BLS algorithm for structural optimization is presented as follows:

**Algorithm 3.1.** Initialize level set function  $\phi^0$  on the entire domain  $D$ . The parameter  $\epsilon, \lambda_1^0, \lambda_2^0$  and tolerance  $\tau^0$  are given. For  $n = 0, 1, 2, \dots$ ,

1. Solve the problem (3.8) using linear finite elements for the state variable  $u^n$ .
2. Compute the sensitivities  $\frac{\partial L}{\partial \phi}(\phi^n, \lambda^n)$  depending on  $u^n$ . Terminate if  $\|\nabla_\phi L(\phi^n, \lambda^n)\| < \tau^n$ .
3. Update the level set function  $\phi^{n+1}$  by (3.15).

4. Update the Lagrange multipliers  $\lambda^{n+1}$  and  $\tau^{n+1}$ .
5. If not convergent, set  $n=n+1$  and go to step 1.

To achieve this algorithm, we also need to determine two things. One is the choice of time step  $\Delta t^n$  which should satisfy the CFL condition

$$\Delta t^n < h / \max \left| \frac{\partial L}{\partial \phi}(\phi^n, \lambda^n) \right|. \quad (3.16)$$

The other is how to update Lagrange multiplier  $\lambda$ . There are several ways to deal with the problem, such as augmented Lagrangian method [6, 18, 34], projection Lagrangian method [9, 36] and so on. No matter which method we use, it is complex and troublesome to update the Lagrange multiplier  $\lambda$  in the numerical examples, since the law of changes of parameter is difficult to accurately describe. In this paper, in order to avoid the update of Lagrange multiplier and taking into account the special nature of the algorithm, we present another more convenient and efficient approach whose details can be found in the next section.

## 4 Acceleration of algorithm

We have the following facts. With piecewise constant level set method for solving shape optimization, the level set function in the region far from the interface converges quickly to piecewise constant. Once the topology structure for the optimal design is determined, the interface is smooth. But it still takes rather many steps for  $\phi$  to form the discontinuity near the interface. Therefore, several algorithms [13, 34, 36] have been proposed to accelerate convergence.

In our approach, level set function  $\phi^n$  is required to be 1 or  $-1$  in each iteration, which means that the function  $\phi^n$  is discontinuous on the entire domain. Therefore, the condition  $H(\phi) = 0$  is directly established. This phenomenon is not automatically generated during the iteration, but we enforce it.

In order to illustrate our algorithm, we first divide the domain  $D$  into quadrilaterals or triangles. And the level set function  $\phi$  equals 1 or  $-1$  on each element. For simplicity, we initialize level set function  $\phi^0 = 1$  on all elements. Since the condition  $H(\phi) = 0$  has been established, the Eq. (3.13) becomes

$$\frac{\partial \tilde{L}}{\partial \phi} = -\frac{1}{4}(1-\epsilon)E\varepsilon(u) : \varepsilon(u) + \beta \nabla \cdot \left( \frac{\nabla \phi}{|\nabla \phi|} \right) + \frac{1}{2}\lambda_1. \quad (4.1)$$

Noticing the update of  $\phi^n$

$$\phi^n = \phi^{n-1} - \Delta t^n \frac{\partial \tilde{L}}{\partial \phi}(\phi^{n-1}, \lambda_1^{n-1}), \quad (4.2)$$

we can see that the level set function  $\phi^n$  is most possibly from 1 to  $-1$  on those elements where the relatively value of  $\Delta t \frac{\partial \bar{L}}{\partial \phi}(\phi^{n-1}, \lambda^{n-1})$  are large. At each iteration, the time step  $\Delta t^n$  and the Lagrange multiplier  $\lambda_1$  are equal in each elements, thus we just need to find the elements which have large value of

$$\frac{\partial Q}{\partial \phi} = -\frac{1}{4}(1-\epsilon)E\varepsilon(u) : \varepsilon(u) + \beta \nabla \cdot \left( \frac{\nabla \phi}{|\nabla \phi|} \right). \quad (4.3)$$

In those elements, we set  $\phi^n = -1$ . We introduce a parameter  $\theta (0 < \theta < 1)$  to control the number of elements on which the value of level set function  $\phi^n$  would convert from 1 to  $-1$ . For example, when we want to turn five percent elements from solid to void, we can set  $\theta = 0.95$ . Then, we list the acceleration of algorithm as follows:

**Algorithm 4.1.** Initialize level set function  $\phi^0 = 1$  on the entire domain  $D$ . The parameter  $\epsilon, \theta^0$  are given. For  $n = 0, 1, 2, \dots$ ,

1. Solve the problem (3.8) using linear finite elements for the state variable  $u^n$ .
2. Compute the sensitivities  $\frac{\partial Q}{\partial \phi}(\phi^n)$  depending on  $u^n$ .
3. Find out  $\theta^n$  times the maximum of  $\frac{\partial Q}{\partial \phi}(\phi^n)$ , then set the corresponding level set function  $\phi^{n+1} = -1$ . Terminate if  $G(\phi) < 0$ .
4. If not convergent, set  $n = n + 1$  and go to step 1.

Usually the value of  $\theta$  cannot be chosen under a uniform criterion for different problems. From the numerical experiments we observe that the parameter  $\theta$  needs to gradually reduce when the iterative number  $n$  increases. For example, we can set  $\theta = 0.95$  at the beginning. After several iterations, we can reduce the parameter  $\theta$  to 0.9 or even smaller. The selection of parameter  $\theta$  directly affects the speed of iteration (the details can be found in the next section). Actually, the choices of the three parameter  $\lambda_1, \lambda_2, \Delta t$  of Algorithm 3.1 are simplified as one parameter  $\theta$  of Algorithm 4.1. This makes the optimization problem much simpler.

## 5 Numerical experiments

In this section we present several examples of topology optimization with a set of 2D minimum compliance problems, which use the finite element method to solve the equilibrium equations. In those examples, the linear elastic material has Young's modulus=1, Poisson's ratio=0.3. We assume for void materials the Young's modulus=0.001. Except for specified cases, a quadrangular mesh is used for discretizations of both the level set function and the displacement. We have also tried other kind of physical problems in [22], which conform the validity and efficiency of our method.

## 5.1 Cantilever beam

As a benchmark problem for the proposed method, stiffness maximization of a cantilever is performed, as illustrated in Fig. 1. The working domain is a rectangle of size  $L=2, H=1$  with zero displacement boundary condition on the left side and a vertical point load  $P=1$  at the center of the right side. The domain is discretized with a  $160 \times 80$  rectangle mesh and the value of the Lagrange multiplier  $\lambda_1$  is fixed at 50 during the optimization process. The volume fraction of solid material is 50%. We set the regularization parameter  $\beta = 0.001$  and acceleration factor  $\theta^0 = 0.95$  at the beginning. The initial design is set to be the structure that is fully distributed with solid material in the whole design domain as showed in Fig. 2(a).

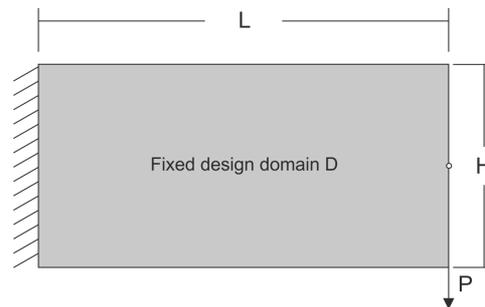


Figure 1: Fixed design domain and boundary condition of a cantilever example.

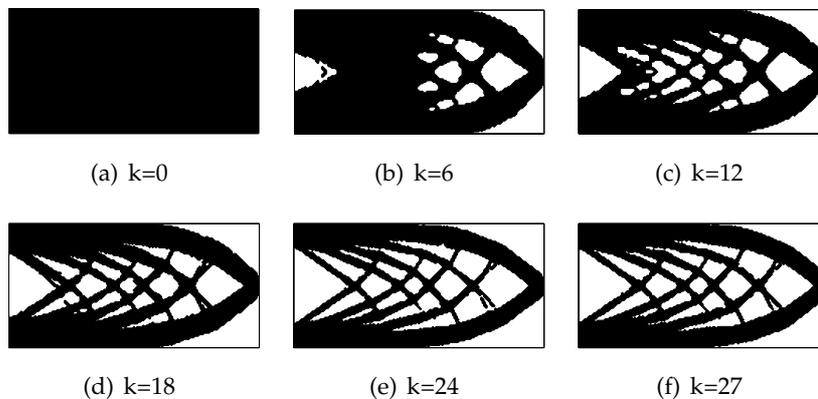


Figure 2: The evolution of the cantilever structure.

Fig. 2 shows the evolution process of the optimal topology of the cantilever beam and Fig. 3 shows the convergent process of the optimization. The same BLS method as PCLS method does not require any re-initialization of the level set function. From Fig. 2 we can clearly see that many new holes are easily produced in the iterative process. It is difficult to achieve in the conventional level set method. The result also confirms the validity of our acceleration of algorithm. Only 27 steps are needed to get the optimal design.

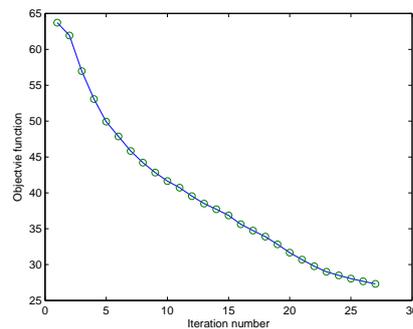


Figure 3: Convergence of the objective function for the cantilever of Fig. 2.

Next, we investigate the effect of acceleration factor  $\theta$  with three different cases (see Fig. 4). The result shows that different values effect not only the topology of the final design but also the speed of convergence. Smaller value leads to smoother boundary, simpler topology and faster convergence. On the contrary, larger value may yield many holes with rough boundary and slower convergence.

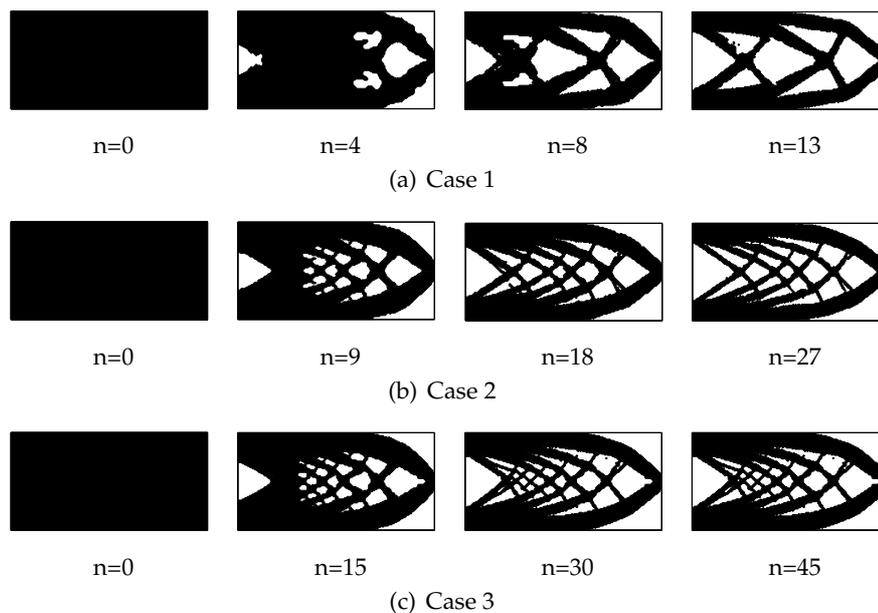


Figure 4: Effect of the different acceleration factor  $\theta$ : (a)  $\theta^0 = 0.92$ ; (b)  $\theta^0 = 0.95$ ; (c)  $\theta^0 = 0.96$ .

## 5.2 Michell-type structure

We consider a Michell-type structure in this example. The working domain is a rectangle of size  $L=2, H=1$  with fixed boundary conditions on the left and right lower edges and a

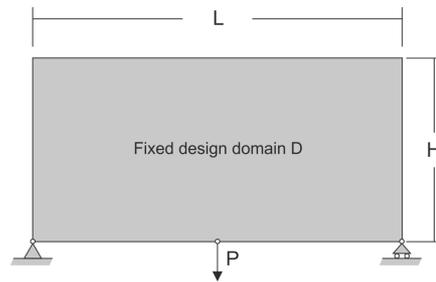


Figure 5: Fixed design domain and boundary condition of a Michell-type structure.

vertical point load  $P=1$  on the center of the bottom (see Fig. 5). The domain is discretized with a  $160 \times 80$  rectangle mesh. The volume fraction of solid material is 40%. We set the regularization parameter  $\beta=0.001$  and acceleration factor  $\theta^0=0.90$  at the beginning. The same initial design as in the cantilever beam structure example is set.

The evolution of optimal topology of the Michell-type structure is showed in Fig. 6. We can see that it converges quickly as it only need 20 iterations for convergence.

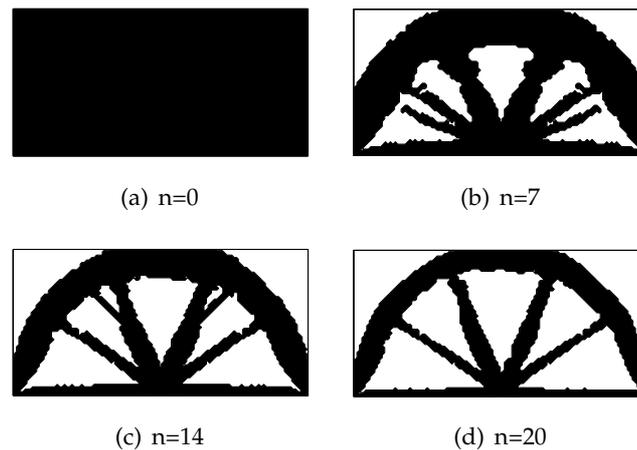


Figure 6: The evolution of the optimal design.

### 5.3 MBB beam

A MBB beam structure optimization problem is considered in this section. The working domain considered is a rectangle of size  $L=4, H=1$  with fixed boundary conditions on the left and right lower edges and a vertical point load  $P=1$  on the center of the top (see Fig. 7). The domain is discretized with a  $240 \times 60$  rectangle mesh. The volume fraction of solid material is 40%. We set the regularization parameter  $\beta=0.001$  and acceleration factor  $\theta^0=0.80$  at the beginning. The initial design is set so as the structure is fully distributed with solid material in the whole design domain.

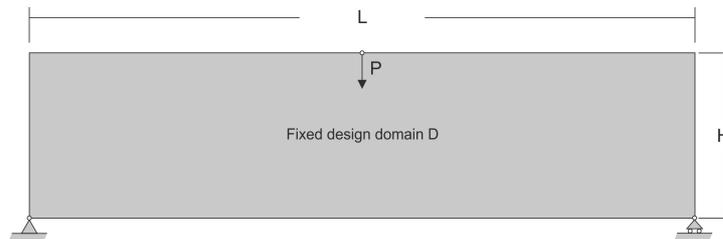


Figure 7: Fixed design domain and boundary condition of a MBB beam.

Fig. 8 shows the evolution of the optimal topology of this MBB beam. The same phenomenon that it converges quickly to the optimal structure as the first two examples can be found in this example.

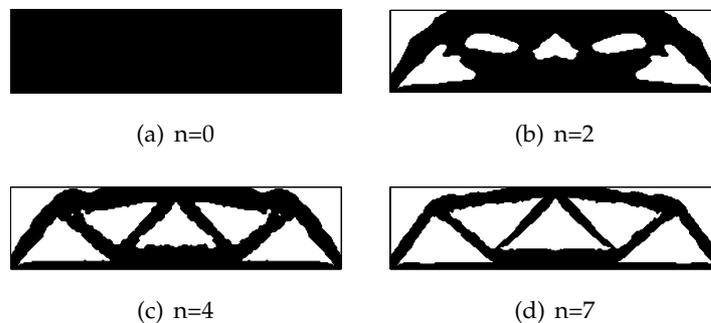


Figure 8: The evolution of the optimal design.

#### 5.4 Michell-type structure with three materials

In this subsection, a numerical example with three materials is presented to confirm the validity of proposed optimization method. At this moment, we only need two level set function  $\phi_1$  and  $\phi_2$  and the material density  $\rho(x)$  is defined by Eq. (2.5). Thus, the structure optimization problem (3.3) is now reformulated as:

$$\begin{aligned} \min \quad & J(u, \phi_1, \phi_2) = \int_D F(u) \rho(\phi) dx + \beta_1 \int_D |\nabla \psi_1| dx + \beta_2 \int_D |\nabla \psi_2| dx + \beta_3 \int_D |\nabla \psi_3| dx \\ \text{s. t.} \quad & G(\phi) = \int_D \psi_i dx - V_i \leq 0, \quad i=1,2,3, \\ & H(\phi) = (\phi_1^2 - 1)(\phi_2^2 - 1) = 0, \end{aligned}$$

where  $\psi_i$  are the characteristic functions of corresponding subregions.

We consider a Michell-type structure with multiple loads at its bottom as show in Fig. 9. The working domain is a rectangular of size  $L=2, H=1$ . The domain is discretized with a  $120 \times 60$  rectangle mesh. In Fig. 9, the structure has a fixed and a simple support

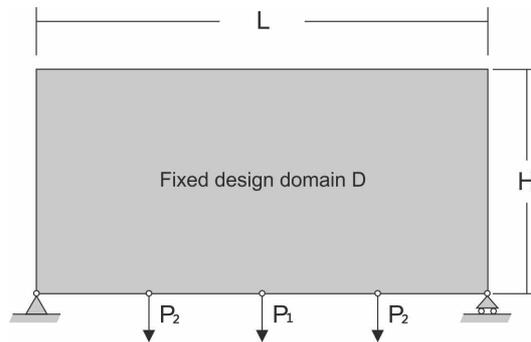


Figure 9: Fixed design domain and boundary condition of a MBB beam.

at the bottom corners with  $P_1 = 30$  and  $P_2 = 5$ . Three materials are used with the modulus of elasticity of  $c_1 = 4$ ,  $c_2 = 2$  and  $c_3 = 1$  separately and with the same Poisson's ratio of 0.3. Their maximum volume ratios are given all as 0.15. We first use Algorithm 4.1 to divide the domain into two parts with one part  $\phi_1 = 1$  and the other  $\phi_1 = -1$ , respectively. Then we use this as the new initial value and use Algorithm 4.1 again to distinguish the other  $\phi_2$  until we obtain the optimal solution. The initial design and some intermediate and the final optimization results are shown in Fig. 10. We further illustrate the changes of the objective function in Fig. 11.

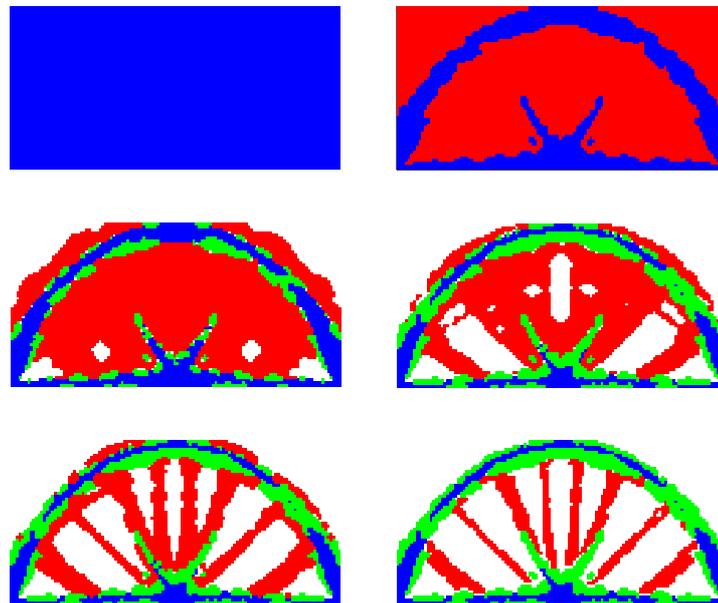


Figure 10: The evolution of the optimal design.

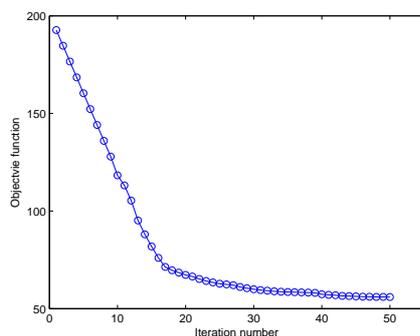


Figure 11: Convergence of the objective function for the cantilever of Fig. 10.

## 6 Conclusions

In this paper, we have presented a variational binary level set method for shape and topology optimization of structural. In this method, different material phases are represented by distinct unions of different level sets. The BLS method has the same advantage as the piecewise constant method. It does not need to re-initialize level set function and can easily create small holes without topological derivatives during the evolution. Compared to the general PCLS method, we need less values of piecewise constant function. This means that BLS method can represent  $2^m$  material phases with  $m$  level set functions, which would substantially reduce the computational complexity. Furthermore, by reducing several parameters to only one parameter, we obtain a numerical technique for efficient and robust implementation of the proposed method. Since we only need to adjust one parameter, we can get the optimal solution much easily and quickly. From numerical computation, we can see that the method we obtained is a mesh-independency scheme and the optimal result is depending on the initial guess of  $\theta$ .

## Acknowledgments

The authors would like to thank the anonymous reviewers for their valuable suggestions on the manuscript. The authors also acknowledge the support of Zhejiang Provincial Natural Science Foundation of China (No. LQ12A01012), Key Project of the Major Research Plan of NSFC (No. 91130004) and the Major Projects on Control and Rectification of Water Body Pollution of China (No. 2009ZX07424-001).

## References

- [1] G. Allaire, F. Jouve, A. M. Toader, Structural optimization using sensitivity analysis and a level-set method, *J. Comput. Phys.*, 194(2004) 363-393.

- [2] M. P. Bendsoe, *Optimization of Structural Topology, Shape and Material*, Springer, Berlin, 1997.
- [3] M. Bendsoe, N. Kikuchi, Generating optimal topologies in structural design using a homogenization method, *Comput. Methods Appl. Mech. Engrg.*, 71(2)(1988) 197-224.
- [4] O. Christiansen, X. C. Tai, Fast implementation of piecewise constant level set methods, *Cam-report-06*, UCLA, *Appl. Math.*, 2005.
- [5] X. C. Tai, H. Li, A piecewise constant level set method for elliptic inverse problems, *Appl. Numer. Math.*, 57(2007) 686-696.
- [6] X. C. Tai, C. H. Yao, Image segmentation by piecewise constant Mumford-Shah model without estimating the constants, *Technical Report 06-18*, Group in Computational and Applied Mathematics, Department of Mathematics, University of California, April 2006.
- [7] A. Z. Diaz, O. Sigmund, Checkerboard patterns in layout optimization, *Struct. Optim.*, 10(1995) 40-45.
- [8] G. S. Jiang, D. P. Peng, Weighted ENO schemes for Hamilton-Jacobi equations, *SIAM J. Sci. Comput.*, 21 (2000) 2126-2143.
- [9] J. Lie, M. Lysaker, X. C. Tai, A binary level set model and some applications to Mumford-Shah image segmentation, *IEEE transactions on image processing*, 15(2006) 1171-1181.
- [10] J. Lie, M. Lysaker, X. C. Tai, A piecewise constant level set framework, *Int. J. Numer. Anal. Model.*, 2 (2005) 422-438.
- [11] J. Lie, M. Lysaker, X. C. Tai, A variant of the level set method and applications to image segmentation, *Math. Comput.*, 75 (2006) 1155-1174.
- [12] H. W. Li, X. C. Tai, Piecewise constant level set method interface problems, *Free boundary problems*, *Int. ser. Numer. Math.*, 154(2007) 307-316.
- [13] L. K. Nielsen, X. C. Tai, S. I. Aanonsen, M. Espedal, A binary level set model for elliptic inverse problems with discontinuous coefficients, *Int. J. Numer. Anal. Model.*, 4 (2007) 74-99.
- [14] S. J. Osher, J. A. Sethian, Front propagating with curvature dependent speed: algorithms based on Hamilton-Jacobi formulations, *J. Comput. Phys.*, 79(1988) 12-49.
- [15] S. J. Osher, C. W. Shu, High-order essentially non-oscillatory schemes for Hamilton-Jacobi equations, *SIAM J. Numer. Anal.*, 28 (1991) 907-922.
- [16] G. Rozvany, *Structural Design via Optimality Criteria*, Kluwer, Dordrecht, 1988.
- [17] G. Rozvany, Aims, scope, methods, history and unified terminology of computer-aided topology optimization in structural mechanics, *Struct. Multidiscip. Optim.*, 21(2)(2001) 90-108.
- [18] R. T. Rockafellar, The multiplier method of Hestenes and Powell applied to convex programming, *J. Optim Theory and Appl.*, 12(1973) 555-562.
- [19] L. I. Rudin, S. J. Osher, E. Fatemi, Nonlinear total variation based noise removal algorithms, *Physica D*, 60(1992) 259-268.
- [20] O. Sigmund, J. Petersson, Numerical instabilities in topology optimization: a survey on procedures dealing with checkerboards, mesh-dependencies and local minima, *Struct. Optim.*, 16(1998) 68-75.
- [21] Y. H. R. Tsai, L. T. Cheng, S. J. Osher, H. K. Zhao, Fast sweeping algorithms for a class of Hamilton-Jacobi equations, *J. Comput. Phys.*, 127(1996) 179-195.
- [22] P. P. Tang, X. X. Dai, Acceleration algorithm of piecewise constant level set method for structural topology optimization, submitted.
- [23] X. C. Tai, H. W. Li, A piecewise constant level set method for elliptic inverse problems, *Appl. Numer. Math.*, 57(2007) 686-696.

- [24] A. Takezawa, S. Nishiwaki, M. Kitamura, Shape and topology optimization based on the phase field method and sensitivity analysis, *J. Comput. Phys.*, 229(2010) 2697-2718.
- [25] L.-L. Wang, Y. Gu, Efficient dual algorithms for image segmentation using TV-Allen-Cahn type models, *Commun. Comput. Phys.*, 9 (2011) 859-877.
- [26] M. Y. Wang, X. M. Wang, "Color" level sets: a multi-phase method for structural topology optimization with multiple materials, *Comput. Methods Appl. Mech. Engrg.*, 193(2004) 469-496.
- [27] M. Y. Wang, X. M. Wang, D. M. Guo, A level set method for structural topology optimization, *Comput. Methods Appl. Mech. Engrg.*, 192(2003) 227-246.
- [28] S. Y. Wang, M. Y. Wang, Structural shape and topology optimization using an implicit free boundary parameterization method, *Comput. Model. in Engrg. Sci.*, 13(2)(2006) 119-147.
- [29] P. Wei, M. Y. Wang, Piecewise constant level set method for structural topology optimization, *Int. J. Numer. Meth. Eng.*, 78 (2009) 379-402.
- [30] Y. M. Xie, G. P. Steven, A simple evolutionary procedure for structural optimization, *Comput. Struct.*, 49(1993) 885-896.
- [31] R. J. Yang, C. H. Chung, Optimal topology design using linear programming, *Comput. Struct.*, 53(1994) 265-275.
- [32] T. Yamada, K. Izui, S. Nishiwaki, A. Takezawa, A topology optimization method based on the level set method incorporating a fictitious interface energy, *Comput. Methods Appl. Mech. Engrg.* (2010), doi:10.1016/j.cma.2010.05.013
- [33] X. Y. Yang, Y. M. Xie, G. P. Steven, O. M. Querin, Topology optimization for frequencies using an evolutionary method, *J. Struct. Engrg.*, 125(1999) 1432-1438.
- [34] Z. F. Zhang, X. L. Cheng, A boundary piecewise constant level set method for boundary control of eigenvalue optimization problems, *J. Comput. Phys.*, 230(2011) 458-473.
- [35] Z. F. Zhang, K.-W. Liang and X. L. Cheng, A monotonic algorithm for eigenvalue optimization in shape design problems of multi-density inhomogeneous materials. *Commun. Comput. Phys.*, 8(2010) 565-584.
- [36] S. F. Zhu, X. X. Dai, C. X. Liu, A variational binary level-set method for elliptic shape optimization problems, to appear in *Int. J. Comput. Math.*