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Second-Order Two-Scale Analysis Method for the Heat Conductive Problem with Radiation Boundary Condition in Periodical Porous Domain

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Abstract. In this paper a second-order two-scale (SOTS) analysis method is developed for a static heat conductive problem in a periodical porous domain with radiation boundary condition on the surfaces of cavities. By using asymptotic expansion for the temperature field and a proper regularity assumption on the macroscopic scale, the cell problem, effective material coefficients, homogenization problem, first-order correctors and second-order correctors are obtained successively. The characteristics of the asymptotic model is the coupling of the cell problems with the homogenization temperature field due to the nonlinearity and nonlocality of the radiation boundary condition. The error estimation is also obtained for the original solution and the SOTS's approximation solution. Finally the corresponding finite element algorithms are developed and a simple numerical example is presented.

AMS subject classifications: 35B27, 74Q05, 83C30, 34E13

Key words: Periodic structure, porous material, radiation boundary condition, second-order two-scale method.

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1 Introduction

Porous materials have many elegant qualities, such as low relative density, heat insulation etc, and have been widely used in high technology engineering. As the materials often have periodic configurations and the coefficients change rapidly in small cells, it is needed to develop new effective numerical methods for predicting the physical and mechanical performance.

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Based on the homogenization method proposed [2–6], the Second-Order Two-Scale Analysis Method (SOTS) is introduced by Cui and Cao [15–18] to predict the physical and mechanical behavior of the materials. By second order corrector, the microscopic fluctuation of physical and mechanical behaviors inside the material can be captured more accurately. In those methods, the original problem can be approximately solved by solving a homogenized problem in original domain without holes and a series of cell problems only in one normalized cell.

As we all known that some extreme conditions are often encountered in the modern engineering. For example, the spacecraft's flying out or reentry into the atmosphere, its surface will bear strong aerodynamic force and heat. Under such conditions, the heat radiation should not be omitted. Because of its nonlinearity, it is difficulty to solve this kind of problems.

In the study of heat transfer model, there are few results concerning the heat radiation. Tiihonen [9] discussed the radiation on non-convex surfaces, and proved the existence and uniqueness of the stationary conduction radiation problem, Bachvalov [7] studied an averaging method on the heat transfer process inside periodic media with radiation and gave the asymptotic expansion of the temperature. Allaire and Ganaoui [8] studied the homogenization method of heat transfer problem with radiation on the surface of the cavities by a scaling hypothesis, and gave the homogenized solutions and first-order two-scale approximate solution, but higher order correctors are not presented.

It should be noted that if substituting the first-order two-scale solution into original equation, one can find that the residual is $\mathcal{O}(1)$ even though H^1 norm of its error is $\mathcal{O}(\varepsilon^{1/2})$. In practical engineering computation, however, ε is a constant less than the structural size L and does not tend to zero. So the local error $\mathcal{O}(1)$ is not accepted for engineer who wants to capture the local behavior of the solution. In this paper, the second-order two-scale approximation solution is discussed even though its convergence order is $\mathcal{O}(\varepsilon^{1/2})$ yet.

The remainder of this paper is organized as follows: The heat transfer model with radiation boundary condition is discussed in Section 2. The second-order two-scale asymptotic analysis for the model is presented in Section 3. The error estimation on the asymptotic solution is analyzed in Section 4. The second-order two-scale algorithm and a simple numerical example are shown in Section 5, followed by conclusions.

Throughout this paper, C (with or without subscripts) denotes a generic positive constant with possibly different values in different contexts. By $\mathcal{O}(\varepsilon^k)$, $k\in\mathbb{N}$, we denote that there exists a constant C independent of ε and $|\mathcal{O}(\varepsilon^k)|\leq C\varepsilon^k$. Also we use convention of summation on repeated indices.

2 Heat transfer model with radiation boundary condition

2.1 Periodical porous materials and Radiative boundary condition

The materials occupy a periodical porous domain in two dimension, let ω be invariant under the shifts by any $z=(z_1,\cdots,z_n)\in\mathbb{Z}^n$.

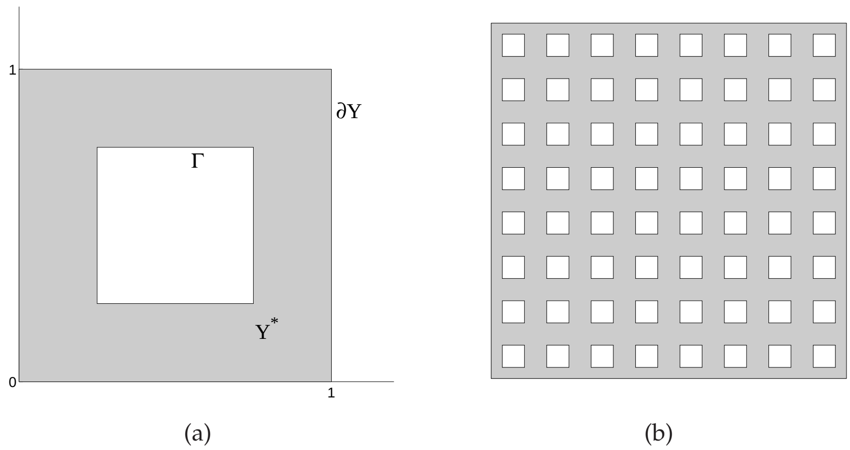

Figure 1: (a) Unit cell Y^*; (b) Periodical porous domain Ω^ε.

Suppose that ω and Y^* satisfy the following conditions:

(A1) ω is a smooth unbounded domain of R^n with a 1-periodic structure.

(A2) As the Fig. 1(a), the cell with periodicity is $Y^*=\omega\bigcap Y$, $Y=[0,1]^n$ is a domain with a Lipschitz boundary ∂Y. The boundary of Y^* is $\partial Y^*=\partial Y\bigcup\Gamma$, where Γ is the surface of the cavity.

(A3) The surface Γ of the cavity does not intersect the boundary ∂Y.

(A4) The porous domain Ω^ε has the form $\Omega^\varepsilon=\Omega\bigcap\varepsilon\omega$, where Ω is a homogeneous convex domain with Lipschitz boundary $\partial\Omega$. Besides we assume Ω^ε is composed of entire cells,

$$\Omega^\varepsilon=\bigcup_{z\in I}(\varepsilon(Y^*+z)),$$

where $I=\{z\in\mathbb{Z}^n|\varepsilon(Y^*+z)\subset\Omega\}$ is the index set. So the boundary of Ω^ε is the union of $\partial\Omega$ and the surface $\Gamma^\varepsilon=\bigcup_{z\in I_z}\Gamma_z^\varepsilon$ of the cavities, i.e.

$$\partial\Omega^\varepsilon=\partial\Omega\bigcup\Gamma^\varepsilon.$$

Let us take the surface Γ in the normalized cell Y^* to consider the radiative boundary condition. Suppose that the surface is not perfectly black, which implies that it partly reflects the projected radiation. To simplify the treatment of reflections, we assume that the surface is a grey-diffuse surface, i.e., it emits, absorbs and reflects radiation in the same manner in all directions. The radiation then is characterized by its emissivity e, $0<e\leq 1$. We also assume that the medium in the cavities is transparent (neither conduction nor absorption of radiation).

Denote the temperature by T and the intensity of emitted radiation by $R(x)$, i.e. the radiosity. For $x\in\Gamma$, $R(x)$ can be expressed as

$$R(x)=e\sigma T^4(x)+(1-e)J(R)(x), \tag{2.1}$$

where σ is the Stefan-Boltzmann constant and J is a functional denoting the energy of projected radiation on Γ. For any $x\in\Gamma$, J can be written by

$$J(R)(x)=\int_\Gamma k(x,s)R(s)d\Gamma_s, \tag{2.2}$$

where $k(x,s)$ is the view factor between two different points x and s of Γ. For convex three dimensional enclosures the formula of view factor is

$$k(x,s)=\frac{n_s\cdot(x-s)n_x\cdot(s-x)}{\pi|s-x|^4}, \tag{2.3}$$

where n_s denotes the unit normal at the point x. In two dimensional case the view factor is

$$k(x,s)=\frac{n_s\cdot(x-s)n_x\cdot(s-x)}{2|s-x|^3}. \tag{2.4}$$

According to [8], there are following results.

Lemma 2.1. *The function k defined in* (2.3) *or* (2.4) *satisfies the following properties: For any $x,s\in\Gamma$*

$$k(x,s)\geq 0,\quad k(x,s)=k(s,x),\quad \int_\Gamma k(x,s)d\Gamma_s=1. \tag{2.5}$$

Lemma 2.2. *The function J defined in* (2.2) *is an operator from $L^p(\Gamma)$ to $L^p(\Gamma)$, $1\leq p\leq\infty$ and has the following properties:*

- $J(c)=c,\ \forall c\in\mathbb{R}.$
- $\|J\|\leq 1.$
- *J is nonnegative: $\forall f\in L^p(\Gamma),\ f>0\rightarrow J(f)\geq 0.$*
- *J is symmetric (self adjoint for $p=2$) in the following sense*

$$\int_\Gamma J(\varphi)\psi d\Gamma=\int_\Gamma J(\psi)\varphi d\Gamma,\quad \forall\varphi\in L^p(\Gamma),\quad \psi\in L^{p'}(\Gamma),\quad \textit{with}\quad \frac{1}{p}+\frac{1}{p'}=1.$$

Denote by q the heat flux transmitted by conduction from Y^* to the cavity. From Tiihonen [9] and Allaire [8], q is expressed as

$$q=G(\sigma T^4)=(I-J)(I-(I-E)J)^{-1}E(\sigma T^4), \tag{2.6}$$

where G is a linear nonlocal operator defined by

$$G(\varphi)=(I-J)(I-(I-E)J)^{-1}E(\varphi),\forall\varphi\in L^p(\Omega). \tag{2.7}$$

E is the operator induced by multiplier e and I is the Identity operator. From the properties of J, G is also symmetric and nonnegative. Particularly, when $e=1$ (black body radiation), q is simply

$$q=(I-J)(\sigma T^4)=\sigma T^4(x)-\sigma\int_\Gamma k(x,s)T^4(s)d\Gamma_s. \tag{2.8}$$

For the radiation condition acting on Γ^ε we denote the operator J_ε and G_ε instead of J and G.

2.2 Conductive-radiative heat transfer problem

The heat conduction model with radiation is firstly studied by Bachvalov [7], in which the radiation boundary condition in a closed cavity is essentially expressed as

$$-a_{ij}(\frac{x}{\varepsilon})\frac{\partial T_\varepsilon}{\partial x_j}n_i=(I-J)(\sigma T_\varepsilon^4)\quad \text{on } \Gamma^\varepsilon, \tag{2.9}$$

and the homogenized solution and the first order correctors are obtained successively. Allaire [8] considered the radiation condition with ε^{-1} scaling on Γ^ε such that the radiation behavior can be represented in the homogenized problem. The homogenized equation obtained is like Rosseland equation [10,20] which is a conductive-radiative model for the high porosity material. In this paper the heat conductive-radiative model with ε^{-1} scaling is also considered, even though it is not completely correct for the materials with low porosity.

The static heat conduction problem with ε^{-1} scaling radiation boundary condition on the surface of cavities reads as follows:

$$\begin{cases} -\dfrac{\partial}{\partial x_i}\Big(a_{ij}\Big(\dfrac{x}{\varepsilon}\Big)\dfrac{\partial T_\varepsilon}{\partial x_j}\Big)=f & \text{in } \Omega^\varepsilon, \\ -a_{ij}\Big(\dfrac{x}{\varepsilon}\Big)\dfrac{\partial T_\varepsilon}{\partial x_j}n_i=\varepsilon^{-1}G_\varepsilon(\sigma T_\varepsilon^4) & \text{on } \Gamma^\varepsilon, \\ T_\varepsilon=\widetilde{T} & \text{on } \partial\Omega, \end{cases} \tag{2.10}$$

where T_ε is temperature field, $\widetilde{T}$ is the prescribed temperature on $\partial\Omega$, f is the source term, $\mathbf{n}=(n_i)$ is the outward unit normal to Ω^ε and ε is a small scale parameter.

Suppose that the following conditions are satisfied:

(B1) Let $y=\frac{x}{\varepsilon}$, the coefficients $a_{ij}(y)$ be 1-periodic in y.

(B2) For $v\in\mathbb{R}^n$, there exist $0<\mu_1\le\mu_2$ such that $\mu_1|v|^2\le a_{ij}(y)v_iv_j\le\mu_2|v|^2$.

(B3) $a_{ij}(y)=a_{ji}(y)$.

(B4) $a_{ij}(y)\in L^\infty(\Omega^\varepsilon)$.

(B5) $f\in L^\infty(\Omega)$, $f\ge 0$, still by f we denote its restriction on Ω^ε.

(B6) $\widetilde{T}\in C^{0,1}(\bar{\Omega})$, $\widetilde{T}\in[\tau_1,\tau_2]$. τ_1,τ_2 are two constants and $\tau_1>0$.

Define the following spaces

$$V(\Omega^\varepsilon)=\big\{v^\varepsilon\in H^1(\Omega^\varepsilon)\big|\, v^\varepsilon=\widetilde{T} \text{ on } \partial\Omega\big\}, \tag{2.11a}$$

$$V_0(\Omega^\varepsilon)=\big\{v^\varepsilon\in H^1(\Omega^\varepsilon)\big|\, v^\varepsilon=0 \text{ on } \partial\Omega\big\}. \tag{2.11b}$$

We can see that $T_\varepsilon\in V(\Omega^\varepsilon)$ and the weak form of (2.10) can be obtained as

$$\int_{\Omega^\varepsilon}a_{ij}(\frac{x}{\varepsilon})\frac{\partial T_\varepsilon}{\partial x_j}\frac{\partial\varphi^\varepsilon}{\partial x_i}dx+\varepsilon^{-1}\sigma\int_{\Gamma^\varepsilon}T_\varepsilon^4G_\varepsilon(\varphi^\varepsilon)d\Gamma^\varepsilon=\int_{\Omega^\varepsilon}f\varphi^\varepsilon dx,\quad \forall\varphi^\varepsilon\in V_0(\Omega^\varepsilon). \tag{2.12}$$

Theorem 2.1. *Suppose the conditions* (B1)-(B6) *are satisfied, there exists a unique solution* $T_\varepsilon \in V(\Omega^\varepsilon)$.

Proof. We refer to [9] for the proof. □

From this weak form, Allaire and Ganaoui [8,12] use the two-scale convergence method to obtain the homogenized problem and effective conductivity. In this paper based on the form (2.10) we apply the SOTS to obtaining the asymptotic solutions for the black body radiation, i.e. $e=1$. Note that the main difficulty is the nonlinear and nonlocal radiation boundary condition, in the next section we will make use of Taylor expansion on Γ^ε to overcome it. The advantage to use this method is that we can find the higher correctors and obtain a more accurate solution.

3 Second-order two-scale asymptotic analysis

3.1 Asymptotic expansion for the temperature field and Taylor expansion on the boundary

Now we use the SOTS method to analyze the problem (2.10). Firstly assume that the temperature T_ε can be formally expanded as follows:

$$T_\varepsilon(x)=T(x,\frac{x}{\varepsilon})=T(x,y)=T_0(x)+\varepsilon T_1(x,y)+\varepsilon^2 T_2(x,y)+\mathcal{O}(\varepsilon^3), \tag{3.1}$$

where x is in non-perforated domain Ω, y is in Y^* and $T_0(x)$ is homogenized solution that does not depend on the microscopic variable y. For T_ε^4, we have

$$\begin{aligned} T_\varepsilon^4 &= (T_0+\varepsilon T_1+\varepsilon^2 T_2+\cdots)^4 \\ &= T_0^4+\varepsilon(4T_0^3T_1)+\varepsilon^2(6T_0^2T_1^2+4T_0^3T_2)+\mathcal{O}(\varepsilon^3). \end{aligned} \tag{3.2}$$

Respecting

$$\frac{\partial T_\varepsilon}{\partial x_i}(x)=\frac{\partial T}{\partial x_i}(x,y)+\frac{1}{\varepsilon}\frac{\partial T}{\partial y_i}(x,y), \tag{3.3}$$

and denoting by A_ε the operator

$$A_\varepsilon=-\frac{\partial}{\partial x_i}\Big(a_{ij}(y)\frac{\partial}{\partial x_j}\Big), \tag{3.4}$$

consequently from (3.3) we can write

$$-\frac{\partial}{\partial x_i}\Big(a_{ij}(y)\frac{\partial T_\varepsilon}{\partial x_j}\Big)=A_\varepsilon T_\varepsilon=(\varepsilon^{-2}A_0+\varepsilon^{-1}A_1+A_2)T(x,y), \tag{3.5}$$

where

$$\begin{cases} A_0 = -\dfrac{\partial}{\partial y_i}\Big(a_{ij}(y)\dfrac{\partial}{\partial y_j}\Big), \\ A_1 = -\dfrac{\partial}{\partial y_i}\Big(a_{ij}(y)\dfrac{\partial}{\partial x_j}\Big) - \dfrac{\partial}{\partial x_i}\Big(a_{ij}(y)\dfrac{\partial}{\partial y_j}\Big), \\ A_2 = -\dfrac{\partial}{\partial x_i}\Big(a_{ij}(y)\dfrac{\partial}{\partial x_j}\Big). \end{cases} \tag{3.6}$$

Using (3.1), (3.5) and equating the power-like terms of ε, we can obtain the following system of equations:

$$A_0 T_0 = 0, \tag{3.7a}$$
$$A_0 T_1 + A_1 T_0 = 0, \tag{3.7b}$$
$$A_0 T_2 + A_1 T_1 + A_2 T_0 = f, \tag{3.7c}$$
$$A_0 T_3 + A_1 T_2 + A_2 T_1 = 0, \ \cdots. \tag{3.7d}$$

Denote by B_ε the operator

$$B_\varepsilon = -a_{ij}(y)\frac{\partial}{\partial x_j} n_i, \tag{3.8}$$

from (3.3), we can write

$$B_\varepsilon = \varepsilon^{-1} B_0 + B_1, \tag{3.9}$$

where

$$B_0 = -a_{ij}(y)\frac{\partial}{\partial y_j} n_i, \qquad B_1 = -a_{ij}(y)\frac{\partial}{\partial x_j} n_i. \tag{3.10}$$

Our aim is to expand the operator G_ε in ε power series. On one cavity Γ_i^ε, G_ε reads as

$$G_\varepsilon(\sigma T_\varepsilon^4) = \sigma(I-J)(T_\varepsilon^4) = \sigma\Big(T_\varepsilon^4(x) - \int_{\Gamma_i^\varepsilon} k(x,s) T_\varepsilon^4(s) d\Gamma_s^\varepsilon\Big). \tag{3.11}$$

Note that $T_\varepsilon^4(s)$ in the integrand is different from $T_\varepsilon^4(x)$ in that s is on Γ_ε other than x. We expand this term as in (3.2)

$$\begin{aligned} T_\varepsilon^4(s) = & T_0^4(s) + \varepsilon(4T_0^3(s) T_1(s,\lambda)) \\ & + \varepsilon^2(6T_0^2(s) T_1^2(s,\lambda) + 4T_0^3(s) T_2(s,\lambda)) + \mathcal{O}(\varepsilon^3), \end{aligned} \tag{3.12}$$

where $\lambda = \frac{s}{\varepsilon}$. It is observed that double integration (both for s and λ) has to be performed. From the idea of [7, 8], to separate the microscopic variable — s from the macroscopic variable — λ, we assume that T_ε is smooth enough in Ω, and make the Taylor expansion on x for the terms of $T_\varepsilon^4(s)$ as follows:

$$
\begin{aligned}
&T_0^4(s)=T_0^4(x)+4T_0^3\frac{\partial T_0(x)}{\partial x_i}(s_i-x_i)+\frac{\partial}{\partial x_j}\Big(2T_0^3(x)\frac{\partial T_0(x)}{\partial x_i}\Big)(s_i-x_i)(s_j-x_j)+\cdots,\\
&4T_0^3(s)T_1(s,\lambda)=4T_0(x)^3T_1(x,\lambda)+\frac{\partial}{\partial x_i}\big(4T_0^3(x)T_1(x,\lambda)\big)(s_i-x_i)+\cdots,\\
&6T_0^2(s)T_1^2(s,\lambda)+4T_0^3(s)T_2(s,\lambda)=6T_0^2(x)T_1^2(x,\lambda)+4T_0^3(x)T_2(x,\lambda)+\cdots.
\end{aligned}
$$

Taking these into (3.12) and making a change of variable from s to λ such that Γ_i^ε on Ω^ε becomes Γ on cell Y^ε, we rewrite the expansion of $\int_{\Gamma_i^\varepsilon}k(x,s)T_\varepsilon^4(s)d\Gamma_s^\varepsilon$ as

$$
\int_{\Gamma_i^\varepsilon}k(x,s)T_\varepsilon^4(s)d\Gamma_s^\varepsilon=\int_\Gamma k(y,\lambda)(T_0^4(x)+\varepsilon T_{1,\lambda}^4+\varepsilon^2T_{2,\lambda}^4)d\Gamma_\lambda+\mathcal{O}(\varepsilon^3), \tag{3.13}
$$

where

$$
\begin{aligned}
T_{1,\lambda}^4(x,\lambda)=&4T_0^3(x)\Big(T_1(x,\lambda)+\frac{\partial T_0(x)}{\partial x_i}(\lambda_i-y_i)\Big), &(3.14a)\\
T_{2,\lambda}^4(x,\lambda)=&6T_0^2(x)T_2^2(x,\lambda)+4T_0^3(x)T_1(x,\lambda)+\frac{\partial}{\partial x_i}\big(4T_0^3(x)T_1(x,\lambda)\big)(\lambda_i-y_i)\\
&+\frac{\partial}{\partial x_i}\Big(2T_0^3(x)\frac{\partial T_0(x)}{\partial x_j}\Big)(\lambda_i-y_i)(\lambda_j-y_j). &(3.14b)
\end{aligned}
$$

Substituting them into (2.10), then

$$
\begin{aligned}
B_\varepsilon T_\varepsilon=&\sigma\Big\{4T_0^3(x)T_1(x,y)-\int_\Gamma k(y,\lambda)T_{1,\lambda}^4(x,\lambda)d\Gamma_\lambda\Big\}\\
&+\varepsilon\sigma\Big\{6T_0^2(x)T_1^2(x,y)+4T_0^3(x)T_2(x,y)\Big\}-\varepsilon\sigma\int_\Gamma k(y,\lambda)T_{2,\lambda}^4(x,\lambda)d\Gamma_\lambda. \qquad (3.15)
\end{aligned}
$$

Using (3.1) and (3.9) and also equating the power-like terms of ε, we can obtain

$$
\begin{aligned}
&B_0T_0=0, &(3.16a)\\
&B_0T_1+B_1T_0=\sigma\Big\{4T_0^3(x)T_1(x,y)-\int_\Gamma k(y,\lambda)T_{1,\lambda}^4(x,\lambda)d\Gamma_\lambda\Big\}, &(3.16b)\\
&B_0T_2+B_1T_1=\sigma\Big\{6T_0^2(x)T_1^2(x,y)+4T_0^3(x)T_2(x,y)\Big\}\\
&\qquad\qquad\qquad-\sigma\int_\Gamma k(y,\lambda)T_{2,\lambda}^4(x,\lambda)d\Gamma_\lambda,\ \cdots. &(3.16c)
\end{aligned}
$$

From (3.7a)-(3.7d) and (3.16a)-(3.16c), $T_0(x)$ has already satisfied (3.7a) and (3.16a), so it is sufficient to solve the following systems of equations:

$$\begin{cases} -\dfrac{\partial}{\partial y_i}\Big(a_{ij}\dfrac{\partial T_1}{\partial y_j}\Big)=\dfrac{\partial}{\partial x_i}\Big(a_{ij}\dfrac{\partial T_0}{\partial y_j}\Big)+\dfrac{\partial}{\partial y_i}\Big(a_{ij}\dfrac{\partial T_0}{\partial x_j}\Big) & \text{in } Y^*,\\ -a_{ij}\dfrac{\partial T_1}{\partial y_j}n_i=a_{ij}\dfrac{\partial T_0}{\partial x_j}n_i+\sigma 4T_0^3T_1-\sigma\displaystyle\int_\Gamma k(y,\lambda)T_{1,\lambda}^4 d\Gamma_\lambda & \text{on } \Gamma,\\ T_1(x,y)\ \text{1-periodic in } y, \end{cases} \tag{3.17}$$

$$\begin{cases} -\dfrac{\partial}{\partial y_i}\Big(a_{ij}\dfrac{\partial T_2}{\partial y_j}\Big)=f+\dfrac{\partial}{\partial y_i}\Big(a_{ij}\dfrac{\partial T_1}{\partial x_j}\Big)+\dfrac{\partial}{\partial x_i}\Big(a_{ij}\Big(\dfrac{\partial T_1}{\partial y_j}+\dfrac{\partial T_0}{\partial x_j}\Big)\Big) & \text{in } Y^*,\\ -a_{ij}\dfrac{\partial T_2}{\partial y_j}n_i=a_{ij}\dfrac{\partial T_1}{\partial x_j}n_i+\sigma(6T_0^2T_1^2+4T_0^3T_2)-\sigma\displaystyle\int_\Gamma k(y,\lambda)T_{2,\lambda}^4 d\Gamma_\lambda & \text{on } \Gamma,\\ T_2(x,y)\ \text{1-periodic in } y. \end{cases} \tag{3.18}$$

3.2 Cell problem, homogenized problem and homogenized conductivity

From (3.17), since $T_0(x)$ is independent of y, it can be rewritten as

$$\begin{cases} -\dfrac{\partial}{\partial y_i}\Big(a_{ij}\dfrac{\partial T_1}{\partial y_j}\Big)=\dfrac{\partial a_{ij}}{\partial y_i}\dfrac{\partial T_0}{\partial x_j} & \text{in } Y^*,\\ -a_{ij}\dfrac{\partial T_1}{\partial y_j}n_i=a_{ij}\dfrac{\partial T_0}{\partial x_j}n_i+4\sigma T_0^3T_1(x,y) & \\ \qquad\qquad -4\sigma T_0^3\displaystyle\int_\Gamma k(y,\lambda)\Big(T_1(x,\lambda)+\dfrac{\partial T_0}{\partial x_i}(\lambda_i-y_i)\Big)d\Gamma_\lambda & \text{on } \Gamma,\\ T_1(x,y)\ \text{1-periodic in } y. \end{cases} \tag{3.19}$$

Setting $T_1(x,y)$ formally

$$T_1(x,y)=N_{\alpha_1}(T_0,y)\frac{\partial T_0}{\partial x_{\alpha_1}}, \tag{3.20}$$

where $\alpha_1=1,\cdots,n$ and substituting it into (3.19), we have

$$\begin{cases} -\dfrac{\partial}{\partial y_i}\Big(a_{ij}\dfrac{\partial N_{\alpha_1}}{\partial y_j}\Big)\dfrac{\partial T_0}{\partial x_{\alpha_1}}=\dfrac{\partial a_{i\alpha_1}}{\partial y_i}\dfrac{\partial T_0}{\partial x_{\alpha_1}} & \text{in } Y^*,\\ -a_{ij}\dfrac{\partial N_{\alpha_1}}{\partial y_j}n_i\dfrac{\partial T_0}{\partial x_{\alpha_1}}=a_{i\alpha_1}n_i\dfrac{\partial T_0}{\partial x_{\alpha_1}}+4\sigma T_0^3N_{\alpha_1}\dfrac{\partial T_0}{\partial x_{\alpha_1}} & \\ \qquad\qquad -4\sigma T_0^3\displaystyle\int_\Gamma k(y,\lambda)(N_{\alpha_1}+\lambda_{\alpha_1}-y_{\alpha_1})d\Gamma_\lambda\dfrac{\partial T_0}{\partial x_{\alpha_1}} & \text{on } \Gamma,\\ T_1(x,y)\ \text{1-periodic in } y. \end{cases} \tag{3.21}$$

For the boundary condition on Γ, from Lemma 2.1

$$
\begin{aligned}
&4\sigma T_0^3 N_{\alpha_1}\frac{\partial T_0}{\partial x_{\alpha_1}}-4\sigma T_0^3\int_\Gamma k(y,\lambda)(N_{\alpha_1}+\lambda_{\alpha_1}-y_{\alpha_1})d\Gamma_\lambda\frac{\partial T_0}{\partial x_{\alpha_1}}\\
=&4\sigma T_0^3\frac{\partial T_0}{\partial x_{\alpha_1}}\Big(N_{\alpha_1}-\int_\Gamma k(y,\lambda)(N_{\alpha_1}+\lambda_{\alpha_1}-y_{\alpha_1})d\Gamma_\lambda\Big)\\
=&4\sigma T_0^3\frac{\partial T_0}{\partial x_{\alpha_1}}\Big(N_{\alpha_1}+y_{\alpha_1}-\int_\Gamma k(y,\lambda)(N_{\alpha_1}+\lambda_{\alpha_1})d\Gamma_\lambda\Big)\\
=&4\sigma T_0^3\frac{\partial T_0}{\partial x_{\alpha_1}}(I-J)(N_{\alpha_1}+y_{\alpha_1})=4\sigma T_0^3\frac{\partial T_0}{\partial x_{\alpha_1}}G(N_{\alpha_1}+y_{\alpha_1}),
\end{aligned}
\tag{3.22}
$$

then we define $N_{\alpha_1}(T_0,y)$ that satisfies the following cell problem

$$
\begin{cases}
-\dfrac{\partial}{\partial y_i}\Big(a_{ij}\dfrac{\partial N_{\alpha_1}}{\partial y_j}\Big)=\dfrac{\partial a_{i\alpha_1}}{\partial y_i} & \text{in } Y^*,\\
-a_{ij}\dfrac{\partial N_{\alpha_1}}{\partial y_j}n_i=a_{i_{\alpha_1}}n_i+4\sigma T_0^3 G(N_{\alpha_1}+y_{\alpha_1}) & \text{on } \Gamma,\\
N_{\alpha_1}(T_0,y)\ \text{1-periodic in } y.
\end{cases}
\tag{3.23}
$$

Theorem 3.1. *For $T_0>0$, the cell problem* (3.23) *admits a unique solution $N_{\alpha_1}(T_0,y)$ in $W_{per}(Y^*)$, where*

$$
W_{per}(Y^*)=\Big\{v\in H^1_{per}(Y^*);\int_{Y^*}vdy=0\Big\},
\tag{3.24}
$$

and $H^1_{per}(Y)$ is the closure of subset $C^\infty(\mathbb{R}^n)$ of 1-periodic functions for the H^1-norm.

Proof. The variational problem of (3.23) is

$$
\begin{cases}
\text{find } N_{\alpha_1}(T_0,y)\in W_{per}(Y^*) \text{ such that}\\
a_{Y^*}(N_{\alpha_1},\varphi)=(g_1(T_0),\varphi)\quad \forall\varphi\in W_{per}(Y^*),
\end{cases}
\tag{3.25}
$$

where

$$
a_{Y^*}(N_{\alpha_1},\varphi)=\int_{Y^*}a_{ij}\frac{\partial N_{\alpha_1}}{\partial y_j}\frac{\partial\varphi}{\partial y_i}dy+4\sigma T_0^3\int_\Gamma G(N_{\alpha_1})\varphi d\Gamma_y,
\tag{3.26a}
$$

$$
(g_1(T_0),\varphi)=-\int_{Y^*}a_{i_{\alpha_1}}\frac{\partial\varphi}{\partial y_i}dy-4\sigma T_0^3\int_\Gamma G(y_{\alpha_1})\varphi d\Gamma_y.
\tag{3.26b}
$$

Since $T_0>0$ and $\|J\|\le 1$, we have

$$
\begin{aligned}
a_{Y^*}(\varphi,\varphi)=&\int_{Y^*}a_{ij}\frac{\partial\varphi}{\partial y_j}\frac{\partial\varphi}{\partial y_i}dy+4\sigma T_0^3\int_\Gamma G(\varphi)\varphi d\Gamma_y\\
&\ge\mu_1\|\varphi\|^2_{H^1(Y^*)}+4\sigma T_0^3\int_\Gamma (I-J)(\phi)\phi d\Gamma_y\ge\mu_1\|\varphi\|^2_{H^1(Y^*)}.
\end{aligned}
\tag{3.27}
$$

Hence a_{Y^*} is coercive on Y^*. By Lax-Milgram theorem, (3.23) admits a unique solution $N_{\alpha_1}(T_0,y)$ in $W_{per}(Y^*)$. ∎

For the problem (3.18), we make a integral average over Y^*. Since $T_1(x,y)$ has the form (3.20) and $T_2(x,y)$ is 1-periodic in y, we get

$$-\frac{\partial}{\partial x_i}\left\{\frac{1}{|Y^*|}\int_{Y^*}\Big(a_{ij}+a_{ik}\frac{\partial N_j}{\partial y_k}\Big)dy\frac{\partial T_0}{\partial x_j}\right\}+I_1+I_2+I_3=f, \tag{3.28}$$

where

$$\begin{aligned}
I_1&=\frac{\sigma}{|Y^*|}\int_\Gamma\Big\{6T_0^2T_1^2+4T_0^3T_2-\int_\Gamma k(y,\lambda)(6T_0^2T_1^2(x,\lambda)+4T_0^3T_2(x,\lambda))d\Gamma_\lambda\Big\}d\Gamma_y,\\
I_2&=-\frac{\sigma}{|Y^*|}\int_\Gamma\int_\Gamma k(y,\lambda)\frac{\partial}{\partial x_i}(4T_0^3T_1(x,\lambda)(\lambda_i-y_i))d\Gamma_\lambda d\Gamma_y,\\
I_3&=\frac{\sigma}{|Y^*|}\int_\Gamma\int_\Gamma k(y,\lambda)\frac{\partial}{\partial x_i}(2T_0^3\frac{\partial T_0}{\partial x_j})(\lambda_i-y_i)(\lambda_i-y_j)d\Gamma_\lambda d\Gamma_y.
\end{aligned}$$

From the properties of $k(y,\lambda)$ in (2.1)

$$\begin{aligned}
I_1=&0,\\
I_2=&-\frac{\sigma}{|Y^*|}\int_\Gamma\int_\Gamma k(y,\lambda)N_j(T_0(x),\lambda)(\lambda_i-y_i)d\Gamma_\lambda d\Gamma_y\frac{\partial}{\partial x_i}\Big(4T_0^3\frac{\partial T_0}{\partial x_j}\Big)\\
&-\frac{\sigma}{|Y^*|}\int_\Gamma\int_\Gamma k(y,\lambda)\frac{\partial N_j(T_0(x),\lambda)}{\partial x_i}(\lambda_i-y_i)d\Gamma_\lambda d\Gamma_y 4T_0^3\frac{\partial T_0}{\partial x_j}\\
=&\frac{\sigma}{|Y^*|}\int_\Gamma y_i\Big[N_j(x,y)-\int_\Gamma(k(y,\lambda)N_j(T_0(x),\lambda))d\Gamma_\lambda\Big]d\Gamma_y\frac{\partial}{\partial x_i}\Big(4T_0^3\frac{\partial T_0}{\partial x_j}\Big)\\
&+\frac{\sigma}{|Y^*|}\int_\Gamma y_i\Big[\frac{\partial N_j(T_0(x),y)}{\partial x_i}-\int_\Gamma(k(y,\lambda)\frac{\partial N_j(T_0(x),\lambda)}{\partial x_i})d\Gamma_\lambda\Big]d\Gamma_y 4T_0^3\frac{\partial T_0}{\partial x_j}\\
=&\frac{\sigma}{|Y^*|}\int_\Gamma y_iG(N_j(T_0(x),y))d\Gamma_y\frac{\partial}{\partial x_i}\Big(4T_0^3\frac{\partial T_0}{\partial x_j}\Big)\\
&+\frac{\sigma}{|Y^*|}\int_\Gamma y_iG\Big(\frac{\partial N_j(T_0(x),y)}{\partial x_i}\Big)d\Gamma_y 4T_0^3\frac{\partial T_0}{\partial x_j},\\
I_3=&\frac{\sigma}{|Y^*|}\frac{\partial}{\partial x_i}\Big(4T_0^3\frac{\partial T_0}{\partial x_j}\Big)\int_\Gamma y_i\Big[y_j-\int_\Gamma k(y,\lambda)\lambda_j d\Gamma_\lambda\Big]d\Gamma_y\\
=&\frac{\sigma}{|Y^*|}\int_\Gamma y_iG(y_j)d\Gamma_y\frac{\partial}{\partial x_i}\Big(4T_0^3\frac{\partial T_0}{\partial x_j}\Big).
\end{aligned}$$

Take I_1, I_2, I_3 into (3.28) and then we obtain the homogenized problem

$$\begin{cases}
-\dfrac{\partial}{\partial x_i}\Big(a_{ij}^0(T_0)\dfrac{\partial T_0}{\partial x_j}\Big)=f & \text{in } \Omega,\\
T_0=\widetilde{T} & \text{on } \partial\Omega,
\end{cases} \tag{3.29}$$

where the homogenized conductivity $a_{ij}^0(T_0)$ is

$$a_{ij}^0(T_0)=\frac{1}{|Y^*|}\left[\int_{Y^*}\left(a_{ij}+a_{ik}\frac{\partial N_j}{\partial y_k}\right)dy+4\sigma T_0^3\int_{\Gamma}y_iG(N_j+y_j)d\Gamma_y\right]. \tag{3.30}$$

Theorem 3.2. *For $T_0>0$, the homogenized conductivity $a_{ij}^0(T_0)$ is symmetric and coercive on Ω.*

Proof. In fact, from (3.23), taking $\varphi=N_i$ into the variational form (3.25), we have

$$\int_{Y^*}a_{mn}\frac{\partial N_i}{\partial y_m}\frac{\partial(N_j+y_j)}{\partial y_n}dy+4\sigma T_0^3\int_{\Gamma}N_iG(N_j+y_j)d\Gamma_y=0.$$

Take the expression into (3.30)

$$a_{ij}^0(T_0)=\frac{1}{|Y^*|}\left[\int_{Y^*}a_{mn}\frac{\partial(N_i+y_i)}{\partial y_m}\frac{\partial(N_j+y_j)}{\partial y_n}dy+4\sigma T_0^3\int_{\Gamma}(N_i+y_i)G(N_j+y_j)d\Gamma_y\right]. \tag{3.31}$$

From the symmetry and nonnegativity of G, it follows that

$$a_{ij}^0(T_0)=a_{ji}^0(T_0),\quad a_{ji}^0(T_0)\xi_i\xi_j\geq\mu_1|\xi|^2,\quad \forall\xi=(\xi_1,\cdots,\xi_n)\in\mathbb{R}^n. \tag{3.32}$$

This completes the proof of the theorem. □

It directly follows that

Theorem 3.2.1. *For $f\in L^\infty(\Omega)$, $f\geq 0$, the homogenized problem* (3.29) *admits a unique solution in $V(\Omega)=\{\varphi\in H^1(\Omega)|\varphi=\widetilde{T}$ on $\partial\Omega\}$.*

From the cell problem (3.23) and homogenized problem (3.29), it is observed that they are coupling with each other. In the sequel we will prove the existence of the coupling system $(N_{\alpha_1}(T_0,y)$, $T_0(x))$.

According to [12], we have the following two lemmas.

Lemma 3.1. *For $T_0\to\infty$, N_{α_1} is approaching to $N_{\alpha_1}^0$, which satisfies the following limit problem*

$$\begin{cases} -\dfrac{\partial}{\partial y_i}\left(a_{ij}(y)\dfrac{\partial N_{\alpha_1}^0}{\partial y_j}\right)=\dfrac{\partial a_{i\alpha_1}}{\partial y_i} & \text{in } Y^*,\\ N_{\alpha_1}^0=-y_{\alpha_1} & \text{on } \Gamma,\\ N_{\alpha_1}^0(T_0,y)\ \text{1-periodic in } y. \end{cases} \tag{3.33}$$

Lemma 3.2. *For $T_0\to\infty$, the homogenized conductivity $a_{ij}^0(T_0)$ has the following limit*

$$\lim_{T_0\to\infty}a_{ij}^0(T_0)=\frac{1}{|Y^*|}\int_{Y^*}a_{mn}\frac{\partial(N_i^0+y_i)}{\partial y_m}\frac{\partial(N_j^0+y_j)}{\partial y_n}dy. \tag{3.34}$$

In the sequel we always consider $T_0(x)$ in a closed positive interval, then it is obvious that both $N_{\alpha_1}(T_0,y)$ and $a_{ij}^0(T_0)$ are uniformly bounded.

Lemma 3.3. *For two different T_{k_1}, $T_{k_2}\in[M_1,M_2]$, where M_1 and M_2 are two constants and $M_1>0$, we have*

(a) *Two cell solutions $N_{\alpha_1}(T_{k_1},y)$, $N_{\alpha_1}(T_{k_2},y)$ satisfy*

$$\|N_{\alpha_1}(T_{k_1},y)-N_{\alpha_1}(T_{k_2},y)\|_{H^1(Y^*)}\leq C\|T_{k_1}-T_{k_2}\|_{C^0(\Omega)}, \tag{3.35}$$

(b) *The homogenized coefficients $a_{ij}^0(T_{k_1})$, $a_{ij}^0(T_{k_2})$ satisfy*

$$|a_{ij}^0(T_{k_1})-a_{ij}^0(T_{k_2})|\leq C\|T_{k_1}-T_{k_2}\|_{C^0(\Omega)}, \tag{3.36}$$

where the constant C is independent y, T_{k_1} and T_{k_2}.

Proof. (a) For T_{k_1} and T_{k_2}, we have the following two weak forms

$$a_{Y^*}(N_{\alpha_1}(T_{k_1},y),\varphi)=(g_1(T_{k_1},\varphi)\quad\forall\varphi\in W_{per}(Y^*), \tag{3.37a}$$
$$a_{Y^*}(N_{\alpha_1}(T_{k_2},y),\varphi)=(g_1(T_{k_2},\varphi)\quad\forall\varphi\in W_{per}(Y^*). \tag{3.37b}$$

By subtracting of the two equations and taking $\varphi=N_{\alpha_1}(T_{k_1},y)-N_{\alpha_1}(T_{k_2},y)$, we obtain

$$\begin{aligned}
&\int_{Y^*}a_{ij}\frac{\partial(N_{\alpha_1}(T_{k_1},y)-N_{\alpha_1}(T_{k_2},y))}{\partial y_j}\frac{\partial(N_{\alpha_1}(T_{k_1},y)-N_{\alpha_1}(T_{k_2},y))}{\partial y_i}dy\\
=&-4\sigma(T_{k_1}^3-T_{k_2}^3)\int_\Gamma G(y_{\alpha_1})(N_{\alpha_1}(T_{k_1},y)-N_{\alpha_1}(T_{k_2},y))d\Gamma_y\\
&-4\sigma T_{k_1}^3\int_\Gamma G(N_{\alpha_1}(T_{k_1},y))(N_{\alpha_1}(T_{k_1},y)-N_{\alpha_1}(T_{k_2},y))d\Gamma_y\\
&+4\sigma T_{k_2}^3\int_\Gamma G(N_{\alpha_1}(T_{k_2},y))(N_{\alpha_1}(T_{k_1},y)-N_{\alpha_1}(T_{k_2},y))d\Gamma_y\\
=&-4\sigma(T_{k_1}^3-T_{k_2}^3)\int_\Gamma G(y_j)(N_{\alpha_1}(T_{k_1},y)-N_{\alpha_1}(T_{k_2},y))d\Gamma_y\\
&-4\sigma T_{k_1}^3\int_\Gamma G(N_{\alpha_1}(T_{k_1},y)-N_{\alpha_1}(T_{k_2},y))(N_{\alpha_1}(T_{k_1},y)-N_{\alpha_1}(T_{k_2},y))d\Gamma_y\\
&-4\sigma(T_{k_1}^3-T_{k_2}^3)\int_\Gamma G(N_{\alpha_1}(T_{k_2},y))(N_{\alpha_1}(T_{k_1},y)-N_{\alpha_1}(T_{k_2},y))d\Gamma_y.
\end{aligned}\tag{3.38}$$

Then

$$\begin{aligned}
&\int_{Y^*}a_{ij}\frac{\partial(N_{\alpha_1}(T_{k_1},y)-N_{\alpha_1}(T_{k_2},y))}{\partial y_j}\frac{\partial(N_{\alpha_1}(T_{k_1},y)-N_{\alpha_1}(T_{k_2},y))}{\partial y_i}dy\\
&\quad+4\sigma T_{k_1}^3\int_\Gamma G(N_{\alpha_1}(T_{k_1},y)-N_{\alpha_1}(T_{k_2},y))(N_{\alpha_1}(T_{k_1},y)-N_{\alpha_1}(T_{k_2},y))d\Gamma_y\\
=&-4\sigma(T_{k_1}^3-T_{k_2}^3)\int_\Gamma G(N_{\alpha_1}(T_{k_2},y)+y_j)(N_{\alpha_1}(T_{k_1},y)-N_{\alpha_1}(T_{k_2},y))d\Gamma_y.
\end{aligned}\tag{3.39}$$

Since $T_{k_1}>0$

$$\begin{aligned}&\int_{Y^*} a_{ij}\frac{\partial(N_{\alpha_1}(T_{k_1},y)-N_{\alpha_1}(T_{k_2},y))}{\partial y_j}\frac{\partial(N_{\alpha_1}(T_{k_1},y)-N_{\alpha_1}(T_{k_2},y))}{\partial y_i}dy\\&\quad+4\sigma T_{k_1}^3\int_\Gamma G(N_{\alpha_1}(T_{k_1},y)-N_{\alpha_1}(T_{k_2},y))(N_{\alpha_1}(T_{k_1},y)-N_{\alpha_1}(T_{k_2},y))d\Gamma_y\\&\geq \mu_1\|N_{\alpha_1}(T_{k_1},y)-N_{\alpha_1}(T_{k_2},y)\|^2_{H^1(Y^*)}.\end{aligned}\tag{3.40}$$

From the boundedness of T_{k_1}, T_{k_2}, $N_{\alpha_1}(T_{k_1},y)$ and $N_{\alpha_1}(T_{k_2},y)$

$$\begin{aligned}&-4\sigma(T_{k_1}^3-T_{k_2}^3)\int_\Gamma G(N_{\alpha_1}(T_{k_2},y)+y_{\alpha_1})(N_{\alpha_1}(T_{k_1},y)-N_{\alpha_1}(T_{k_2},y))d\Gamma_y\\&\leq C(\Gamma,M)\|T_{k_1}-T_{k_2}\|_{C^0(\Omega)}\|N_{\alpha_1}(T_{k_1},y)-N_{\alpha_1}(T_{k_2},y)\|_{H^1(Y^*)}.\end{aligned}\tag{3.41}$$

Therefore we obtain the result.

(b) From (3.30) we have

$$\begin{aligned}&(a_{ij}^0(T_{k_1})-a_{ij}^0(T_{k_2}))|Y^*|\\=&\int_{Y^*} a_{ik}\frac{\partial}{\partial y_k}(N_j(T_{k_1},y)-N_j(T_{k_2},y))dy\\&+4\sigma T_{k_1}^3\int_\Gamma y_iG(N_j(T_{k_1},y)+y_j)d\Gamma_y-4\sigma T_{k_2}^3\int_\Gamma y_iG(N_j(T_{k_2},y)+y_j)d\Gamma_y\\=&\int_{Y^*} a_{ik}\frac{\partial}{\partial y_k}(N_j(T_{k_1},y)-N_{\alpha_1}(T_{k_2},y))dy+4\sigma T_{k_1}^3\int_\Gamma y_iG(N_j(T_{k_1},y)-N_j(T_{k_2},y))d\Gamma_y\\&+4\sigma(T_{k_1}^3-T_{k_2}^3)\int_\Gamma y_iG(N_j(T_{k_2},y))d\Gamma_y+4\sigma(T_{k_1}^3-T_{k_2}^3)\int_\Gamma y_iG(y_j)d\Gamma_y.\end{aligned}\tag{3.42}$$

From the boundedness of T_{k_1}, T_{k_2}, $N_j(T_{k_1},y)$ and $N_j(T_{k_2},y)$ and Lemma 3.3, we have

$$\begin{aligned}&\int_{Y^*} a_{ik}\frac{\partial}{\partial y_k}(N_j(T_{k_1},y)-N_j(T_{k_2},y))dy\\&\quad\leq C_1\|N_j(T_{k_1},y)-N_j(T_{k_2},y)\|_{H^1(Y^*)}\leq C_2\|T_{k_1}-T_{k_2}\|_{C^0(\Omega)},\end{aligned}\tag{3.43a}$$

$$\begin{aligned}&4\sigma T_{k_1}^3\int_\Gamma y_iG(N_j(T_{k_1},y)-N_j(T_{k_2},y))d\Gamma_y\\&\quad\leq C_3\|N_j(T_{k_1},y)-N_j(T_{k_2},y)\|_{H^1(Y^*)}\leq C_4\|T_{k_1}-T_{k_2}\|_{C^0(\Omega)},\end{aligned}\tag{3.43b}$$

$$4\sigma(T_{k_1}^3-T_{k_2}^3)\int_\Gamma y_iG(N_j(T_{k_2},y))d\Gamma_y\leq C_5\|T_{k_1}-T_{k_2}\|_{C^0(\Omega)},\tag{3.43c}$$

$$4\sigma(T_{k_1}^3-T_{k_2}^3)\int_\Gamma y_iG(y_j)d\Gamma_y\leq C_6\|T_{k_1}-T_{k_2}\|_{C^0(\Omega)}.\tag{3.43d}$$

Then we have the inequality (3.36). □

To prove the existence of the coupled problem for $N_{\alpha_1}(T_0,y)$ and T_0, we use the following fixed point lemma.

Lemma 3.4 (Corollary 11.2 [13]). *Let $\mathfrak{C}$ be a closed convex set in a Banach space $\mathfrak{B}$ and let $\mathcal{L}$ be a continuous mapping of $\mathfrak{C}$ into itself such that the image $\mathcal{L}\mathfrak{C}$ is precompact. Then $\mathcal{L}$ has a fixed point.*

From the classical global boundedness estimation of elliptic equations we have the following lemma.

Lemma 3.5. *For $v\in[\tau_1,\tau_2+C_0]$, $\forall C_0>0$, $f\in L^\infty(\Omega), f\geq 0$ and $\tau_1\leq\widetilde{T}\leq\tau_2$, ω is the weak solution of the following equation*

$$\begin{cases} -\dfrac{\partial}{\partial x_i}\Big(a_{ij}^0(v)\dfrac{\partial\omega}{\partial x_j}\Big)=f & \text{in } \Omega,\\ \omega=\widetilde{T} & \text{on } \partial\Omega. \end{cases} \tag{3.44}$$

Then we have

$$\tau_1\leq\omega\leq\tau_2+C_1, \tag{3.45}$$

where the constant C_1 is independent of C_0.

Proof. 1. Since $a_{ij}^0(v)$ is symmetric and coercive on Ω, ω exists and satisfies

$$\int_\Omega a_{ij}^0(v)\frac{\partial\omega}{\partial x_j}\frac{\partial\varphi}{\partial x_i}dx=\int_\Omega f\varphi dx \quad \forall x\in H_0^1(\Omega). \tag{3.46}$$

For $a_{ij}^0(v)$ is fixed, (3.44) is simply an elliptic equation. Set

$$(\omega-\tau_1)_-=\begin{cases} \omega-\tau_1, & \omega-\tau_1<0,\\ 0, & \omega-\tau_1\geq 0, \end{cases} \tag{3.47}$$

since $\omega=\widetilde{T}\geq\tau_1$ on $\partial\Omega$, $(\omega-\tau_1)_-\in H_0^1(\Omega)$. Let $\varphi=(\omega-\tau_1)_-$ and use the Poincaré's inequality, we have

$$\int_\Omega a_{ij}^0(v)\frac{\partial(\omega-\tau_1)_-}{\partial x_j}\frac{\partial(\omega-\tau_1)_-}{\partial x_i}dx\geq C\int_\Omega|(\omega-\tau_1)_-|^2dx\geq 0, \tag{3.48a}$$

$$\int_\Omega f(\omega-\tau_1)_-dx\leq 0, \tag{3.48b}$$

which implies $(\omega-\tau_1)_-=0$ *a.e.* in Ω. From the definition of $(\omega-\tau_1)_-$,

$$\omega\geq\tau_1. \tag{3.49}$$

2. Use De Giorgi iteration to estimate the upper bound. Take $k>\tau_2$ and $\varphi=(\omega-k)_+$ with

$$(\omega-k)_+=\begin{cases} \omega-k, & \omega-k>0,\\ 0, & \omega-k\leq 0. \end{cases} \tag{3.50}$$

Since $\omega=\widetilde{T}\leq\tau_2$ on $\partial\Omega$, $(\omega-k)_+\in H_0^1(\Omega)$. From Poincaré's inequality

$$\begin{aligned}&\int_\Omega a_{ij}^0(v)\frac{\partial\omega}{\partial x_j}\frac{\partial(\omega-k)_+}{\partial x_i}dx=\int_\Omega a_{ij}^0(v)\frac{\partial(\omega-k)_+}{\partial x_j}\frac{\partial(\omega-k)_+}{\partial x_i}dx\\ \geq& C_1\int_\Omega\frac{\partial(\omega-k)_+}{\partial x_i}\frac{\partial(\omega-k)_+}{\partial x_i}dx\geq C_1C(n,\Omega)\|(\omega-k)_+\|^2_{H^1(\Omega)}.\end{aligned}\tag{3.51}$$

According to imbedding theorem

$$\|(\omega-k)_+\|^2_{H^1(\Omega)}\geq C(n,\Omega)\|(\omega-k)_+\|^2_{L^p(A(k))},\tag{3.52}$$

where $A(k)=\{x\in\Omega;\omega>k\}$ and

$$2<p<\begin{cases}+\infty, & n=1,2,\\ \dfrac{2n}{n-2}, & n\geq 3,\end{cases}\tag{3.53}$$

in other words

$$\Big(\int_{A(k)}|(\omega-k)_+|^p dx\Big)^{\frac{2}{p}}\leq C(n,\Omega)\int_{A(k)}|f(\omega-k)_+|dx.\tag{3.54}$$

From this, using the Hölder inequality

$$\int_{A(k)}|f(\omega-k)_+|dx\leq\Big(\int_{A(k)}|(\omega-k)_+|^p dx\Big)^{\frac{1}{p}}\Big(\int_{A(k)}|f|^q dx\Big)^{\frac{1}{q}},\tag{3.55}$$

where $\frac{1}{p}+\frac{1}{q}=1$, we obtain

$$\Big(\int_{A(k)}|(\omega-k)_+|^p dx\Big)^{\frac{1}{p}}\leq C\Big(\int_{A(k)}|f|^q dx\Big)^{\frac{1}{q}}.\tag{3.56}$$

Since $h>k$ implies $A(h)\subset A(k)$, $|A(k)|$ is a nonnegative and nonincreasing function, $(\omega-k)_+>(h-k)$ on $A(h)$, we have

$$\int_{A(k)}|(\omega-k)_+|^p dx\geq\int_{A(h)}|(\omega-k)_+|^p dx\geq(h-k)^p|A(h)|.\tag{3.57}$$

This combined with (3.56) gives

$$(h-k)|A(h)|^{\frac{1}{p}}\leq C\|f\|_{L^\infty(\Omega)}|A(k)|^{\frac{1}{q}},\tag{3.58}$$

i.e.

$$|A(h)|\leq\Big(\frac{C\|f\|_{L^\infty(\Omega)}}{h-k}\Big)^p|A(k)|^{\frac{p}{q}}.\tag{3.59}$$

Since $p>2$ implies $p>q$, use the De Giorgi iteration lemma, we obtain

$$|A(\tau_2+C_1)|=0, \tag{3.60}$$

where

$$C_1=C(n,\Omega)\|f\|_{L^\infty(\Omega)}|A(\tau_2)|^{\frac{p-q}{pq}}2^{\frac{p}{p-q}}. \tag{3.61}$$

By the definition of $A(k)$, we have

$$\omega\le\tau_2+C_1. \tag{3.62}$$

Since C_1 is independent of C_0, we choose $C_0=C_1=d$.

Define operator $\mathcal{L}v=\omega$, then $\mathcal{L}$ is from $C^0([\tau_1,\tau_2+d])$ to $C^0([\tau_1,\tau_2+d])$.

For fixed v the homogenized conductivity $a_{ij}^0(v)$ is invariant, so we have the global regularity of ω

$$\|\omega\|_{H^2(\Omega)}\le C\Big(\|f\|_{L^2(\Omega)}+\|\widetilde{T}\|_{H^{\frac{3}{2}}(\Omega)}\Big), \tag{3.63}$$

i.e., $\omega\in H^2(\Omega)$. Using imbedding theorem

$$\|\omega\|_{C^{0,\alpha}(\Omega)}\le C\|\omega\|_{H^2(\Omega)}\quad \forall\alpha\in(0,1), \tag{3.64}$$

we obtain $\omega\in C^{0,\alpha}(\Omega)$, $0<\alpha<1$, thus $\mathcal{L}v=\omega$ is precompact.

From Lemma 3.4 it is sufficient to prove the operator $\mathcal{L}$ is continuous.

In fact, for T_{k_1}, $T_{k_2}\in[\tau_1,\tau_2+d]$, let $\mathcal{L}(T_{k_1})=T_{\beta_1}$, $\mathcal{L}(T_{k_2})=T_{\beta_2}$, T_{β_1} and T_{β_2} satisfy

$$\int_\Omega a_{ij}^0(T_{k_1})\frac{\partial T_{\beta_1}}{\partial x_i}\frac{\partial\varphi}{\partial x_j}dx=\int_\Omega f\varphi dx,\qquad \int_\Omega a_{ij}^0(T_{k_2})\frac{\partial T_{\beta_2}}{\partial x_i}\frac{\partial\varphi}{\partial x_j}dx=\int_\Omega f\varphi dx, \tag{3.65a}$$

$$\int_\Omega a_{ij}^0(T_{k_1})\frac{\partial T_{\beta_1}}{\partial x_i}\frac{\partial\varphi}{\partial x_j}dx=\int_\Omega a_{ij}^0(T_{k_2})\frac{\partial T_{\beta_2}}{\partial x_i}\frac{\partial\varphi}{\partial x_j}dx. \tag{3.65b}$$

Moreover

$$\int_\Omega a_{ij}^0(T_{k_1})\frac{\partial(T_{\beta_1}-T_{\beta_2})}{\partial x_i}\frac{\partial\varphi}{\partial x_j}dx=-\int_\Omega(a_{ij}^0(T_{k_1})-a_{ij}^0(T_{k_2}))\frac{\partial T_{\beta_2}}{\partial x_i}\frac{\partial\varphi}{\partial x_j}dx, \tag{3.66}$$

take $\varphi=T_{\beta_1}-T_{\beta_2}$, from the regularity of $T_{\beta_1}-T_{\beta_2}$, Lemma 3.3, we have

$$\|T_{\beta_1}-T_{\beta_2}\|_{H^2(\Omega)}\le C\|T_{k_1}-T_{k_2}\|_{C^0(\Omega)}. \tag{3.67}$$

From the imbedding theorem

$$\|T_{\beta_1}-T_{\beta_2}\|_{C^0(\Omega)}\le C\|T_{k_1}-T_{k_2}\|_{C^0(\Omega)}, \tag{3.68}$$

then $\mathcal{L}$ is continuous, and thus there exists a fixed point. □

Remark 3.1. The homogenized problem (3.29), cell problem (3.23) and the homogenized conductivity (3.31) is actually equivalent to the results of [8], which is obtained by the two-scale convergence method.

Now the first order approximate solution of the temperature is obtained as

$$T_0+\varepsilon N_{\alpha_1}(T_0,y)\frac{\partial T_0}{\partial x_{\alpha_1}}. \tag{3.69}$$

To compare it with the original solution, we take $T-(T_0+\varepsilon T_1)$ into (2.10) and have

$$\begin{aligned} A_\varepsilon(T_\varepsilon-T_0-\varepsilon T_1)&=f-(\varepsilon^{-2}A_0+\varepsilon^{-1}A_1+A_2)(T_0+\varepsilon T_1)\\ &=f-\Big[\varepsilon^{-2}A_0T_0+\varepsilon^{-1}(A_0T_1+A_1T_0)+(A_1T_1+A_2T_0)+\varepsilon A_2T_1\Big]\\ &=f-\frac{\partial}{\partial y_i}\Big(a_{ij}\frac{\partial}{\partial x_j}\Big(N_{\alpha_1}\frac{\partial T_0}{\partial x_{\alpha_1}}\Big)\Big)-\frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial}{\partial y_j}\Big(N_{\alpha_1}\frac{\partial T_0}{\partial x_{\alpha_1}}\Big)\Big)\\ &\quad -a_{ij}\frac{\partial T_0^2}{\partial x_i\partial x_j}-\varepsilon a_{ij}\frac{\partial^2}{\partial x_i\partial x_j}\Big(N_{\alpha_1}\frac{\partial T_0}{\partial x_{\alpha_1}}\Big). \end{aligned} \tag{3.70}$$

Note that residual is of order $\mathcal{O}(1)$ that does not equal to 0. In the practical engineering computation, it can not be omitted for a constant ε, so engineer concludes that the first order approximate solution can not be accepted and the micro-scale fluctuation of the temperature is far from being captured. This is the reason why it is necessary to consider the higher order expansions.

3.3 Higher order expansions

From (3.23), (3.29), it is noted that N_{α_1} depends on T_0 directly, so we have

$$\frac{\partial N_{\alpha_1}}{\partial x_i}=\frac{\partial N_{\alpha_1}}{\partial T_0}\frac{\partial T_0}{\partial x_i}. \tag{3.71}$$

Taking T_0 and $T_1=N_{\alpha_1}(T_0,y)\frac{\partial T_0}{\partial x_{\alpha_1}}$ into (3.18), then

$$\begin{aligned} -\frac{\partial}{\partial y_i}\Big(a_{ij}\frac{\partial T_2}{\partial y_j}\Big)=&\Big[-a_{ij}^0(T_0)+a_{ij}+\frac{\partial}{\partial y_k}(a_{ki}N_j)+a_{ik}\frac{\partial N_j}{\partial y_k}\Big]\frac{\partial^2 T_0}{\partial x_i\partial x_j}\\ &-\frac{1}{|Y^*|}\Big[\int_\Gamma y_iG(N_j+y_j)d\Gamma_y\Big]12\sigma T_0^2\frac{\partial T_0}{\partial x_i}\frac{\partial T_0}{\partial x_j}\\ &-\frac{1}{|Y^*|}\Big[\int_{Y^*}a_{ik}\frac{\partial^2 N_j}{\partial T_0\partial y_k}dy+4\sigma T_0^3\int_\Gamma y_iG\Big(\frac{\partial N_j}{\partial T_0}\Big)d\Gamma_y\Big]\frac{\partial T_0}{\partial x_i}\frac{\partial T_0}{\partial x_j}\\ &+\Big[\frac{\partial}{\partial y_k}\Big(a_{ki}\frac{\partial N_j}{\partial T_0}\Big)+a_{ik}\frac{\partial^2 N_j}{\partial T_0\partial y_k}\Big]\frac{\partial T_0}{\partial x_i}\frac{\partial T_0}{\partial x_j}. \end{aligned} \tag{3.72}$$

The boundary condition on Γ becomes

$$\begin{aligned}-a_{ij}\frac{\partial T_2}{\partial y_j}n_i=&4\sigma T_0^3G(T_2)+a_{ki}N_j\frac{\partial^2 T_0}{\partial x_i\partial x_j}n_k+a_{ki}\frac{\partial N_j}{\partial T_0}\frac{\partial T_0}{\partial x_i}\frac{\partial T_0}{\partial x_j}n_k\\&+6\sigma T_0^2\frac{\partial T_0}{\partial x_i}\frac{\partial T_0}{\partial x_j}G(N_iN_j)-4\sigma T_0^3\frac{\partial T_0}{\partial x_i}\frac{\partial T_0}{\partial x_j}\int_\Gamma k(y,\lambda)\frac{\partial N_j}{\partial T_0}(\lambda_i-y_i)d\Gamma_\lambda\\&-\sigma\frac{\partial}{\partial x_i}\Big(4T_0^3\frac{\partial T_0}{\partial x_j}\Big)\int_\Gamma k(y,\lambda)N_j(\lambda_i-y_i)d\Gamma_\lambda\\&-\sigma\frac{\partial}{\partial x_i}\Big(2T_0^3\frac{\partial T_0}{\partial x_j}\Big)\int_\Gamma k(y,\lambda)(\lambda_i-y_i)(\lambda_j-y_j)d\Gamma_\lambda.\end{aligned}\tag{3.73}$$

Define T_2 as the following form

$$T_2(x,y)=N_{\alpha_1\alpha_2}(T_0,y)\frac{\partial^2 T_0}{\partial x_{\alpha_1}\partial x_{\alpha_2}}+M_{\alpha_1\alpha_2}(T_0,y)\frac{\partial T_0}{\partial x_{\alpha_1}}\frac{\partial T_0}{\partial x_{\alpha_2}},\tag{3.74}$$

where $\alpha_1,\alpha_2=1,\cdots,n$, $N_{\alpha_1\alpha_2}(T_0,y)$ is the solution of the following problem

$$\begin{cases}-\dfrac{\partial}{\partial y_i}\Big(a_{ij}\dfrac{\partial N_{\alpha_1\alpha_2}}{\partial y_j}\Big)=a_{\alpha_1\alpha_2}-a^0_{\alpha_1\alpha_2}+\dfrac{\partial}{\partial y_i}(a_{i\alpha_1}N_{\alpha_2})+a_{\alpha_1 i}\dfrac{\partial N_{\alpha_2}}{\partial y_i} & \text{in } Y^*,\\ -a_{ij}\dfrac{\partial N_{\alpha_1\alpha_2}}{\partial y_j}n_i=4\sigma T_0^3G(N_{\alpha_1\alpha_2})+a_{i\alpha_1}N_{\alpha_2}n_i & \\ \qquad -4\sigma T_0^3\displaystyle\int_\Gamma k(y,\lambda)(\lambda_{\alpha_1}-y_{\alpha_1})N_{\alpha_2}d\Gamma_\lambda & \\ \qquad -2\sigma T_0^3\displaystyle\int_\Gamma k(y,\lambda)(\lambda_{\alpha_1}-y_{\alpha_1})(\lambda_{\alpha_2}-y_{\alpha_2})d\Gamma_\lambda & \text{on } \Gamma,\\ N_{\alpha_1\alpha_2}(T_0,y)\ \text{1-periodic in } y,\end{cases}\tag{3.75}$$

and $M_{\alpha_1\alpha_2}(T_0,y)$ satisfies the following cell problem

$$\begin{cases}-\dfrac{\partial}{\partial y_i}\Big(a_{ij}\dfrac{\partial M_{\alpha_1\alpha_2}}{\partial y_j}\Big)=\dfrac{\partial}{\partial y_i}\Big(a_{i\alpha_1}\dfrac{\partial N_{\alpha_2}}{\partial T_0}\Big)+a_{\alpha_1 i}\dfrac{\partial^2 N_{\alpha_2}}{\partial T_0\partial y_i}-\dfrac{12\sigma T_0^2}{|Y^*|}\displaystyle\int_\Gamma y_{\alpha_1}G(N_{\alpha_2}+y_{\alpha_2})d\Gamma_y & \\ \qquad -\dfrac{1}{|Y^*|}\Big(\displaystyle\int_{Y^*}a_{\alpha_1 i}\dfrac{\partial^2 N_{\alpha_2}}{\partial T_0\partial y_i}dy+4\sigma T_0^3\int_\Gamma y_{\alpha_1}G\Big(\dfrac{\partial N_{\alpha_2}}{\partial T_0}\Big)d\Gamma_y\Big) & \text{in } Y^*,\\ -a_{ij}\dfrac{\partial M_{\alpha_1\alpha_2}}{\partial y_j}n_i=a_{i\alpha_1}\dfrac{\partial N_{\alpha_2}}{\partial T_0}n_i+4\sigma T_0^3G(M_{\alpha_1\alpha_2})+6\sigma T_0^2G(N_{\alpha_1}N_{\alpha_2}) & \\ \qquad -4\sigma T_0^3\displaystyle\int_\Gamma k(y,\lambda)(\lambda_{\alpha_1}-y_{\alpha_1})\dfrac{\partial N_{\alpha_2}}{\partial T_0}d\Gamma_\lambda & \\ \qquad -12\sigma T_0^2\displaystyle\int_\Gamma k(y,\lambda)(\lambda_{\alpha_1}-y_{\alpha_1})N_{\alpha_2}d\Gamma_\lambda & \\ \qquad -6\sigma T_0^2\displaystyle\int_\Gamma k(y,\lambda)(\lambda_{\alpha_1}-y_{\alpha_1})(\lambda_{\alpha_2}-y_{\alpha_2})d\Gamma_\lambda & \text{on } \Gamma,\\ M_{\alpha_1\alpha_2}(T_0,y)\ \text{1-periodic in } y.\end{cases}\tag{3.76}$$

Theorem 3.3. *For $T_0>0$, problem* (3.75) *and* (3.76) *admit a unique solution $N_{\alpha_1\alpha_2}(T_0,y)$, $M_{\alpha_1\alpha_2}(T_0,y)$ in $W_{per}(Y^*)$, respectively.*

Proof. The variational forms of (3.75) and (3.76) are respectively

$$\begin{cases} \text{find } N_{\alpha_1\alpha_2}(T_0,y)\in W_{per}(Y^*) \text{ such that} \\ a_{Y^*}(N_{\alpha_1\alpha_2},\varphi)=(g_2(T_0),\varphi) \quad \forall\varphi\in W_{per}(Y^*), \end{cases} \tag{3.77}$$

$$\begin{cases} \text{find } M_{\alpha_1\alpha_2}(T_0,y)\in W_{per}(Y^*) \text{ such that} \\ a_{Y^*}(M_{\alpha_1\alpha_2},\varphi)=(g_3(T_0),\varphi) \quad \forall\varphi\in W_{per}(Y^*), \end{cases} \tag{3.78}$$

where

$$\begin{aligned}(g_2(T_0),\varphi)=&\int_{Y^*}\Big(a_{\alpha_1\alpha_2}-a^0_{\alpha_1\alpha_2}+a_{\alpha_1 i}\frac{\partial N_{\alpha_2}}{\partial y_i}\Big)\varphi dy-\int_{Y^*}a_{i\alpha_1}N_{\alpha_2}\frac{\partial\varphi}{\partial y_i}dy\\ &+4\sigma T_0^3\int_\Gamma\Big(\int_\Gamma k(y,\lambda)(\lambda_{\alpha_1}-y_{\alpha_1})N_{\alpha_2}d\Gamma_\lambda\Big)\varphi d\Gamma_y\\ &+2\sigma T_0^3\int_\Gamma\Big(\int_\Gamma k(y,\lambda)(\lambda_{\alpha_1}-y_{\alpha_1})(\lambda_{\alpha_2}-y_{\alpha_2})d\Gamma_\lambda\Big)\varphi dy,\end{aligned} \tag{3.79}$$

$$\begin{aligned}(g_3(T_0),\varphi)=&-\int_{Y^*}a_{i\alpha_1}\frac{\partial N_{\alpha_2}}{\partial T_0}\frac{\partial\varphi}{\partial y_i}dy+\int_{Y^*}a_{\alpha_1 i}\frac{\partial^2 N_{\alpha_2}}{\partial T_0\partial y_i}\varphi dy\\ &-\frac{12\sigma T_0^2}{|Y^*|}\int_{Y^*}\Big(\int_\Gamma y_{\alpha_1}G(N_{\alpha_2}+y_{\alpha_2})d\Gamma_y\Big)\varphi dy\\ &-\frac{1}{|Y^*|}\int_{Y^*}\Big(\int_{Y^*}a_{\alpha_1 i}\frac{\partial^2 N_{\alpha_2}}{\partial T_0\partial y_i}dy+4\sigma T_0^3\int_\Gamma y_{\alpha_1}G\Big(\frac{\partial N_{\alpha_2}}{\partial T_0}\Big)d\Gamma_y\Big)\varphi dy\\ &-6\sigma T_0^2\int_\Gamma G(N_{\alpha_1}N_{\alpha_2})\varphi d\Gamma_y+4\sigma T_0^3\int_\Gamma\Big(\int_\Gamma k(y,\lambda)(\lambda_{\alpha_1}-y_{\alpha_1})\frac{\partial N_{\alpha_2}}{\partial T_0}d\Gamma_\lambda\Big)\varphi d\Gamma_y\\ &+12\sigma T_0^2\int_\Gamma\Big(\int_\Gamma k(y,\lambda)(\lambda_{\alpha_1}-y_{\alpha_1})N_{\alpha_2}d\Gamma_\lambda\Big)\varphi d\Gamma_y\\ &+6\sigma T_0^2\int_\Gamma\Big(\int_\Gamma k(y,\lambda)(\lambda_{\alpha_1}-y_{\alpha_1})(\lambda_{\alpha_2}-y_{\alpha_2})d\Gamma_\lambda\Big)\varphi d\Gamma_y.\end{aligned} \tag{3.80}$$

Here a_{Y^*} is the same as (3.26a), so a_{Y^*} is coercive in Y^*. By the same argument in the proof of Theorem 3.1, we obtain the result. □

4 Error estimation

Now we have obtained the first three terms of the expansion, let

$$\widehat{T}^1(x,y)=T_0+\varepsilon T_1=T_0(x)+\varepsilon N_{\alpha_1}(T_0(x),y)\frac{\partial T_0(x)}{\partial x_{\alpha_1}}, \tag{4.1a}$$

$$\begin{aligned}\widehat{T}^2(x,y)=&T_0+\varepsilon T_1+\varepsilon^2 T_2\\ =&T_0(x)+\varepsilon N_{\alpha_1}(T_0(x),y)\frac{\partial T_0(x)}{\partial x_{\alpha_1}}+\varepsilon^2 N_{\alpha_1\alpha_2}(T_0(x),y)\frac{\partial^2 T_0(x)}{\partial x_{\alpha_1}\partial x_{\alpha_2}}\\ &+\varepsilon^2 M_{\alpha_1\alpha_2}(T_0(x),y)\frac{\partial T_0(x)}{\partial x_{\alpha_1}}\frac{\partial T_0(x)}{\partial x_{\alpha_2}}.\end{aligned} \tag{4.1b}$$

The following lemma follows from Oleinik [3, Chapter 1, Lemma 1.5].

Lemma 4.1. *Let Ω be a bounded domain with a smooth boundary and $B_\delta=\{x\in\Omega|\,\rho(x,\partial\Omega)<\delta,\delta>0\}$. Then there exists $\delta_0>0$ such that for every $\delta\in(0,\delta_0)$ and every $v\in H^1(\Omega)$, the following inequality holds*

$$\|v\|_{L^2(B_\delta)}\le C\delta^{\frac{1}{2}}\|v\|_{H^1(\Omega)}, \tag{4.2}$$

where C is a constant independent of δ and v.

Next we prove the following estimation.

Theorem 4.1. *Let T_ε be the weak solution of problem* (2.10), $\widetilde{T}\in H^{\frac{3}{2}}(\Omega)$, $T_0\in H^3(\Omega)$, N_{α_1}, $N_{\alpha_1\alpha_2}$, $M_{\alpha_1\alpha_2}\in H^2([\tau_1,\tau_2+d])\times W_{per}(Y^*)$, *where d is constant given by* (3.61), *the following estimation holds:*

$$\|T_\varepsilon-\widehat{T}^2\|_{H^1(\Omega^\varepsilon)}\le C\varepsilon^{\frac{1}{2}}\|T_0\|_{H^3(\Omega)}, \tag{4.3}$$

where C is a constant independent of ε, T_0.

Proof. Applying the operator A_ε to $T_\varepsilon-\widehat{T}^2$, we obtain that

$$\begin{aligned}A_\varepsilon(T_\varepsilon-\widehat{T}^2)=&A_\varepsilon(T_\varepsilon)-A_\varepsilon(\widehat{T}^2)\\ =&f-(\varepsilon^{-2}A_0+\varepsilon^{-1}A_1+A_2)(\widehat{T}^2)\\ =&f-(\varepsilon^{-2}A_0+\varepsilon^{-1}A_1+A_2)(T_0+\varepsilon T_1+\varepsilon^2 T_2)\\ =&f-\Big[\varepsilon^{-2}A_0T_0+\varepsilon^{-1}(A_0T_1+A_1T_0)\\ &+(A_0T_2+A_1T_1+A_2T_0)+\varepsilon(A_1T_2+A_2T_1)+\varepsilon^2A_2T_2\Big]\\ =&-\varepsilon(A_2T_1+A_1T_2)-\varepsilon^2A_2T_2.\end{aligned} \tag{4.4}$$

From the definition of A_1,A_2,T_1,T_2 and the symmetry of a_{ij}, we have

$$A_2T_1=-a_{ij}\frac{\partial^2 N_{\alpha_1}}{\partial x_i\partial x_j}\frac{\partial T_0}{\partial x_{\alpha_1}}-2a_{ij}\frac{\partial N_{\alpha_1}}{\partial x_i}\frac{\partial^2 T_0}{\partial x_j\partial x_{\alpha_1}}-a_{ij}N_{\alpha_1}\frac{\partial^3 T_0}{\partial x_i\partial x_j\partial x_{\alpha_1}}, \tag{4.5a}$$

$$
\begin{aligned}
A_1T_2=&-a_{ij}\frac{\partial^2 N_{\alpha_1\alpha_2}}{\partial x_i\partial y_j}\frac{\partial^2 T_0}{\partial x_{\alpha_1}\partial x_{\alpha_2}}-a_{ij}\frac{\partial N_{\alpha_1\alpha_2}}{\partial y_j}\frac{\partial^3 T_0}{\partial x_i\partial x_{\alpha_1}\partial x_{\alpha_2}}\\
&-a_{ij}\frac{\partial^2 M_{\alpha_1\alpha_2}}{\partial x_i\partial y_j}\frac{\partial T_0}{\partial x_{\alpha_1}}\frac{\partial T_0}{\partial x_{\alpha_2}}-a_{ij}\frac{\partial M_{\alpha_1\alpha_2}}{\partial y_j}\frac{\partial}{\partial x_i}\Big(\frac{\partial T_0}{\partial x_{\alpha_1}}\frac{\partial T_0}{\partial x_{\alpha_2}}\Big)\\
&-\frac{\partial}{\partial y_i}\Big(a_{ij}\frac{\partial N_{\alpha_1\alpha_2}}{\partial x_j}\Big)\frac{\partial^2 T_0}{\partial x_{\alpha_1}\partial x_{\alpha_2}}-\frac{\partial}{\partial y_i}(a_{ij}N_{\alpha_1\alpha_2})\frac{\partial^3 T_0}{\partial x_j\partial x_{\alpha_1}\partial x_{\alpha_2}}\\
&-\frac{\partial}{\partial y_i}\Big(a_{ij}\frac{\partial M_{\alpha_1\alpha_2}}{\partial x_j}\Big)\frac{\partial T_0}{\partial x_{\alpha_1}}\frac{\partial T_0}{\partial x_{\alpha_2}}-\frac{\partial}{\partial y_i}(a_{ij}M_{\alpha_1\alpha_2})\frac{\partial}{\partial x_j}\Big(\frac{\partial T_0}{\partial x_{\alpha_1}}\frac{\partial T_0}{\partial x_{\alpha_2}}\Big),
\end{aligned}\tag{4.5b}
$$

$$
\begin{aligned}
A_2T_2=&-a_{ij}\frac{\partial^2 N_{\alpha_1\alpha_2}}{\partial x_i\partial x_j}\frac{\partial^2 T_0}{\partial x_{\alpha_1}\partial x_{\alpha_2}}-2a_{ij}\frac{\partial N_{\alpha_1\alpha_2}}{\partial x_j}\frac{\partial^3 T_0}{\partial x_i\partial x_{\alpha_1}\partial x_{\alpha_2}}\\
&-a_{ij}N_{\alpha_1\alpha_2}\frac{\partial^4 T_0}{\partial x_i\partial x_j\partial x_{\alpha_1}\partial x_{\alpha_2}}-a_{ij}\frac{\partial^2 M_{\alpha_1\alpha_2}}{\partial x_i\partial x_j}\frac{\partial T_0}{\partial x_{\alpha_1}}\frac{\partial T_0}{\partial x_{\alpha_2}}\\
&-2a_{ij}\frac{\partial M_{\alpha_1\alpha_2}}{\partial x_j}\frac{\partial}{\partial x_i}\Big(\frac{\partial T_0}{\partial x_{\alpha_1}}\frac{\partial T_0}{\partial x_{\alpha_2}}\Big)-a_{ij}M_{\alpha_1\alpha_2}\frac{\partial^2}{\partial x_i\partial x_j}\Big(\frac{\partial T_0}{\partial x_{\alpha_1}}\frac{\partial T_0}{\partial x_{\alpha_2}}\Big).
\end{aligned}\tag{4.5c}
$$

For the last four terms of A_1T_2, we have

$$
\begin{aligned}
&-\frac{\partial}{\partial y_i}\Big(a_{ij}\frac{\partial N_{\alpha_1\alpha_2}}{\partial x_j}\Big)\frac{\partial^2 T_0}{\partial x_{\alpha_1}\partial x_{\alpha_2}}\\
=&-\varepsilon\frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial N_{\alpha_1\alpha_2}}{\partial x_j}\frac{\partial^2 T_0}{\partial x_{\alpha_1}\partial x_{\alpha_2}}\Big)+\varepsilon a_{ij}\frac{\partial N_{\alpha_1\alpha_2}}{\partial x_j}\frac{\partial^3 T_0}{\partial x_i\partial x_{\alpha_1}\partial x_{\alpha_2}},
\end{aligned}\tag{4.6a}
$$

$$
\begin{aligned}
&-\frac{\partial}{\partial y_i}(a_{ij}N_{\alpha_1\alpha_2})\frac{\partial^3 T_0}{\partial x_j\partial x_{\alpha_1}\partial x_{\alpha_2}}\\
=&-\varepsilon\frac{\partial}{\partial x_i}\Big(a_{ij}N_{\alpha_1\alpha_2}\frac{\partial^3 T_0}{\partial x_j\partial x_{\alpha_1}\partial x_{\alpha_2}}\Big)+\varepsilon a_{ij}N_{\alpha_1\alpha_2}\frac{\partial^4 T_0}{\partial x_i\partial x_j\partial x_{\alpha_1}\partial x_{\alpha_2}},
\end{aligned}\tag{4.6b}
$$

$$
\begin{aligned}
&-\frac{\partial}{\partial y_i}\Big(a_{ij}\frac{\partial M_{\alpha_1\alpha_2}}{\partial x_j}\Big)\frac{\partial T_0}{\partial x_{\alpha_1}}\frac{\partial T_0}{\partial x_{\alpha_2}}\\
=&-\varepsilon\frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial M_{\alpha_1\alpha_2}}{\partial x_j}\frac{\partial T_0}{\partial x_{\alpha_1}}\frac{\partial T_0}{\partial x_{\alpha_2}}\Big)+\varepsilon a_{ij}\frac{\partial M_{\alpha_1\alpha_2}}{\partial x_j}\frac{\partial}{\partial x_i}\Big(\frac{\partial T_0}{\partial x_{\alpha_1}}\frac{\partial T_0}{\partial x_{\alpha_2}}\Big),
\end{aligned}\tag{4.6c}
$$

$$
\begin{aligned}
&-\frac{\partial}{\partial y_i}(a_{ij}M_{\alpha_1\alpha_2})\frac{\partial}{\partial x_j}\Big(\frac{\partial T_0}{\partial x_{\alpha_1}}\frac{\partial T_0}{\partial x_{\alpha_2}}\Big)\\
=&-\varepsilon\frac{\partial}{\partial x_i}\Big(a_{ij}M_{\alpha_1\alpha_2}\frac{\partial}{\partial x_j}\Big(\frac{\partial T_0}{\partial x_{\alpha_1}}\frac{\partial T_0}{\partial x_{\alpha_2}}\Big)\Big)+\varepsilon a_{ij}M_{\alpha_1\alpha_2}\frac{\partial^2}{\partial x_i\partial x_j}\Big(\frac{\partial T_0}{\partial x_{\alpha_1}}\frac{\partial T_0}{\partial x_{\alpha_2}}\Big).
\end{aligned}\tag{4.6d}
$$

Then (4.4) finally becomes

$$
A_\varepsilon(T_\varepsilon-\widehat{T}^2)=\varepsilon F_0^\varepsilon+\varepsilon^2\frac{\partial}{\partial x_i}F_i^\varepsilon+\varepsilon^2 W^\varepsilon,\tag{4.7}
$$

where

$$
\begin{aligned}
F_0^{\varepsilon}=&a_{ij}\frac{\partial^2 N_{\alpha_1}}{\partial x_i\partial x_j}\frac{\partial T_0}{\partial x_{\alpha_1}}+2a_{ij}\frac{\partial N_{\alpha_1}}{\partial x_i}\frac{\partial^2 T_0}{\partial x_j\partial x_{\alpha_1}}+a_{ij}N_{\alpha_1}\frac{\partial^3 T_0}{\partial x_i\partial x_j\partial x_{\alpha_1}}\\
&+a_{ij}\frac{\partial^2 N_{\alpha_1\alpha_2}}{\partial x_i\partial y_j}\frac{\partial^2 T_0}{\partial x_{\alpha_1}\partial x_{\alpha_2}}+a_{ij}\frac{\partial N_{\alpha_1\alpha_2}}{\partial y_j}\frac{\partial^3 T_0}{\partial x_i\partial x_{\alpha_1}\partial x_{\alpha_2}}\\
&+a_{ij}\frac{\partial^2 M_{\alpha_1\alpha_2}}{\partial x_i\partial y_j}\frac{\partial T_0}{\partial x_{\alpha_1}}\frac{\partial T_0}{\partial x_{\alpha_2}}+a_{ij}\frac{\partial M_{\alpha_1\alpha_2}}{\partial y_j}\frac{\partial}{\partial x_i}\Big(\frac{\partial T_0}{\partial x_{\alpha_1}}\frac{\partial T_0}{\partial x_{\alpha_2}}\Big),
\end{aligned}
\tag{4.8a}
$$

$$
\begin{aligned}
F_i^{\varepsilon}=&a_{ij}\frac{\partial N_{\alpha_1\alpha_2}}{\partial x_j}\frac{\partial^2 T_0}{\partial x_{\alpha_1}\partial x_{\alpha_2}}+a_{ij}N_{\alpha_1\alpha_2}\frac{\partial^3 T_0}{\partial x_j\partial x_{\alpha_1}\partial x_{\alpha_2}}\\
&+a_{ij}\frac{\partial M_{\alpha_1\alpha_2}}{\partial x_j}\frac{\partial T_0}{\partial x_{\alpha_1}}\frac{\partial T_0}{\partial x_{\alpha_2}}+a_{ij}M_{\alpha_1\alpha_2}\frac{\partial}{\partial x_j}\Big(\frac{\partial T_0}{\partial x_{\alpha_1}}\frac{\partial T_0}{\partial x_{\alpha_2}}\Big),
\end{aligned}
\tag{4.8b}
$$

$$
\begin{aligned}
W^{\varepsilon}=&a_{ij}\frac{\partial^2 N_{\alpha_1\alpha_2}}{\partial x_i\partial x_j}\frac{\partial^2 T_0}{\partial x_{\alpha_1}\partial x_{\alpha_2}}+a_{ij}\frac{\partial N_{\alpha_1\alpha_2}}{\partial x_j}\frac{\partial^3 T_0}{\partial x_i\partial x_{\alpha_1}\partial x_{\alpha_2}}\\
&+a_{ij}\frac{\partial^2 M_{\alpha_1\alpha_2}}{\partial x_i\partial x_j}\frac{\partial T_0}{\partial x_{\alpha_1}}\frac{\partial T_0}{\partial x_{\alpha_2}}+a_{ij}\frac{\partial M_{\alpha_1\alpha_2}}{\partial x_j}\frac{\partial}{\partial x_i}\Big(\frac{\partial T_0}{\partial x_{\alpha_1}}\frac{\partial T_0}{\partial x_{\alpha_2}}\Big).
\end{aligned}
\tag{4.8c}
$$

It is noted that

$$
\frac{\partial N_{\alpha_1}}{\partial x_j}=\frac{\partial N_{\alpha_1}}{\partial T_0}\frac{\partial T_0}{\partial x_j},
\tag{4.9a}
$$

$$
\frac{\partial^2 N_{\alpha_1}}{\partial x_i\partial x_j}=\frac{\partial}{\partial x_i}\Big(\frac{\partial N_{\alpha_1}}{\partial T_0}\frac{\partial T_0}{\partial x_j}\Big)=\frac{\partial^2 N_{\alpha_1}}{\partial T_0^2}\frac{\partial T_0}{\partial x_i}\frac{\partial T_0}{\partial x_j}+\frac{\partial N_{\alpha_1}}{\partial T_0}\frac{\partial^2 T_0}{\partial x_i\partial x_j},
\tag{4.9b}
$$

and so are $N_{\alpha_1\alpha_2}$ and $M_{\alpha_1\alpha_2}$.

On Γ_i^{ε}, applying the operator B_{ε} to $T_{\varepsilon}-\widetilde{T}$, we have

$$
\begin{aligned}
B_{\varepsilon}(T_{\varepsilon}-\widehat{T}^2)=&B_{\varepsilon}(T_{\varepsilon})-B_{\varepsilon}(\widehat{T}^2)\\
=&\varepsilon^{-1}\sigma\Big[T_{\varepsilon}^4(x)-\int_{\Gamma_i^{\varepsilon}}k(x,s)T_{\varepsilon}^4(s)d\Gamma_s\Big]-(\varepsilon^{-1}B_0+B_1)(\widehat{T}^2)\\
=&\varepsilon^{-1}\sigma\Big[T_{\varepsilon}^4(x)-\int_{\Gamma_i^{\varepsilon}}k(x,s)T_{\varepsilon}^4(s)d\Gamma_s\Big]-(\varepsilon^{-1}B_0+B_1)(T_0+\varepsilon T_1+\varepsilon^2T_2)\\
=&\varepsilon^{-1}\sigma\Big[T_{\varepsilon}^4(x)-\int_{\Gamma_i^{\varepsilon}}k(x,s)T_{\varepsilon}^4(s)d\Gamma_s\Big]\\
&-[\varepsilon^{-1}B_0T_0+(B_1T_0+B_0T_1)+\varepsilon(B_0T_2+B_1T_1)+\varepsilon^2B_1T_2]\\
=&\varepsilon^{-1}\sigma\Big[(T_{\varepsilon}^4(x)-T_0^4-\varepsilon(4T_0^3T_1)-\varepsilon^2(6T_0^2T_1^2+4T_0^3T_2)\Big]\\
&-\varepsilon^{-1}\sigma\int_{\Gamma}k(y,\lambda)(T_{\varepsilon}^4(s)-T_0^4-\varepsilon T_{1,\lambda}^4-\varepsilon^2T_{2,\lambda}^4)d\Gamma_{\lambda}+\varepsilon^2n_iF_i^{\varepsilon}.
\end{aligned}
\tag{4.10}
$$

From (3.2) and (3.12), we have

$$
B_{\varepsilon}(T_{\varepsilon}-\widehat{T}^2)=\mathcal{O}(\varepsilon^2)+\varepsilon^2n_iF_i^{\varepsilon}.
\tag{4.11}
$$

On the outer boundary $\partial\Omega$, we have

$$T_\varepsilon - \widehat{T}^2 = G^\varepsilon, \tag{4.12}$$

where

$$G^\varepsilon = -\varepsilon N_{\alpha_1}\frac{\partial T_0}{\partial x_{\alpha_1}} - \varepsilon^2\Big(N_{\alpha_1\alpha_2}\frac{\partial^2 T_0}{\partial x_{\alpha_1}\partial x_{\alpha_2}} + M_{\alpha_1\alpha_2}\frac{\partial T_0}{\partial x_{\alpha_1}}\frac{\partial T_0}{\partial x_{\alpha_2}}\Big). \tag{4.13}$$

So we conclude that $T_\varepsilon - \widehat{T}^2$ is a solution of the following boundary value problem:

$$\begin{cases} A_\varepsilon(T_\varepsilon - \widehat{T}^2) = \varepsilon F_0^\varepsilon + \varepsilon^2\dfrac{\partial}{\partial x_i}F_i^\varepsilon + \varepsilon^2 W^\varepsilon & \text{in } \Omega^\varepsilon, \\ B_\varepsilon(T_\varepsilon - \widehat{T}^2) = \mathcal{O}(\varepsilon^2) + \varepsilon^2 n_i F_i^\varepsilon & \text{in } \Gamma^\varepsilon, \\ T_\varepsilon - \widehat{T}^2 = G^\varepsilon & \text{on } \partial\Omega. \end{cases} \tag{4.14}$$

By using Lemma 4.1 and the argument of cut-off function on the $\partial\Omega$ from the idea of Oleinik [3,4], we have

$$\|G^\varepsilon\|_{H^{\frac{1}{2}}(\partial\Omega)} \le C\varepsilon^{\frac{1}{2}}\|T_0\|_{H^3(\Omega)}. \tag{4.15}$$

From the regularity of N_{α_1}, $N_{\alpha_1\alpha_2}$, $M_{\alpha_1\alpha_2}$ and T_0, one obtains

$$\|F_0^\varepsilon\|_{L^2(\Omega^\varepsilon)} \le C_1\|T_0\|_{H^3(\Omega)}, \tag{4.16a}$$

$$\sum_{k=1}^{n}\|F_k^\varepsilon\|_{L^2(\Omega^\varepsilon)} \le C_2\|T_0\|_{H^3(\Omega)}, \tag{4.16b}$$

$$\|W^\varepsilon\|_{L^2(\Omega^\varepsilon)} \le C_3\|T_0\|_{H^3(\Omega)}, \tag{4.16c}$$

where the constants C_1, C_2 and C_3 is independent of ε. Note that after integration and summation over all cells, we obtain a remainder term for $\mathcal{O}(\varepsilon^2)$ given by

$$\sum_{z\in I_z}|\Gamma_z^\varepsilon|\mathcal{O}(\varepsilon^2) = \mathcal{O}(\varepsilon^{-n})\mathcal{O}(\varepsilon^{n-1})\mathcal{O}(\varepsilon^2) = \mathcal{O}(\varepsilon). \tag{4.17}$$

Then the term $\mathcal{O}(\varepsilon^2)$ is of order 1. From the assumption of Theorem 4.1 and a prior estimation of elliptic problem, (4.3) is obtained. □

We can only obtain the convergence order of $\mathcal{O}(\varepsilon^{1/2})$ because of the residual (4.13) on the boundary $\partial\Omega$, but inside Ω^ε $\widehat{T}^2(x,y)$ is more accurate than $\widehat{T}^1(x,y)$.

5 FE Algorithms for SOTS method and numerical example

5.1 FE Algorithms For SOTS method

Since the coupling occurs between the homogenized solution T_0 and N_{α_1}, which needs the iterative computation. It is noted that for the same temperature T_0 at different x in Ω, we can get the same cell solution. Then we propose following algorithms.

1. FE computation of N_{α_1} for a range of temperature parameter $T\in[\overline{T}_1,\overline{T}_2]$.
   From the weak formulation (3.25), $N_{\alpha_1}^{h_1}$ can be obtained by solving

$$\begin{aligned}&\int_{Y^*} a_{ij}\frac{\partial N_{\alpha_1}^{h_1}}{\partial y_j}\frac{\partial \varphi}{\partial y_i}dy+4\sigma T^3\int_\Gamma G(N_{\alpha_1}^{h_1})\varphi dy\\ =&-4\sigma T^3\int_\Gamma G(y_{\alpha_1})\varphi dy-\int_{Y^*} a_{\alpha_1 k}\frac{\partial \varphi}{\partial y_k}dy \quad \forall \varphi\in S^{h_1}(Y^*),\end{aligned} \tag{5.1}$$

where $S^{h_1}(Y^*)$ denotes the FE space of Y^*, and Y^* is partitioned into FE set S^{h_1}, with the mesh size h_1.

2. Computation of homogenized conductivity $a_{ij}^{0,h_1}(T)$.

$$a_{ij}^{0,h_1}(T)=\frac{1}{|Y^*|}\Big[\int_{Y^*}\Big(a_{ij}+a_{ik}\frac{\partial N_j^{h_1}}{\partial y_k}\Big)dy+4\sigma T^3\int_\Gamma y_i G(N_j^{h_1}+y_j)dy\Big]. \tag{5.2}$$

3. FE computation of $N_{\alpha_1\alpha_2}^{h_1}$ and $M_{\alpha_1\alpha_2}^{h_1}$ in terms of a range of $T\in[\overline{T}_1,\overline{T}_2]$ and cell solution $N_{\alpha_1}^h$.
   From the weak formulation (3.77) and (3.78), $N_{\alpha_1\alpha_2}^{h_1}$ and $M_{\alpha_1\alpha_2}^{h_1}$ can be obtained by solving

$$\begin{aligned}&\int_{Y^*} a_{ij}\frac{\partial N_{\alpha_1\alpha_2}^{h_1}}{\partial y_j}\frac{\partial \varphi}{\partial y_i}dy+4\sigma T^3\int_\Gamma G(N_{\alpha_1\alpha_2}^{h_1})\varphi dy\\ =&\int_{Y^*}\Big(a_{\alpha_1\alpha_2}-a_{\alpha_1\alpha_2}^{0,h_1}+a_{\alpha_1 i}\frac{\partial N_{\alpha_2}^{h_1}}{\partial y_i}\Big)\varphi dy-\int_{Y^*}a_{i\alpha_1}N_{\alpha_2}^{h_1}\frac{\partial\varphi}{\partial y_i}dy\\ &+4\sigma T^3\int_\Gamma\Big(\int_\Gamma k(y,\lambda)(\lambda_{\alpha_1}-y_{\alpha_1})N_{\alpha_2}^{h_1}d\Gamma_\lambda\Big)\varphi d\Gamma_y\\ &+2\sigma T^3\int_\Gamma\Big(\int_\Gamma k(y,\lambda)(\lambda_{\alpha_1}-y_{\alpha_1})(\lambda_{\alpha_2}-y_{\alpha_2})d\Gamma_\lambda\Big)\varphi dy \quad \forall\varphi\in S^{h_1}(Y^*),\end{aligned} \tag{5.3a}$$

$$\begin{aligned}&\int_{Y^*} a_{ij}\frac{\partial M_{\alpha_1\alpha_2}^{h_1}}{\partial y_j}\frac{\partial \varphi}{\partial y_i}dy+4\sigma T^3\int_\Gamma G(M_{\alpha_1\alpha_2}^{h_1})\varphi dy\\ =&-\int_{Y^*}a_{i\alpha_1}\frac{\partial N_{\alpha_2}^{h_1}}{\partial T}\frac{\partial\varphi}{\partial y_i}dy+\int_{Y^*}a_{\alpha_1 i}\frac{\partial^2 N_{\alpha_2}^{h_1}}{\partial T\partial y_i}\varphi dy-\frac{12\sigma T^2}{|Y^*|}\int_{Y^*}\Big(\int_\Gamma y_{\alpha_1}G(N_{\alpha_2}^{h_1}+y_{\alpha_2})d\Gamma_y\Big)\varphi dy\\ &-\frac{1}{|Y^*|}\int_{Y^*}\Big(\int_{Y^*}a_{\alpha_1 i}\frac{\partial^2 N_{\alpha_2}^{h_1}}{\partial T\partial y_i}dy+4\sigma T^3\int_\Gamma y_{\alpha_1}G\Big(\frac{\partial N_{\alpha_2}^{h_1}}{\partial T}\Big)d\Gamma_y\Big)\varphi dy\\ &-6\sigma T^2\int_\Gamma G(N_{\alpha_1}^{h_1}N_{\alpha_2}^{h_1})\varphi d\Gamma_y+4\sigma T^3\int_\Gamma\Big(\int_\Gamma k(y,\lambda)(\lambda_{\alpha_1}-y_{\alpha_1})\frac{\partial N_{\alpha_2}^{h_1}}{\partial T}d\Gamma_\lambda\Big)\varphi d\Gamma_y\\ &+12\sigma T^2\int_\Gamma\Big(\int_\Gamma k(y,\lambda)(\lambda_{\alpha_1}-y_{\alpha_1})N_{\alpha_2}^{h_1}d\Gamma_\lambda\Big)\varphi d\Gamma_y\\ &+6\sigma T^2\int_\Gamma\Big(\int_\Gamma k(y,\lambda)(\lambda_{\alpha_1}-y_{\alpha_1})(\lambda_{\alpha_2}-y_{\alpha_2})d\Gamma_\lambda\Big)\varphi d\Gamma_y \quad \forall\varphi\in S^h(Y^*).\end{aligned} \tag{5.3b}$$

To approximate $\frac{\partial N_{\alpha_1}^{h_1}}{\partial T}$ and $\frac{\partial^2 N_{\alpha_1}^{h_1}}{\partial T\partial y_i}$, we use the difference quotient

$$\frac{\partial N_{\alpha_1}^{h_1}}{\partial T}\approx\frac{N_{\alpha_1}^{h_1}(T,y)-N_{\alpha_1}^{h_1}(T+\Delta T,y)}{\Delta T}, \tag{5.4a}$$

$$\frac{\partial^2 N_{\alpha_1}^{h_1}}{\partial T \partial y_i} \approx \frac{\frac{\partial N_{\alpha_1}^{h_1}(T,y)}{\partial y_i} - \frac{\partial N_{\alpha_1}^{h_1}(T+\Delta T,y)}{\partial y_i}}{\Delta T}. \tag{5.4b}$$

4. FE computation of the nonlinear homogenized problem (3.29) on Ω using the fixed-point algorithm.

Firstly for $n=0$, choose the initial temperate $T_{(0)}$ and get the cell solutions $N_{\alpha_1}^{h_1}(T_{(0)},y)$, $N_{\alpha_1\alpha_2}^{h_1}(T_{(0)},y)$, $M_{\alpha_1\alpha_2}^{h_1}(T_{(0)},y)$ and homogenized conductivity $a_{ij}^{0,h_1}(T_{(0)})$.

Secondly set $(a_{ij}^{0,h_1})_n = a_{ij}^{0,h_1}(T_{(0)})$ and solve the following linearized weak form at the step n

$$\int_\Omega (a_{ij}^{0,h_1})_n \frac{\partial (T_0^{h_0})_n}{\partial x_j} \frac{\partial \varphi}{\partial x_i} = \int_\Omega f \varphi dx \quad \forall \varphi \in S^{h_0}(\Omega), \tag{5.5}$$

where $S^{h_0}(\Omega)$ is the FE space with mesh size h_0 on homogenized domain Ω.

Compute the average temperature $(T_0^{h_0})^*$ and find the temperature $T\in[\overline{T}_1,\overline{T}_2]$ such that $T_{\alpha_1}\leq (T_0^{h_0})^* \leq T_{\alpha_2}$. Then Compute the homogenized conductivity $a_{ij}^{0,h}(T_{(n+1)})$ by

$$(a_{ij}^{0,h_1})_{n+1} = \eta a_{ij}^{0,h_1}(T_{\alpha_1}) + (1-\eta) a_{ij}^{0,h_1}(T_{\alpha_2}) \quad \forall 0<\eta<1. \tag{5.6}$$

Set $n=n+1$, solve the (5.5) and iterate the above procedure until the following convergence is satisfied

$$\|(T_0^{h_0})_n - (T_0^{h_0})_{n+1}\|_{H^1(\Omega)} \leq \epsilon, \tag{5.7}$$

where ϵ is tolerant error.

5. Computation of the $\widehat{T}^{L,h}$.

For $L=1$,

$$\widehat{T}^{1,h} = T_0^{h_0} + \varepsilon N_{\alpha_1}^{h_1} \frac{\partial T_0^{h_0}}{\partial x_{\alpha_1}}. \tag{5.8}$$

For $L=2$

$$\widehat{T}^{2,h} = \widehat{T}^{1,h} + \varepsilon^2 \Big(N_{\alpha_1\alpha_2}^{h_1} \frac{\partial^2 T_0^{h_0}}{\partial x_{\alpha_1} \partial x_{\alpha_1}} + M_{\alpha_1\alpha_2}^{h_1} \frac{\partial T_0^{h_0}}{\partial x_{\alpha_1}} \frac{\partial T_0^{h_0}}{\partial x_{\alpha_2}} \Big). \tag{5.9}$$

On each cell, we compute the average temperature

$$T_m(x) = \frac{1}{|\varepsilon Y^*|} \int_{\varepsilon Y^*} T_0^{h_0} dy,$$

and find the temperature $T\in[\overline{T}_1,\overline{T}_2]$ such that $T_{\beta_1} \leq T_m(x) \leq T_{\beta_2}$.

The solution $N_{\alpha_1}^{h_1}$ is evaluated as

$$N_{\alpha_1}^{h_1} = \eta N_{\alpha_1}^{h_1}(T_{\beta_1},y) + (1-\eta) N_{\alpha_1}^{h_1}(T_{\beta_2},y) \quad \forall 0<\eta<1, \tag{5.10}$$

and so are $N_{\alpha_1\alpha_2}^{h_1}$ and $M_{\alpha_1\alpha_2}^{h_1}$.

6. Computation of the thermal gradient.

$$\frac{\partial \widehat{T}^{1,h}}{\partial x_i}=\frac{\partial T_0^{h_0}}{\partial x_i}+\frac{\partial N_{\alpha_1}^{h_1}}{\partial y_i}\frac{\partial T_0^{h_0}}{\partial x_{\alpha_1}}+\varepsilon\left(\frac{\partial N_{\alpha_1}^{h_1}}{\partial T_0^{h_0}}\frac{\partial T_0^{h_0}}{\partial x_i}\frac{\partial T_0^{h_0}}{\partial x_{\alpha_1}}+N_{\alpha_1}^{h_1}\frac{\partial^2 T_0^{h_0}}{\partial x_{\alpha_1}\partial x_i}\right), \tag{5.11a}$$

$$\begin{aligned}\frac{\partial \widehat{T}^{2,h}}{\partial x_i}=&\frac{\partial \widehat{T}^{1,h}}{\partial x_i}+\varepsilon\left(\frac{\partial N_{\alpha_1\alpha_2}^{h_1}}{\partial y_i}\frac{\partial^2 T_0^{h_0}}{\partial x_{\alpha_1}\partial x_{\alpha_2}}+\frac{\partial M_{\alpha_1\alpha_2}^{h_1}}{\partial y_i}\frac{\partial T_0^{h_0}}{\partial x_{\alpha_1}}\frac{\partial T_0^{h_0}}{\partial x_{\alpha_2}}\right)\\ &+\varepsilon^2\left(\frac{\partial N_{\alpha_1\alpha_2}^{h_1}}{\partial T_0^{h_0}}\frac{\partial T_0^{h_0}}{\partial x_i}\frac{\partial^2 T_0^{h_0}}{\partial x_{\alpha_1}\partial x_{\alpha_2}}+\frac{\partial M_{\alpha_1\alpha_2}^{h_1}}{\partial T_0^{h_0}}\frac{\partial T_0^{h_0}}{\partial x_i}\frac{\partial T_0^{h_0}}{\partial x_{\alpha_1}}\frac{\partial T_0^{h_0}}{\partial x_{\alpha_2}}\right)\\ &+\varepsilon^2\left(N_{\alpha_1\alpha_2}^{h_1}\frac{\partial^3 T_0^{h_0}}{\partial x_{\alpha_1}\partial x_{\alpha_2}\partial x_i}+M_{\alpha_1\alpha_2}^{h_1}\left(\frac{\partial^2 T_0^{h_0}}{\partial x_{\alpha_1}\partial x_i}\frac{\partial T_0^{h_0}}{\partial x_{\alpha_2}}+\frac{\partial T_0^{h_0}}{\partial x_{\alpha_1}}\frac{\partial^2 T_0^{h_0}}{\partial x_{\alpha_2}\partial x_i}\right)\right),\end{aligned} \tag{5.11b}$$

where

$$\frac{\partial N_{\alpha_1}^{h_1}}{\partial y_i},\quad N_{\alpha_1\alpha_2}^{h_1},\quad M_{\alpha_1\alpha_2}^{h_1},\quad \frac{\partial N_{\alpha_1\alpha_2}^{h_1}}{\partial y_i}\quad \text{and}\quad \frac{\partial M_{\alpha_1\alpha_2}^{h_1}}{\partial y_i}$$

are evaluated the same way as (5.10). The terms

$$\frac{\partial N_{\alpha_1}^{h_1}}{\partial T_0^{h_0}},\quad \frac{\partial N_{\alpha_1\alpha_2}^{h_1}}{\partial T_0^{h_0}}\quad \text{and}\quad \frac{\partial M_{\alpha_1\alpha_2}^{h_1}}{\partial T_0^{h_0}}$$

are evaluated the way in (5.4a) and (5.10).

5.2 Numerical example

Here we make FE computations to show the effectiveness of the SOTS method for the conductivity-radiative heat transfer problem. Take the dimension $n=2$ and $\varepsilon=\frac{1}{8}$. The periodical porous domain Ω^ε and the normalized cell domain Y^* are shown in Fig. 2(a) and (b), respectively and the homogeneous domain $\Omega=[0,1]^2$.

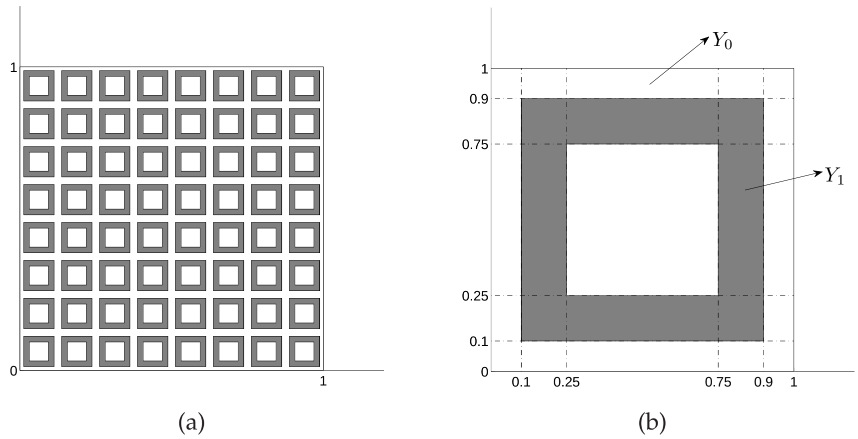

Figure 2: (a) Domain Ω^ε; (b) Unit cell Y^*.

The cell is composed of two kinds of materials — Y_0 and Y_1, each of which is homogeneous and isotropic, and the heat conductivity a_{ij} are respectively

$$a_{ij}\Big|_{Y_0}=\begin{pmatrix} 50 & 0 \\ 0 & 50 \end{pmatrix}, \quad a_{ij}\Big|_{Y_1}=\begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}. \tag{5.12}$$

The information of the FE meshes is listed in Table 1.

Table 1: Mesh information.

	Refined meshes	Cell mesh	Homogenized mesh
Number of elements	38272	15702	5862
Number of nodes	20673	8151	3032

According to the algorithms, for different temperature parameter, we can compute the corresponding homogenized conductivity a^0_{ij}. From the symmetry of our domains, $a^0_{11}=a^0_{22}$, $a^0_{12}=a^0_{21}=0$. Fig. 3 shows the homogenized conductivity in terms of T. We can see that, at low temperature the effect of radiation is not so important, but as the temperature become higher, it plays a prominent role in the heat transfer.

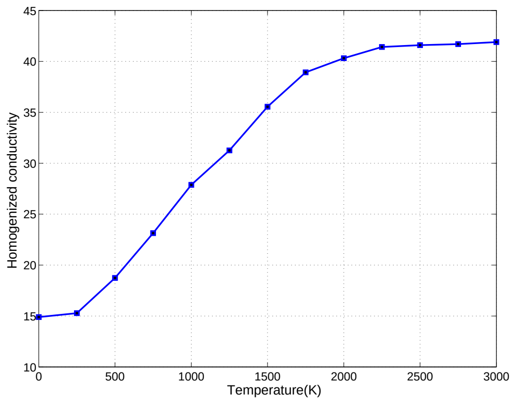

Figure 3: Homogenized Conductivity in terms of T.

Let the heat source $f=10000$ and the temperature $\widetilde{T}=500$ on the outer boundary $\partial\Omega$, and black body radiation on the surfaces Γ^ε. Since the exact solutions of (2.10) can not be obtained, we take $T_\varepsilon^{h_2}$ as the finite element approximate solutions in the fine mesh, and compare the asymptotic solution with it. Here h_2 is the size of the refined mesh, which is quite small.

Fig. 4 shows FE solution of temperature on refined mesh and asymptotic solutions.

Denote by e_0,e_1,e_2 the error between the refined solution and the asymptotic solution as follows:

$$e_0=T_\varepsilon^{h_2}-T_0^{h_0}, \quad e_1=T_\varepsilon^{h_2}-\widehat{T}^{1,h}, \quad e_2=T_\varepsilon^{h_2}-\widehat{T}^{2,h}.$$

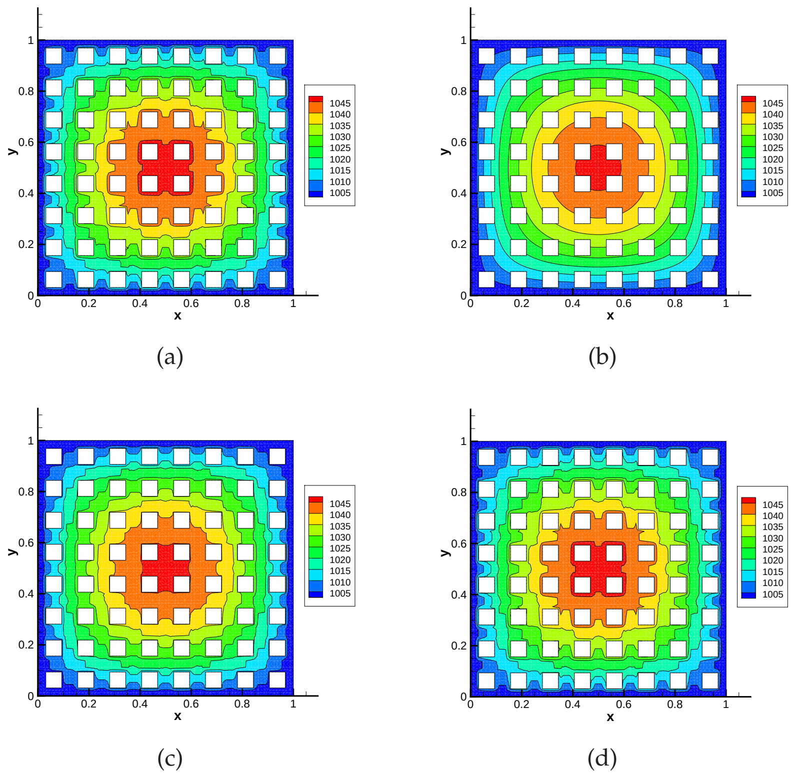

Figure 4: (a) FE solution $T_{\varepsilon}^{h_2}$ on refined mesh. Asymptotic solutions: (b) $T_0^{h_0}$; (c) $\widehat{T}^{1,h}$; (d) $\widehat{T}^{2,h}$.

Table 2 shows the computed relative errors, where $|\cdot|_{H^1}$ denotes the semi-norm.

Table 2: Comparison of computation results.

$\frac{\|e_0\|_{L^2}}{\|T_\varepsilon\|_{L^2}}$	$\frac{\|e_1\|_{L^2}}{\|T_\varepsilon\|_{L^2}}$	$\frac{\|e_2\|_{L^2}}{\|T_\varepsilon\|_{L^2}}$	$\frac{\vert e_0\vert_{H^1}}{\vert T_\varepsilon\vert_{H^1}}$	$\frac{\vert e_1\vert_{H^1}}{\vert T_\varepsilon\vert_{H^1}}$	$\frac{\vert e_2\vert_{H^1}}{\vert T_\varepsilon\vert_{H^1}}$
0.0193521	0.0144525	0.005306	0.72689	0.567339	0.0210051

From the tables and figures, it can be seen that:

1. The computational amount of the SOTS FE algorithm is much less than the classical FE computation with refined mesh. For the SOTS FE algorithm the radiation boundary condition is nonlocal but become linear, which is easier compared with the original problem. On the other hand, we need not to deal with this condition on each cavity, just in the reference cell domain.

2. The homogenized solution gives the original problem an asymptotic behavior, which is not enough for ε that is not so small. So the correctors are necessary. Table 2 shows that the second correctors give much better approximation of the temperature and its gradient.

6 Conclusions

The second-order two-scale approximate expression for the solution of heat conductive problem with the black body radiation on the surfaces of cavities in periodical porous domains is presented. By scaling ε^{-1} on Γ^ε, the homogenized conductivity and homogenized equation with radiation are obtained. The main difficulty lies in expansion of the integrand of the view factor $\int_{\Gamma_i^\varepsilon} k(x,s)T^4(s)d\Gamma_s^\varepsilon$. We make use of proper Taylor expansions so that the two scales, i.e x and y are handled separately. The continuity of $N_{\alpha_1}(T_0,y)$ with respect to T_0 and the existence of the coupled system (T_0,N_{α_1}) are given to insure the regularity of the solutions. The uniqueness of the coupled problem is difficulty to prove, and will be studied in our future paper.

As is previously stated that the first-order two-scale approximation is not enough to capture the microscopic behavior of the solution, the second order expansion is developed such that more detail information of radiation effect on the surfaces of the cavities can be acquired. The H^1-norm error estimation is followed by [3, 4] by the regularity assumption of the homogenized solution and the correctors. Numerical example shows the SOTS method is effective and only by adding the second order corrector can we more precisely obtain the local oscillation of the solution.

For the general case of $0<e<1$, the paper in Allaire [8] says, "The rigorous convergence of the homogenization process for the nonlinear model is an open problem", it is also quite difficult to expand the T^ε to higher order because of the complex form of G_ε. Future work will concern higher expansion for $0<e<1$ and the coupling with the mechanical behavior.

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