

Space-Time Discontinuous Galerkin Method for Maxwell's Equations

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Abstract. A fully discrete discontinuous Galerkin method is introduced for solving time-dependent Maxwell's equations. Distinguished from the Runge-Kutta discontinuous Galerkin method (RKDG) and the finite element time domain method (FETD), in our scheme, discontinuous Galerkin methods are used to discretize not only the spatial domain but also the temporal domain. The proposed numerical scheme is proved to be unconditionally stable, and a convergent rate $\mathcal{O}((\Delta t)^{r+1} + h^{k+1/2})$ is established under the L^2 -norm when polynomials of degree at most r and k are used for temporal and spatial approximation, respectively. Numerical results in both 2-D and 3-D are provided to validate the theoretical prediction. An ultra-convergence of order $(\Delta t)^{2r+1}$ in time step is observed numerically for the numerical fluxes w.r.t. temporal variable at the grid points.

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1 Introduction

Finite element methods, including edge element methods and discontinuous Galerkin methods, have been widely used to solve time-harmonic Maxwell's equations [2,4,28] as well as time-dependent Maxwell's equations [8,9,11,15,20–27], due to their high order accuracy and flexibility in handling complicated domains. Traditionally, they were only used to discretize the spatial domain to produce a system of ordinary differential equations (in time t), which was then solved by the finite difference or Runge-Kutta methods [8,11,15,25,27]. Towards this end, Makridakis and Monk proposed a fully discrete finite element method for Maxwell's equations and investigated the corresponding error estimates in [26]. Their approach resulted in a coupled non-symmetric and indefinite linear algebraic system involving both electric and magnetic fields. Later, Ciarlet Jr. and Zou [9] analyzed a fully discrete finite element approach for a second-order electric field equation derived from Maxwell's equations by eliminating the magnetic field. Both optimal energy-norm error estimate and optimal L^2 -norm error estimate were obtained. When dispersive media were involved, Li proposed some fully discrete numerical schemes. Both mixed finite element method [20–22] and interior penalty discontinuous Galerkin method [23] are considered for spatial discretization. Since Maxwell's equations are a coupled system, a fully discrete scheme was proposed by Ma [25], aimed to reduce the computational cost by denoting the magnetic field explicitly in the numerical scheme.

The idea to discretize the temporal domain by finite element method is not something new in the literature. Actually it was proposed as early as in late 60's by Argyris and Scharpf [1], and Oden [30]. Since then the space-time finite element methods have been widely used to solve a variety of differential equations, e.g., see [3,5,16] for the implementation of time-continuous Galerkin finite element schemes. Some works on space-time finite element method for solving hyperbolic equations are available, see [29,32]. Recently, Tu et al., proposed a space-time discontinuous Galerkin cell vertex scheme to solve conservation law and time dependent diffusion equations [33]. This scheme is conceptually simpler than other existing DG-type methods. Nevertheless, to the best of our knowledge, the finite element method has not been used to discretize the temporal domain in fully discrete scheme for Maxwell's equations up to now.

On the other hand, time-discontinuous Galerkin methods were originally developed for the first order hyperbolic equations [19,31] and have been successfully applied to various hyperbolic and parabolic equations (see [12,16] and the references therein). They usually lead to some stable and higher-order accurate numerical schemes. Actually in [18,19], the time-discontinuous Galerkin method was first shown to be an A-stable, higher-order accurate ordinary differential equation solver. Furthermore, the time-discontinuous Galerkin framework seems conducive for the rigorous justification of the error estimates [18].

In [34] we introduced a semi-discrete locally divergence-free DG method for solving Maxwell's equations in dispersive media under a unified framework. After the discretization of the spatial domain, we obtained a Volterra integro-differential system in

time t . Then a continuous Galerkin method was used to solve this reduced system. The numerical results are surprisingly good! The scheme is stable even when the time step size Δt is larger than the spatial mesh size h ! Indeed, the scheme is essentially implicit and places no restriction on the time step size. This advantage over many explicit schemes, which have a so-called CFL condition on the time step size Δt , makes the scheme worthwhile to be studied. Unfortunately, theoretical analysis of this mixed scheme (DG method in space and continuous Galerkin finite element method in time) seems to be very difficult.

Therefore, we want to seek for a numerical scheme, which not only is unconditionally stable, but also is accessible for theoretical justification. Since the DG method is applied to discretize the spatial domain in our earlier work [34], a natural consideration is to use it to treat the temporal domain as well. Hence, we propose here a space-time DG scheme, in which two different DG methods are used to discretize the spatial and temporal domains, respectively. Fortunately, we are able to prove that the new scheme is unconditionally stable. Again, we obtain very accurate numerical solutions even when the time step size Δt is larger than the spatial mesh size h , as expected. Furthermore, we prove the convergence rate $\mathcal{O}((\Delta t)^{r+1} + h^{k+\frac{1}{2}})$ in the L^2 -norm when the r -th and k -th order polynomials are used in temporal discretization and spatial discretization respectively. Comparing with finite difference methods used in [9, 14, 20–23, 25, 26], our space-time DG method is a high-order scheme in temporal variable.

The situation is even better in our numerical experiments, where the optimal convergence rate $\mathcal{O}(h^{k+1})$ in the spatial step is shown. Moreover, an ultra-convergence rate $\mathcal{O}((\Delta t)^{2r+1})$ in the time step is observed for the numerical fluxes with respect to the temporal variable at the grid points. This is another significant advantage of our approach over many existing numerical methods.

The outline of this paper is as follows. The model problem and our proposed space-time DG scheme are introduced in Section 2. Both the L^2 -stability and L^2 -error estimate are proved in Section 3, where an operator splitting is introduced to decompose the error into temporal part and spatial part. Numerical examples are provided in Section 4. Finally, some possible future works and concluding remarks are presented in Section 5.

2 Space-time discontinuous Galerkin method

2.1 Model problem

We consider Maxwell's equations in simple homogeneous media as follows:

$$\mu \frac{\partial \mathbf{H}}{\partial t} = -\nabla \times \mathbf{E}, \quad (\mathbf{x}, t) \in \Omega \times I, \quad (2.1)$$

$$\epsilon \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{H}, \quad (\mathbf{x}, t) \in \Omega \times I, \quad (2.2)$$

where ϵ and μ are the electric permittivity and magnetic permeability respectively, Ω is a Lipschitz polyhedron and $I = [0, T]$. Moreover a simple initial condition

$$\mathbf{H}(\mathbf{x}, 0) = \mathbf{H}_0(\mathbf{x}), \quad \mathbf{E}(\mathbf{x}, 0) = \mathbf{E}_0(\mathbf{x}) \quad \text{in } \Omega \tag{2.3}$$

and a perfect conduct boundary condition

$$\mathbf{n} \times \mathbf{E} = 0 \quad \text{on } \partial\Omega \times I \tag{2.4}$$

are imposed. Here \mathbf{n} is the outward normal of Ω , \mathbf{E}_0 and \mathbf{H}_0 are given functions with \mathbf{H}_0 satisfying [27]

$$\nabla \cdot (\mu \mathbf{H}_0) = 0 \quad \text{in } \Omega, \quad \mathbf{H}_0 \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega. \tag{2.5}$$

Then (2.5), together with (2.1), implies

$$\nabla \cdot (\mu \mathbf{H}) = 0 \quad \text{in } \Omega \times I, \tag{2.6}$$

which is usually a statement of Maxwell's equations. Furthermore the second condition in (2.5), combined with (2.1) and (2.2), leads to [27]

$$\mathbf{H} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \times (0, T]. \tag{2.7}$$

As in [11], we rewrite (2.1)-(2.3) into the conservative form

$$\mathbf{Q} \mathbf{U}_t + \nabla \cdot \mathbf{f}(\mathbf{U}) = 0, \quad (\mathbf{x}, t) \in \Omega \times I, \tag{2.8}$$

$$\mathbf{U}(\mathbf{x}, 0) = \mathbf{U}_0(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{2.9}$$

where

$$\mathbf{U} = \begin{pmatrix} \mathbf{H} \\ \mathbf{E} \end{pmatrix}, \quad \mathbf{U}_0(\mathbf{x}) = \begin{pmatrix} \mathbf{H}_0(\mathbf{x}) \\ \mathbf{E}_0(\mathbf{x}) \end{pmatrix}, \quad \mathbf{f}(\mathbf{U}) = [\mathbf{f}_1(\mathbf{U}), \mathbf{f}_2(\mathbf{U}), \mathbf{f}_3(\mathbf{U})]^T, \tag{2.10}$$

$$\mathbf{Q} = \begin{pmatrix} \mu I_{3 \times 3} & 0 \\ 0 & \epsilon I_{3 \times 3} \end{pmatrix}, \quad \mathbf{f}_i(\mathbf{U}) = \begin{pmatrix} \mathbf{e}_i \times \mathbf{E} \\ -\mathbf{e}_i \times \mathbf{H} \end{pmatrix}. \tag{2.11}$$

The following notations in Soblev space will be used later. Denote $H^k(\Omega)$ and $H^k(I)$ the standard Soblev spaces equipped with norms $\|\cdot\|_{k,\Omega}$ and $\|\cdot\|_{k,I}$, respectively. Further, define

$$L^2(I, H^k(\Omega)) = \left\{ u \mid u(\cdot, t) \in H^k(\Omega), \forall t \in I, \text{ and } \int_I \|u(\cdot, t)\|_{k,\Omega}^2 dt < \infty \right\}, \tag{2.12}$$

equipped with norm

$$\| \|u\| \|_{k,0} = \left(\int_I \|u(\cdot, t)\|_{k,\Omega}^2 dt \right)^{\frac{1}{2}}; \tag{2.13}$$

and

$$H^k(I, L^2(\Omega)) = \left\{ u \mid u(\mathbf{x}, \cdot) \in H^k(I), \forall \mathbf{x} \in \Omega, \text{ and } \int_{\Omega} \|u(\mathbf{x}, \cdot)\|_{k,I}^2 d\mathbf{x} < \infty \right\}, \tag{2.14}$$

equipped with norm

$$\|u\|_{0,k} = \left(\int_{\Omega} \|u(\mathbf{x}, \cdot)\|_{k,I}^2 d\mathbf{x} \right)^{\frac{1}{2}}. \tag{2.15}$$

The corresponding vector function spaces are denoted by $(H^k(\Omega))^3$, $(H^k(I))^3$, $(L^2(I, H^k(\Omega)))^3$ and $(H^k(I, L^2(\Omega)))^3$. For $\mathbf{U} = (\mathbf{H}, \mathbf{E})^T$ and $\mathbf{H}(\cdot, t), \mathbf{E}(\cdot, t) \in (H^k(\Omega))^3, \forall t \in I$, define

$$\|\mathbf{U}(\cdot, t)\|_{k,\Omega} = (\|\mathbf{H}(\cdot, t)\|_{k,\Omega}^2 + \|\mathbf{E}(\cdot, t)\|_{k,\Omega}^2)^{1/2}, \tag{2.16}$$

and

$$\|\mathbf{Q}^{1/2}\mathbf{U}(\cdot, t)\|_{k,\Omega} = (\mu\|\mathbf{H}(\cdot, t)\|_{k,\Omega}^2 + \epsilon\|\mathbf{E}(\cdot, t)\|_{k,\Omega}^2)^{1/2}. \tag{2.17}$$

2.2 Numerical scheme

Assume that \mathcal{T}_h is a triangulation of the domain Ω with the element denoted by K , the edge by e , and the outward normal by \mathbf{n}_K . We assume that every element K of the triangulation \mathcal{T}_h is affine equivalent, see [10, Section 2.3]. For each element K , we denote by h_K the diameter of K and by ρ_K the diameter of the biggest ball included in K . Set $h = \max_K \{\text{the radius of the largest circle within } K\}$. We also denote by \mathcal{E}_I the union of all interior faces of \mathcal{T}_h , by \mathcal{E}_D the union of all boundary faces of \mathcal{T}_h , and by $\mathcal{E} = \mathcal{E}_I \cup \mathcal{E}_D$ the union of all faces of \mathcal{T}_h . The triangulation we consider has to be regular, i.e. there exists a positive constant C such that

$$\frac{h_K}{\rho_K} \leq C, \quad \forall K \in \mathcal{T}_h, \tag{2.18}$$

see [10, Section 3.1]. Moreover, let $0 = t_0 < t_1 < \dots < t_n = T$ be a uniform triangulation of the interval I with elements denoted by $I_j = [t_j, t_{j+1}]$, $j = 0, 1, \dots, n-1$ and the time step size by $\Delta t = t_{j+1} - t_j$.

Let $P^k(K)$ (or $P^k(I_j)$) denote the space of polynomials in K (or I_j) of degree at most k . Then the DG finite element space for the spatial discretization is

$$\mathbf{S}_{h,\Omega}^k = \{v \in L^2(\Omega) : v|_K \in P^k(K), K \in \mathcal{T}_h\}.$$

On the other hand, the DG finite element space for the temporal discretization is

$$\mathbf{S}_{h,I}^r = \{v \in L^2(I) : v|_{I_j} \in P^r(I_j), j = 0, 1, \dots, n-1\}.$$

Since our approach is to discretize both the spatial and temporal domains by the DG methods, the space-time discontinuous finite element space is defined by

$$\mathbf{V}_h^{r,k} = \bar{\mathbf{V}}_h^{r,k} \oplus \tilde{\mathbf{V}}_h^{r,k},$$

where

$$\bar{\mathbf{V}}_h^{r,k} = (\mathbf{S}_{h,\Omega}^k \otimes \mathbf{S}_{h,I}^r)^3.$$

In fact, each component of the element in $\bar{\mathbf{V}}_h^{r,k}$ is the product of two elements from $\mathbf{S}_{h,\Omega}^k$ and $\mathbf{S}_{h,I}^r$.

It is well known that the choice of the numerical fluxes plays a crucial role in the design of discontinuous Galerkin schemes. In order to define numerical fluxes, we need to introduce some notations first. Let e be an interior face belonging to element K . We denote

$$\mathbf{v}^{int(K)}(\mathbf{x}) = \lim_{\delta \rightarrow 0^-} \mathbf{v}(\mathbf{x} + \delta \mathbf{n}_K), \quad \mathbf{v}^{ext(K)}(\mathbf{x}) = \lim_{\delta \rightarrow 0^+} \mathbf{v}(\mathbf{x} + \delta \mathbf{n}_K) \quad \forall \mathbf{x} \in e.$$

Then we define the average and tangential jump of \mathbf{v} on any interior face e as follows:

$$\bar{\mathbf{v}} = \frac{\mathbf{v}^{int(K)} + \mathbf{v}^{ext(K)}}{2}, \quad [\mathbf{v}]_T = \mathbf{n}_K \times \mathbf{v}^{int(K)} - \mathbf{n}_K \times \mathbf{v}^{ext(K)}.$$

For a boundary face $e \in \mathcal{E}_D$ which belongs to the element K , we denote

$$\mathbf{v}^{int}(\mathbf{x}) = \mathbf{v}^{int(K)}(\mathbf{x}) \quad \forall \mathbf{x} \in e.$$

In addition, we define

$$\mathbf{v}(t_j^+) = \lim_{t \rightarrow t_j+0} \mathbf{v}(t), \quad \mathbf{v}(t_j^-) = \lim_{t \rightarrow t_j-0} \mathbf{v}(t).$$

Now we are ready for the definition of the numerical scheme. Multiplying (2.8)-(2.9) by a test function \mathbf{v} , integrating over each $Q_j^K = K \times I_j$, and then integrating by parts, we obtain

$$\begin{aligned} & - \int_{I_j} \int_K \mathbf{Q}\mathbf{U} \cdot \mathbf{v}_t dxdt + \int_{I_j} \int_{\partial K} (\mathbf{f}(\mathbf{U}) \cdot \mathbf{n}_K) \cdot \mathbf{v} dsdt \\ & - \int_{I_j} \int_K \mathbf{f}(\mathbf{U}) \cdot \nabla \mathbf{v} dxdt + \int_K (\mathbf{Q}\mathbf{U} \cdot \mathbf{v})|_{t_j}^{t_{j+1}} dx = 0. \end{aligned} \tag{2.19}$$

Then the DG scheme based on (2.19) is to find $\mathbf{U}_h \in \mathbf{V}_h^{r,k}$ such that

$$\begin{aligned} & - \int_{I_j} \int_K \mathbf{Q}\mathbf{U}_h \cdot (\mathbf{v}_h)_t dxdt + \int_{I_j} \int_{\partial K} (\widehat{\mathbf{f}(\mathbf{U}_h)} \cdot \mathbf{n}_K) \cdot \mathbf{v}_h dsdt \\ & - \int_{I_j} \int_K \mathbf{f}(\mathbf{U}_h) \cdot \nabla \mathbf{v}_h dxdt + \int_K (\mathbf{Q}\hat{\mathbf{U}}_h \cdot \mathbf{v}_h)|_{t_j}^{t_{j+1}} dx = 0 \end{aligned} \tag{2.20}$$

for all $\mathbf{v}_h \in \mathbf{V}_h^{r,k}$ and all elements $Q_j^K = K \times I_j$, $K \in \mathcal{T}_h$, $j = 0, 1, \dots, n-1$. Here $(\widehat{\mathbf{f}(\mathbf{U}_h)} \cdot \mathbf{n}_K)$ and $\hat{\mathbf{U}}_h$ are the numerical fluxes on the face $e \in \mathcal{E}$ and at the nodal points $t_j, j = 0, 1, \dots, n$, respectively. Motivated by [11] and [34], we take

$$\widehat{\mathbf{f}(\mathbf{U}_h)} \cdot \mathbf{n}_K = \begin{pmatrix} \mathbf{n}_K \times (\bar{\mathbf{E}}_h - \frac{Z}{2} [\mathbf{H}_h]_T) \\ -\mathbf{n}_K \times (\bar{\mathbf{H}}_h + \frac{1}{2Z} [\mathbf{E}_h]_T) \end{pmatrix} \tag{2.21}$$

on an interior face and

$$\widehat{\mathbf{f}(\mathbf{U}_h)} \cdot \mathbf{n} = \begin{pmatrix} \mathbf{0}_{3 \times 1} \\ -\mathbf{n} \times (\mathbf{H}_h^{int} + \frac{1}{Z} \mathbf{n} \times \mathbf{E}_h^{int}) \end{pmatrix} \tag{2.22}$$

on a boundary face $e = \partial K \cap \mathcal{E}_D$, where $Z = \sqrt{\mu/\epsilon}$ denotes the impedance of the medium. Obviously this numerical flux is consistent with $\mathbf{f}(\mathbf{U}) \cdot \mathbf{n}_K$. On the other hand, we take

$$\hat{\mathbf{U}}_h(\mathbf{x}, t_j) = \mathbf{U}_h(\mathbf{x}, t_j^-), \quad j = 1, 2, \dots, n, \quad \hat{\mathbf{U}}_h(\mathbf{x}, 0) = \mathbf{P}_h \mathbf{U}_0, \tag{2.23}$$

where \mathbf{P}_h is the element-wise L^2 projection operator and will be defined later.

3 L^2 -stability and L^2 -error estimate

In this section, both the L^2 -stability and error estimate in L^2 norm of our numerical scheme will be analyzed. The fact that the DG methods are used to discretize spatial and temporal domains simultaneously will facilitate the theoretical justification. Actually the corresponding theoretical analysis can be done under the framework of standard Galerkin finite element method based on an operator decomposition technique.

3.1 L^2 -stability

We first focus on the L^2 -stability of our numerical scheme. Set

$$\begin{aligned} \mathbf{B}_{I_j, K}(\mathbf{U}_h, \mathbf{v}_h) = & - \int_{I_j} \int_K \mathbf{Q} \mathbf{U}_h \cdot (\mathbf{v}_h)_t dx dt + \int_{I_j} \int_{\partial K} (\widehat{\mathbf{f}(\mathbf{U}_h)} \cdot \mathbf{n}_K) \cdot \mathbf{v}_h ds dt \\ & - \int_{I_j} \int_K \mathbf{f}(\mathbf{U}_h) \cdot \nabla \mathbf{v}_h dx dt + \int_K (\mathbf{Q} \hat{\mathbf{U}}_h \cdot \mathbf{v}_h) |_{t_j}^{t_{j+1}} dx, \end{aligned} \tag{3.1}$$

and

$$\mathbf{B}_{I_j}(\mathbf{U}_h, \mathbf{v}_h) = \sum_{K \in \mathcal{T}_h} \mathbf{B}_{I_j, K}(\mathbf{U}_h, \mathbf{v}_h). \tag{3.2}$$

According to the DG scheme (2.20), $\mathbf{B}_{I_j, K}(\mathbf{U}_h, \mathbf{v}_h) = 0$, for all $\mathbf{v}_h \in \mathbf{V}_h^{r, k}$. Then the solution \mathbf{U}_h satisfies

$$\mathbf{B}_{I_j}(\mathbf{U}_h, \mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h^{r, k}, \quad j = 0, 1, \dots, n-1. \tag{3.3}$$

We have the following result for the stability of the DG scheme (2.20).

Theorem 3.1. Assume that $\mathbf{U}_h = (\mathbf{H}_h, \mathbf{E}_h)^T$ is a solution of (2.20), then

$$\|\mathbf{Q}^{1/2} \mathbf{U}_h(\cdot, T^-)\|_{0, \Omega}^2 + \Theta_{T, \mathcal{T}_h}(\mathbf{U}_h) \leq \|\mathbf{Q}^{1/2} \mathbf{U}_0\|_{0, \Omega}^2,$$

where

$$\Theta_{T, \mathcal{T}_h}(\mathbf{U}_h) = \sum_{j=0}^{n-1} \Theta_{I_j, \mathcal{T}_h}(\mathbf{U}_h) \tag{3.4}$$

and

$$\begin{aligned} & \Theta_{I_j, \mathcal{T}_h}(\mathbf{U}_h) \\ &= \int_{I_j} \sum_{e \in \mathcal{E}_T} \int_e \left(Z |[\mathbf{U}_h, \mathbf{H}]_T|^2 + \frac{1}{Z} |[\mathbf{U}_h, \mathbf{E}]_T|^2 \right) ds dt + 2 \int_{I_j} \sum_{e \in \mathcal{E}_D} \int_e \frac{1}{Z} |\mathbf{n} \times \mathbf{U}_{h, \mathbf{E}}^{int}|^2 ds dt. \end{aligned}$$

Proof. Let $\mathbf{v}_h = \mathbf{U}_h$ in (3.2), we have

$$\begin{aligned} \mathbf{B}_{I_j}(\mathbf{U}_h, \mathbf{U}_h) &= - \sum_{K \in \mathcal{T}_h} \int_{I_j} \int_K \mathbf{Q} \mathbf{U}_h \cdot (\mathbf{U}_h)_t dx dt + \sum_{K \in \mathcal{T}_h} \int_{I_j} \int_{\partial K} (\widehat{\mathbf{f}(\mathbf{U}_h)} \cdot \mathbf{n}_K) \cdot \mathbf{U}_h ds dt \\ &\quad - \sum_{K \in \mathcal{T}_h} \int_{I_j} \int_K \mathbf{f}(\mathbf{U}_h) \cdot \nabla \mathbf{U}_h dx dt + \sum_{K \in \mathcal{T}_h} \int_K (\mathbf{Q} \hat{\mathbf{U}}_h \cdot \mathbf{U}_h)|_{t_j}^{t_{j+1}} dx. \end{aligned} \tag{3.5}$$

By the definition of the numerical flux in (2.23) and Schwartz inequality, direct calculation shows that

$$\begin{aligned} & - \sum_{K \in \mathcal{T}_h} \int_{I_j} \int_K \mathbf{Q} \mathbf{U}_h \cdot (\mathbf{U}_h)_t dx dt + \sum_{K \in \mathcal{T}_h} \int_K (\mathbf{Q} \hat{\mathbf{U}}_h \cdot \mathbf{U}_h)|_{t_j}^{t_{j+1}} dx \\ & \geq \frac{1}{2} \|\mathbf{Q}^{1/2} \mathbf{U}_h(\cdot, t_{j+1}^-)\|_{0, \Omega}^2 - \frac{1}{2} \|\mathbf{Q}^{1/2} \mathbf{U}_h(\cdot, t_j^-)\|_{0, \Omega}^2, \quad j = 1, 2, \dots, n-1 \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} & - \sum_{K \in \mathcal{T}_h} \int_{I_0} \int_K \mathbf{Q} \mathbf{U}_h \cdot (\mathbf{U}_h)_t dx dt + \sum_{K \in \mathcal{T}_h} \int_K (\mathbf{Q} \hat{\mathbf{U}}_h \cdot \mathbf{U}_h)|_{t_0}^{t_1} dx \\ & \geq \frac{1}{2} \|\mathbf{Q}^{1/2} \mathbf{U}_h(\cdot, t_1^-)\|_{0, \Omega}^2 - \frac{1}{2} \|\mathbf{Q}^{1/2} \mathbf{U}_0\|_{0, \Omega}^2, \end{aligned} \tag{3.7}$$

where $\|\mathbf{P}_h \mathbf{U}\|_0 \leq \|\mathbf{U}\|_0$ is used according to the definition of the projection operator \mathbf{P}_h in Subsection 3.2. Moreover, by following the same strategy as that in the proof of Lemma 3.1 in [34], we obtain a similar identity, i.e.,

$$\begin{aligned} & \sum_{K \in \mathcal{T}_h} \int_{I_j} \int_{\partial K} (\widehat{\mathbf{f}(\mathbf{U}_h)} \cdot \mathbf{n}_K) \cdot \mathbf{U}_h ds dt - \sum_{K \in \mathcal{T}_h} \int_{I_j} \int_K \mathbf{f}(\mathbf{U}_h) \cdot \nabla \mathbf{U}_h dx dt \\ &= \frac{1}{2} \Theta_{I_j, \mathcal{T}_h}(\mathbf{U}_h). \end{aligned} \tag{3.8}$$

The combination of (3.3) and (3.5)-(3.8) leads to

$$\frac{1}{2} \|\mathbf{Q}^{1/2} \mathbf{U}_h(\cdot, t_{j+1}^-)\|_{0, \Omega}^2 - \frac{1}{2} \|\mathbf{Q}^{1/2} \mathbf{U}_h(\cdot, t_j^-)\|_{0, \Omega}^2 + \frac{1}{2} \Theta_{I_j, \mathcal{T}_h}(\mathbf{U}_h) \leq 0 \tag{3.9}$$

for $j = 1, 2, \dots, n-1$ and

$$\frac{1}{2} \|\mathbf{Q}^{1/2} \mathbf{U}_h(\cdot, t_1^-)\|_{0, \Omega}^2 - \frac{1}{2} \|\mathbf{Q}^{1/2} \mathbf{U}_0\|_{0, \Omega}^2 + \frac{1}{2} \Theta_{I_0, \mathcal{T}_h}(\mathbf{U}_h) \leq 0. \tag{3.10}$$

Summing up (3.9) and (3.10) over j from 0 to $n-1$, we obtain the desired result. \square

3.2 Error estimate

Now we turn to the L^2 error estimate of the space-time DG solution. For this purpose, we introduce two element-wise projection operators Π_h and \mathbf{P}_h and give the corresponding approximation results which will be used in the proof of the L^2 -error estimate later. First we introduce a projection operator $\Pi_h: H^{r+1}(I) \rightarrow S_{h,I}^r$ such that

$$\Pi_h u(t_{j+1}^-) = u(t_{j+1}^-), \tag{3.11}$$

$$\int_{I_j} (u - \Pi_h u) v dt = 0, \quad \forall v \in P^{r-1}(I_j), \quad j=0,1,\dots,n-1, \quad r \geq 1. \tag{3.12}$$

Furthermore, we have the following error estimate [6].

Lemma 3.1. *For any $u \in H^{r+1}(I_j)$, we have*

$$\|u - \Pi_h u\|_{0,I_j} \leq C(\Delta t)^{r+1} |u|_{r+1,I_j}. \tag{3.13}$$

Moreover, we also need the element-wise L^2 -projection operator $\mathbf{P}_h: H^{k+1}(\Omega) \rightarrow \mathbf{S}_{h,\Omega}^k$, such that

$$\int_K (u - \mathbf{P}_h u) v d\mathbf{x} = 0, \quad \forall v \in P^k(K), \quad \forall K \in \mathcal{T}_h. \tag{3.14}$$

For this L^2 projection operator, we have the following approximation lemma.

Lemma 3.2. *Let $u \in H^{k+1}(K)$. Then*

$$\|u - \mathbf{P}_h u\|_{0,K} \leq Ch^{k+1} |u|_{k+1,K}, \quad \|u - \mathbf{P}_h u\|_{0,\partial K} \leq Ch^{k+1/2} |u|_{k+1,K}.$$

The error analysis of numerical methods for time-dependent problems is often more difficult than that for the time-independent ones. Actually, the main difficulty is how to decompose the error into the temporal part and spatial part which can be handled independently. In our work we introduce an operator decomposition as follows:

$$I - \Pi_h \otimes \mathbf{P}_h = (I - \Pi_h) + (I - \mathbf{P}_h) - (I - \Pi_h) \otimes (I - \mathbf{P}_h). \tag{3.15}$$

In fact, this technique was first introduced in the analysis of the finite element method for multi-dimensional elliptic problems by Douglas, Dupont and Wheeler in [13]. Then it was implemented to analyze the convergence property of the finite element methods for parabolic and hyperbolic problems by Chen (see Chapter 3 in [7] for more details). In terms of the orthogonality relations in (3.12) and (3.14), we have

$$\int_{I_j} [(I - \Pi_h) \otimes (I - \mathbf{P}_h) u] v = 0, \quad \forall v \in P^{k-1}(I_j), \quad k \geq 1, \tag{3.16}$$

$$\int_K [(I - \Pi_h) \otimes (I - \mathbf{P}_h) u] v = 0, \quad \forall v \in P^k(K), \quad K \in \mathcal{T}_h. \tag{3.17}$$

Actually (3.16) can be proved by a straightforward implementation of (3.12). On the other hand, (3.17) can be obtained immediately based on the fact that $I - \Pi_h$ and $I - P_h$ are independent of each other.

It is noted that the numerical fluxes defined in (2.21), (2.22) and (2.23) are consistent except for $\widehat{U_h(0)} = P_h U_0$. Set $\mathbf{e} = \mathbf{U} - U_h$. According to (2.19) and (2.20), we have

$$\mathbf{B}_{I_j, K}(\mathbf{e}, \mathbf{v}_h) = 0, \forall \mathbf{v}_h \in \mathbf{V}_h^{r, k}, \quad j = 1, 2, \dots, n-1. \tag{3.18}$$

On the other hand, by (2.20),

$$\begin{aligned} & \mathbf{B}_{I_0, K}(U_h, \mathbf{v}_h) \\ &= - \int_{I_0} \left[\int_K \mathbf{Q}U_h \cdot (\mathbf{v}_h)_t dx + \int_{\partial K} (\mathbf{f}(\widehat{U_h}) \cdot \mathbf{n}_K) \cdot \mathbf{v}_h ds - \int_K \mathbf{f}(U_h) \cdot \nabla \mathbf{v}_h dx \right] dt \\ & \quad + \int_K [\mathbf{Q}U_h(\mathbf{x}, t_1^-) \cdot \mathbf{v}_h(\mathbf{x}, t_1^-) - \mathbf{Q}P_h U_0 \cdot \mathbf{v}_h(\mathbf{x}, 0^+)] dx \\ &= - \int_{I_0} \left[\int_K \mathbf{Q}U_h \cdot (\mathbf{v}_h)_t dx + \int_{\partial K} (\mathbf{f}(\widehat{U_h}) \cdot \mathbf{n}_K) \cdot \mathbf{v}_h ds - \int_K \mathbf{f}(U_h) \cdot \nabla \mathbf{v}_h dx \right] dt \\ & \quad + \int_K [\mathbf{Q}U_h(\mathbf{x}, t_1^-) \cdot \mathbf{v}_h(\mathbf{x}, t_1^-) - \mathbf{Q}U_0 \cdot \mathbf{v}_h(\mathbf{x}, 0^+)] dx. \end{aligned} \tag{3.19}$$

Taking $\mathbf{v} = \mathbf{v}_h$ in (2.19) and then subtracting (3.19) from (2.19) with $j=0$, we obtain

$$\begin{aligned} & - \int_{I_0} \int_K \mathbf{Q}\mathbf{e} \cdot (\mathbf{v}_h)_t dx dt + \int_{I_0} \int_{\partial K} (\mathbf{f}(\widehat{\mathbf{e}}) \cdot \mathbf{n}_K) \cdot \mathbf{v}_h ds dt \\ & \quad - \int_{I_0} \int_K \mathbf{f}(\mathbf{e}) \cdot \nabla \mathbf{v}_h dx dt + \int_K \mathbf{Q}\mathbf{e}(\mathbf{x}, t_1^-) \cdot \mathbf{v}_h(\mathbf{x}, t_1^-) dx = 0. \end{aligned} \tag{3.20}$$

Denote the left-hand side of (3.20) by $\bar{\mathbf{B}}_{I_0, K}(\mathbf{e}, \mathbf{v}_h)$ and let

$$\bar{\mathbf{B}}_{I_j, K}(\mathbf{e}, \mathbf{v}_h) = \mathbf{B}_{I_j, K}(\mathbf{e}, \mathbf{v}_h) \quad \text{for } j = 1, 2, \dots, n-1.$$

Then we obtain the following error equation

$$\bar{\mathbf{B}}_{I_j}(\mathbf{e}, \mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h^{r, k}, \quad j = 0, 1, \dots, n-1, \tag{3.21}$$

where

$$\bar{\mathbf{B}}_{I_j}(\mathbf{e}, \mathbf{v}_h) = \sum_{K \in \mathcal{T}_h} \bar{\mathbf{B}}_{I_j, K}(\mathbf{e}, \mathbf{v}_h). \tag{3.22}$$

The error \mathbf{e} can be decomposed into

$$\mathbf{e} = \mathbf{U} - \Pi_h \otimes P_h \mathbf{U} - (\mathbf{U}_h - \Pi_h \otimes P_h \mathbf{U}) = \mathbf{R} - \theta, \tag{3.23}$$

where

$$\mathbf{R} = \mathbf{U} - \Pi_h \otimes P_h \mathbf{U}, \quad \theta = \mathbf{U}_h - \Pi_h \otimes P_h \mathbf{U} \in \mathbf{V}_h^{r, k}. \tag{3.24}$$

Substituting $\mathbf{e} = \mathbf{R} - \theta$ into (3.21) and taking $\mathbf{v}_h = \theta$, we have

$$\bar{\mathbf{B}}_{I_j}(\mathbf{R}, \theta) = \bar{\mathbf{B}}_{I_j}(\theta, \theta). \tag{3.25}$$

Lemma 3.3. *In terms of $\bar{\mathbf{B}}_{I_j}(\theta, \theta)$, we have the following estimate*

$$\bar{\mathbf{B}}_{I_j}(\theta, \theta) \geq \frac{1}{2} \|\mathbf{Q}^{1/2} \theta(\cdot, t_{j+1}^-)\|_{0, \Omega}^2 - \frac{1}{2} \|\mathbf{Q}^{1/2} \theta(\cdot, t_j^-)\|_{0, \Omega}^2 + \frac{1}{2} \Theta_{I_j, \mathcal{T}_h}(\theta), \tag{3.26}$$

for $j = 1, 2, \dots, n-1$ and

$$\bar{\mathbf{B}}_{I_0}(\theta, \theta) = \frac{1}{2} \|\mathbf{Q}^{1/2} \theta(\cdot, t_1^-)\|_{0, \Omega}^2 + \frac{1}{2} \|\mathbf{Q}^{1/2} \theta(\cdot, 0^+)\|_{0, \Omega}^2 + \frac{1}{2} \Theta_{I_0, \mathcal{T}_h}(\theta). \tag{3.27}$$

Proof. By the definition of $\bar{\mathbf{B}}_{I_j}(\mathbf{e}, \mathbf{v}_h)$, we have

$$\begin{aligned} \bar{\mathbf{B}}_{I_j}(\theta, \theta) = & - \sum_{K \in \mathcal{T}_h} \int_{I_j} \int_K \mathbf{Q} \theta \cdot (\theta)_t dx dt + \sum_{K \in \mathcal{T}_h} \int_{I_j} \int_{\partial K} (\widehat{\mathbf{f}(\theta)} \cdot \mathbf{n}_K) \cdot \theta ds dt \\ & - \sum_{K \in \mathcal{T}_h} \int_{I_j} \int_K \mathbf{f}(\theta) \cdot \nabla \theta dx dt + \sum_{K \in \mathcal{T}_h} \int_K (\mathbf{Q} \hat{\theta} \cdot \theta)|_{t_j}^{t_{j+1}} dx, \end{aligned} \tag{3.28}$$

for $j = 1, 2, \dots, n-1$. Similar to the proof of the stability in Theorem 3.1, we have

$$\begin{aligned} & - \sum_{K \in \mathcal{T}_h} \int_{I_j} \int_K \mathbf{Q} \theta \cdot (\theta)_t dx dt + \sum_{K \in \mathcal{T}_h} \int_K (\mathbf{Q} \hat{\theta} \cdot \theta)|_{t_j}^{t_{j+1}} dx \\ & \geq \frac{1}{2} \|\mathbf{Q}^{1/2} \theta(\cdot, t_{j+1}^-)\|_{0, \Omega}^2 - \frac{1}{2} \|\mathbf{Q}^{1/2} \theta(\cdot, t_j^-)\|_{0, \Omega}^2, \end{aligned} \tag{3.29}$$

for $j = 1, 2, \dots, n-1$. Following the same strategy as those in the proof of Theorem 3.1, it is easy to obtain

$$\sum_{K \in \mathcal{T}_h} \int_{I_j} \int_{\partial K} (\widehat{\mathbf{f}(\theta)} \cdot \mathbf{n}_K) \cdot \theta ds dt - \sum_{K \in \mathcal{T}_h} \int_{I_j} \int_K \mathbf{f}(\theta) \cdot \nabla \theta dx dt = \frac{1}{2} \Theta_{I_j, \mathcal{T}_h}(\theta). \tag{3.30}$$

The combination of (3.28), (3.29) and (3.30) yields (3.26).

According to (3.19), we have

$$\begin{aligned} \bar{\mathbf{B}}_{I_0}(\theta, \theta) = & - \sum_{K \in \mathcal{T}_h} \int_{I_0} \int_K \mathbf{Q} \theta \cdot (\theta)_t dx dt + \sum_{K \in \mathcal{T}_h} \int_{I_0} \int_{\partial K} (\widehat{\mathbf{f}(\theta)} \cdot \mathbf{n}_K) \cdot \theta ds dt \\ & - \sum_{K \in \mathcal{T}_h} \int_{I_0} \int_K \mathbf{f}(\theta) \cdot \nabla \theta dx dt + \sum_{K \in \mathcal{T}_h} \int_K \mathbf{Q} \theta(\mathbf{x}, t_1^-) \cdot \theta(\mathbf{x}, t_1^-) dx \\ = & \frac{1}{2} \|\mathbf{Q}^{1/2} \theta(\cdot, t_1^-)\|_{0, \Omega}^2 + \frac{1}{2} \|\mathbf{Q}^{1/2} \theta(\cdot, 0^+)\|_{0, \Omega}^2 + \frac{1}{2} \Theta_{I_0, \mathcal{T}_h}(\theta). \end{aligned} \tag{3.31}$$

The proof is complete. □

Now the key point is to estimate $\bar{\mathbf{B}}_{I_0}(\mathbf{R}, \theta)$ and $\mathbf{B}_{I_j}(\mathbf{R}, \theta)$, $j = 1, 2, \dots, n-1$. In terms of (3.1), (3.20), we have

$$\begin{aligned} \bar{\mathbf{B}}_{I_j}(\mathbf{R}, \theta) = & - \sum_{K \in \mathcal{T}_h} \int_{I_j} \int_K \mathbf{Q} \mathbf{R} \cdot \theta_t dx dt + \sum_{K \in \mathcal{T}_h} \int_{I_j} \int_{\partial K} (\widehat{\mathbf{f}(\mathbf{R})} \cdot \mathbf{n}_K) \cdot \theta ds dt \\ & - \sum_{K \in \mathcal{T}_h} \int_{I_j} \int_K \mathbf{f}(\mathbf{R}) \cdot \nabla \theta dx dt + \sum_{K \in \mathcal{T}_h} \int_K (\mathbf{Q} \hat{\mathbf{R}} \cdot \theta)|_{t_j}^{t_{j+1}} dx, \end{aligned} \tag{3.32}$$

for $j=1,2,\dots,n-1$, and

$$\begin{aligned} \bar{\mathbf{B}}_{I_0}(\mathbf{R},\theta) = & - \sum_{K \in \mathcal{T}_h} \int_{I_0} \int_K \mathbf{QR} \cdot \theta_t d\mathbf{x} dt + \sum_{K \in \mathcal{T}_h} \int_{I_0} \int_{\partial K} \left(\widehat{\mathbf{f}(\mathbf{R})} \cdot \mathbf{n}_K \right) \cdot \theta ds dt \\ & - \sum_{K \in \mathcal{T}_h} \int_{I_0} \int_K \mathbf{f}(\mathbf{R}) \cdot \nabla \theta d\mathbf{x} dt + \sum_{K \in \mathcal{T}_h} \int_K \mathbf{QR}(\mathbf{x}, t_1^-) \cdot \theta(\mathbf{x}, t_1^-) d\mathbf{x}. \end{aligned} \quad (3.33)$$

Now (3.15) is used to decompose \mathbf{R} into three parts, i.e.,

$$\begin{aligned} \mathbf{R} &= (I - \mathbf{\Pi}_h \otimes \mathbf{P}_h) \mathbf{U} \\ &= (I - \mathbf{\Pi}_h) \mathbf{U} + (I - \mathbf{P}_h) \mathbf{U} - (I - \mathbf{\Pi}_h) \otimes (I - \mathbf{P}_h) \mathbf{U}. \end{aligned} \quad (3.34)$$

Denoted by

$$\xi = (I - \mathbf{\Pi}_h) \mathbf{U}, \quad \eta = (I - \mathbf{P}_h) \mathbf{U}, \quad \rho = -(I - \mathbf{\Pi}_h) \otimes (I - \mathbf{P}_h) \mathbf{U},$$

then

$$\mathbf{R} = \xi + \eta + \rho. \quad (3.35)$$

According to the properties of projection $\mathbf{\Pi}_h$ shown in (3.11), we have

$$\xi(\mathbf{x}, t_j^-) = 0, \quad \rho(\mathbf{x}, t_j^-) = 0, \quad \forall \mathbf{x} \in \Omega, \quad j=1,2,\dots,n. \quad (3.36)$$

Due to $\theta, \theta_t \in \mathbf{V}_h^{r,k}$, thus $\theta_{\mathbf{H}}(\mathbf{x}, \cdot)|_K, (\theta_{\mathbf{H}})_t(\mathbf{x}, \cdot)|_K, \theta_{\mathbf{E}}(\mathbf{x}, \cdot)|_K, (\theta_{\mathbf{E}})_t(\mathbf{x}, \cdot)|_K \in (P^k(K))^3, (\theta_{\mathbf{H}})_t(\cdot, t)|_{I_j}, (\theta_{\mathbf{E}})_t(\cdot, t)|_{I_j} \in (P^{r-1}(I_j))^3$. According to the definition of the projection operators $\mathbf{\Pi}_h$ and \mathbf{P}_h , (3.16) and (3.17), we have the following orthogonality relations, i.e.,

$$\int_{I_j} \xi(\mathbf{x}, t) \cdot \theta_t(\mathbf{x}, t) dt = 0, \quad \int_{I_j} \rho(\mathbf{x}, t) \cdot \theta_t(\mathbf{x}, t) dt = 0, \quad \forall \mathbf{x} \in \Omega, \quad (3.37a)$$

$$\int_K \eta(\mathbf{x}, t) \cdot \theta_t(\mathbf{x}, t) d\mathbf{x} = 0, \quad \int_K \rho(\mathbf{x}, t) \cdot \theta_t(\mathbf{x}, t) d\mathbf{x} = 0, \quad \forall t \in (0, T], \quad (3.37b)$$

$$\int_K \eta(\mathbf{x}, t) \cdot \theta(\mathbf{x}, t) d\mathbf{x} = 0, \quad \int_K \rho(\mathbf{x}, t) \cdot \theta(\mathbf{x}, t) d\mathbf{x} = 0, \quad \forall t \in (0, T]. \quad (3.37c)$$

As a consequence,

$$\int_{I_j} \int_K \mathbf{QR} \cdot \theta_t d\mathbf{x} dt = \int_{I_j} \int_K \mathbf{Q}(\xi + \eta + \rho) \cdot \theta_t d\mathbf{x} dt = 0, \quad K \in \mathcal{T}_h, \quad (3.38)$$

for $j=0,1,2,\dots,n-1$. On the other hand, by (3.36) and (3.37),

$$\begin{aligned} \int_K (\mathbf{QR} \cdot \theta) \Big|_{t_j^-}^{t_{j+1}^+} d\mathbf{x} &= \int_K \mathbf{Q} \left(\mathbf{R}(\mathbf{x}, t_{j+1}^-) \cdot \theta(\mathbf{x}, t_{j+1}^-) - \mathbf{R}(\mathbf{x}, t_j^-) \cdot \theta(\mathbf{x}, t_j^+) \right) d\mathbf{x} \\ &= \int_K \mathbf{Q} \left(\xi(\mathbf{x}, t_{j+1}^-) + \eta(\mathbf{x}, t_{j+1}^-) + \rho(\mathbf{x}, t_{j+1}^-) \right) \cdot \theta(\mathbf{x}, t_{j+1}^-) d\mathbf{x} \\ &\quad - \int_K \mathbf{Q} \left(\xi(\mathbf{x}, t_j^-) + \eta(\mathbf{x}, t_j^-) + \rho(\mathbf{x}, t_j^-) \right) \cdot \theta(\mathbf{x}, t_j^+) d\mathbf{x} \\ &= 0, \quad j=1,2,\dots,n-1. \end{aligned} \quad (3.39)$$

Furthermore,

$$\int_K \mathbf{QR}(\mathbf{x}, t_1^-) \cdot \theta(\mathbf{x}, t_1^-) d\mathbf{x} = \int_K \mathbf{Q}(\zeta(\mathbf{x}, t_1^-) + \eta(\mathbf{x}, t_1^-) + \rho(\mathbf{x}, t_1^-)) \cdot \theta(\mathbf{x}, t_1^-) d\mathbf{x} = 0. \tag{3.40}$$

Implementing (3.38), (3.39) and (3.40) in (3.32) and (3.33), we have

$$\bar{\mathbf{B}}_{I_j}(\mathbf{R}, \theta) = \sum_{K \in \mathcal{T}_h} \int_{I_j} \left[\int_{\partial K} (\widehat{\mathbf{f}(\mathbf{R})} \cdot \mathbf{n}_K) \cdot \theta ds - \int_K \mathbf{f}(\mathbf{R}) \cdot \nabla \theta d\mathbf{x} \right] dt, \tag{3.41}$$

for $j=0, 1, 2, \dots, n-1$.

Lemma 3.4. *In terms of $\bar{\mathbf{B}}_{I_j}(\mathbf{R}, \theta)$, we have*

$$\begin{aligned} \bar{\mathbf{B}}_{I_j}(\mathbf{R}, \theta) &\leq Ch^{2k+1} \int_{I_j} \|\mathbf{U}(\cdot, t)\|_{k+1, \Omega}^2 dt + \frac{1}{2} \int_{I_j} \|\mathbf{Q}^{1/2} \theta(\cdot, t)\|_{0, \Omega}^2 dt + \frac{1}{2} \Theta_{I_j, \mathcal{T}_h}(\theta) \\ &\quad + C(\Delta t)^{2r+2} \int_{\Omega} \left(\|\nabla \times \mathbf{E}(\mathbf{x}, \cdot)\|_{r+1, I_j}^2 + \|\nabla \times \mathbf{H}(\mathbf{x}, \cdot)\|_{r+1, I_j}^2 \right) d\mathbf{x}. \end{aligned} \tag{3.42}$$

Proof. Substituting the decomposition (3.35) in (3.41), we obtain

$$\begin{aligned} \bar{\mathbf{B}}_{I_j}(\mathbf{R}, \theta) &= \left(\sum_{K \in \mathcal{T}_h} \int_{I_j} \int_{\partial K} (\widehat{\mathbf{f}(\zeta)} \cdot \mathbf{n}_K) \cdot \theta ds dt - \sum_{K \in \mathcal{T}_h} \int_{I_j} \int_K \mathbf{f}(\zeta) \cdot \nabla \theta d\mathbf{x} dt \right) \\ &\quad + \left(\sum_{K \in \mathcal{T}_h} \int_{I_j} \int_{\partial K} (\widehat{\mathbf{f}(\eta)} \cdot \mathbf{n}_K) \cdot \theta ds dt - \sum_{K \in \mathcal{T}_h} \int_{I_j} \int_K \mathbf{f}(\eta) \cdot \nabla \theta d\mathbf{x} dt \right) \\ &\quad + \left(\sum_{K \in \mathcal{T}_h} \int_{I_j} \int_{\partial K} (\widehat{\mathbf{f}(\rho)} \cdot \mathbf{n}_K) \cdot \theta ds dt - \sum_{K \in \mathcal{T}_h} \int_{I_j} \int_K \mathbf{f}(\rho) \cdot \nabla \theta d\mathbf{x} dt \right) \\ &= M_j^1 + M_j^2 + M_j^3. \end{aligned} \tag{3.43}$$

Obviously, $\zeta = \mathbf{U} - \Pi_h \mathbf{U}$ is continuous in space. Therefore the consistency of the numerical flux $\widehat{\mathbf{f}(\mathbf{U}_h)} \cdot \mathbf{n}$ implies

$$\begin{aligned} M_j^1 &= \sum_{K \in \mathcal{T}_h} \int_{I_j} \int_{\partial K} (\mathbf{f}(\zeta) \cdot \mathbf{n}_K) \cdot \theta ds dt - \sum_{K \in \mathcal{T}_h} \int_{I_j} \int_K \mathbf{f}(\zeta) \cdot \nabla \theta d\mathbf{x} dt \\ &= \sum_{K \in \mathcal{T}_h} \int_{I_j} \int_K (\nabla \times \zeta_{\mathbf{E}} \cdot \theta_{\mathbf{H}} - \nabla \times \zeta_{\mathbf{H}} \cdot \theta_{\mathbf{E}}) d\mathbf{x} dt. \end{aligned} \tag{3.44}$$

Using Lemma 3.2 and the Young's inequality, we have

$$\begin{aligned} M_j^1 &\leq \int_{\Omega} \int_{I_j} \left(\frac{1}{2\mu} |\nabla \times \zeta_{\mathbf{E}}|^2 + \frac{1}{2\epsilon} |\nabla \times \zeta_{\mathbf{H}}|^2 + \frac{\mu}{2} |\theta_{\mathbf{H}}|^2 + \frac{\epsilon}{2} |\theta_{\mathbf{E}}|^2 \right) dt d\mathbf{x} \\ &\leq C(\Delta t)^{2r+2} \int_{\Omega} \left(\|\nabla \times \mathbf{E}(\mathbf{x}, \cdot)\|_{r+1, I_j}^2 + \|\nabla \times \mathbf{H}(\mathbf{x}, \cdot)\|_{r+1, I_j}^2 \right) d\mathbf{x} \\ &\quad + \frac{1}{2} \int_{I_j} \|\mathbf{Q}^{1/2} \theta(\cdot, t)\|_{0, \Omega}^2 dt. \end{aligned} \tag{3.45}$$

By the definition of the numerical fluxes in (2.21) and (2.22), a straightforward calculation leads to

$$\begin{aligned}
 M_j^2 &= \sum_{K \in \mathcal{T}_h} \int_{I_{j_e} \subset \partial K \cap \mathcal{E}_I} \int_e \left(\mathbf{n}_K \times (\bar{\eta}_{\mathbf{E}} - \frac{Z}{2} [\eta_{\mathbf{H}}]_T) \cdot \theta_{\mathbf{H}}^{int(K)} - \mathbf{n}_K \times (\bar{\eta}_{\mathbf{H}} + \frac{1}{2Z} [\eta_{\mathbf{E}}]_T) \cdot \theta_{\mathbf{E}}^{int(K)} \right) ds dt \\
 &\quad + \sum_{K \in \mathcal{T}_h} \int_{I_{j_e} \subset \partial K \cap \mathcal{E}_D} \int_e -\mathbf{n} \times \left(\eta_{\mathbf{H}}^{int} + \frac{1}{Z} \mathbf{n} \times \eta_{\mathbf{E}}^{int} \right) \cdot \theta_{\mathbf{E}}^{int} ds dt - \int_{I_j} \sum_{K \in \mathcal{T}_h} \int_K \mathbf{f}(\eta) \cdot \nabla \theta dx dt \\
 &= \int_{I_j} \sum_{e \subset \mathcal{E}_I} \int_e \left(\bar{\eta}_{\mathbf{H}} \cdot [\theta_{\mathbf{E}}]_T + \frac{Z}{2} [\eta_{\mathbf{H}}]_T \cdot [\theta_{\mathbf{H}}]_T \right) ds dt \\
 &\quad + \int_{I_j} \sum_{e \subset \mathcal{E}_I} \int_e \left(-\bar{\eta}_{\mathbf{E}} \cdot [\theta_{\mathbf{H}}]_T + \frac{1}{2Z} [\eta_{\mathbf{E}}]_T \cdot [\theta_{\mathbf{E}}]_T \right) ds dt \\
 &\quad + \int_{I_j} \sum_{e \subset \mathcal{E}_D} \int_e \left(-\mathbf{n} \times \eta_{\mathbf{H}}^{int} \cdot \theta_{\mathbf{E}}^{int} + \frac{1}{Z} \mathbf{n} \times \eta_{\mathbf{E}}^{int} \cdot (\mathbf{n} \times \theta_{\mathbf{E}}^{int}) \right) ds dt \\
 &\quad - \int_{I_j} \sum_{K \in \mathcal{T}_h} \int_K (\nabla \times \theta_{\mathbf{E}} \cdot \eta_{\mathbf{H}} - \nabla \times \theta_{\mathbf{H}} \cdot \eta_{\mathbf{E}}) dx dt. \tag{3.46}
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 M_j^3 &= \int_{I_j} \sum_{e \subset \mathcal{E}_I} \int_e \left(\bar{\rho}_{\mathbf{H}} \cdot [\theta_{\mathbf{E}}]_T + \frac{Z}{2} [\rho_{\mathbf{H}}]_T \cdot [\theta_{\mathbf{H}}]_T \right) ds dt \\
 &\quad + \int_{I_j} \sum_{e \subset \mathcal{E}_I} \int_e \left(-\bar{\rho}_{\mathbf{E}} \cdot [\theta_{\mathbf{H}}]_T + \frac{1}{2Z} [\rho_{\mathbf{E}}]_T \cdot [\theta_{\mathbf{E}}]_T \right) ds dt \\
 &\quad + \int_{I_j} \sum_{e \subset \mathcal{E}_D} \int_e \left(-\mathbf{n} \times \rho_{\mathbf{H}}^{int} \cdot \theta_{\mathbf{E}}^{int} + \frac{1}{Z} \mathbf{n} \times \rho_{\mathbf{E}}^{int} \cdot (\mathbf{n} \times \theta_{\mathbf{E}}^{int}) \right) ds dt \\
 &\quad - \int_{I_j} \sum_{K \in \mathcal{T}_h} \int_K (\nabla \times \theta_{\mathbf{E}} \cdot \rho_{\mathbf{H}} - \nabla \times \theta_{\mathbf{H}} \cdot \rho_{\mathbf{E}}) dx dt. \tag{3.47}
 \end{aligned}$$

Since $\nabla \times \theta \in \mathbf{V}_h^{r,k}$, by the orthogonality property of the projection operator \mathbf{P}_h , we have

$$\int_K (\nabla \times \theta_{\mathbf{E}} \cdot \eta_{\mathbf{H}} - \nabla \times \theta_{\mathbf{H}} \cdot \eta_{\mathbf{E}}) dx = 0, \quad \int_K (\nabla \times \theta_{\mathbf{E}} \cdot \rho_{\mathbf{H}} - \nabla \times \theta_{\mathbf{H}} \cdot \rho_{\mathbf{E}}) dx = 0. \tag{3.48}$$

Implementing Lemma 3.3 and the Young's inequality, we obtain

$$\begin{aligned}
 M_j^2 &\leq \int_{I_j} \sum_{e \subset \mathcal{E}_I} \int_e \left(2Z |\bar{\eta}_{\mathbf{H}}|^2 + \frac{2}{Z} |\bar{\eta}_{\mathbf{E}}|^2 + \frac{Z}{2} |[\eta_{\mathbf{H}}]_T|^2 + \frac{1}{2Z} |[\eta_{\mathbf{E}}]_T|^2 \right) ds dt \\
 &\quad + \int_{I_j} \sum_{e \subset \mathcal{E}_D} \int_e \left(Z |\eta_{\mathbf{H}}^{int}|^2 + \frac{1}{Z} |\eta_{\mathbf{E}}^{int}|^2 \right) ds dt + \frac{1}{2Z} \int_{I_j} \sum_{e \subset \mathcal{E}_D} \int_e |\mathbf{n} \times \theta_{\mathbf{E}}^{int}|^2 ds dt \\
 &\quad + \frac{1}{4} \int_{I_j} \sum_{e \subset \mathcal{E}_I} \int_e \left(Z |[\theta_{\mathbf{H}}]_T|^2 + \frac{1}{Z} |[\theta_{\mathbf{E}}]_T|^2 \right) ds dt
 \end{aligned}$$

$$\begin{aligned} &\leq Ch^{2k+1} \int_{I_j} \|\mathbf{U}(\cdot, t)\|_{k+1, \Omega}^2 dt + \frac{1}{4} \int_{I_j} \sum_{e \in \mathcal{E}_I} \int_e \left(Z |[\theta_{\mathbf{H}}]_T|^2 + \frac{1}{Z} |[\theta_{\mathbf{E}}]_T|^2 \right) ds dt \\ &\quad + \frac{1}{2Z} \int_{I_j} \sum_{e \in \mathcal{E}_D} \int_e |\mathbf{n} \times \theta_{\mathbf{E}}^{int}|^2 ds dt, \end{aligned} \tag{3.49}$$

$$\begin{aligned} M_j^3 &\leq \int_{I_j} \sum_{e \in \mathcal{E}_I} \int_e \left(2Z |\bar{\rho}_{\mathbf{H}}|^2 + \frac{2}{Z} |\bar{\rho}_{\mathbf{E}}|^2 + \frac{Z}{2} |[\rho_{\mathbf{H}}]_T|^2 + \frac{1}{2Z} |[\rho_{\mathbf{E}}]_T|^2 \right) ds dt \\ &\quad + \int_{I_j} \sum_{e \in \mathcal{E}_D} \int_e \left(Z |\rho_{\mathbf{H}}^{int}|^2 + \frac{1}{Z} |\rho_{\mathbf{E}}^{int}|^2 \right) ds dt + \frac{1}{2Z} \int_{I_j} \sum_{e \in \mathcal{E}_D} \int_e |\mathbf{n} \times \theta_{\mathbf{E}}^{int}|^2 ds dt \\ &\quad + \frac{1}{4} \int_{I_j} \sum_{e \in \mathcal{E}_I} \int_e \left(Z |[\theta_{\mathbf{H}}]_T|^2 + \frac{1}{Z} |[\theta_{\mathbf{E}}]_T|^2 \right) ds dt \\ &\leq Ch^{2k+1} \int_{I_j} \|\mathbf{U}(\cdot, t)\|_{k+1, \Omega}^2 dt + \frac{1}{4} \int_{I_j} \sum_{e \in \mathcal{E}_I} \int_e \left(Z |[\theta_{\mathbf{H}}]_T|^2 + \frac{1}{Z} |[\theta_{\mathbf{E}}]_T|^2 \right) ds dt \\ &\quad + \frac{1}{2Z} \int_{I_j} \sum_{e \in \mathcal{E}_D} \int_e |\mathbf{n} \times \theta_{\mathbf{E}}^{int}|^2 ds dt. \end{aligned} \tag{3.50}$$

According to (3.45), (3.49) and (3.50), we have

$$\begin{aligned} \bar{\mathbf{B}}_{I_j}(\mathbf{R}, \theta) &\leq Ch^{2k+1} \int_{I_j} \|\mathbf{U}(\cdot, t)\|_{k+1, \Omega}^2 dt + \frac{1}{2} \int_{I_j} \|\mathbf{Q}^{1/2} \theta(\cdot, t)\|_{0, \Omega}^2 dt + \frac{1}{2} \Theta_{I_j, T_h}(\theta) \\ &\quad + C(\Delta t)^{2r+2} \int_{\Omega} \left(\|\nabla \times \mathbf{E}(\mathbf{x}, \cdot)\|_{r+1, I_j}^2 + \|\nabla \times \mathbf{H}(\mathbf{x}, \cdot)\|_{r+1, I_j}^2 \right) d\mathbf{x}. \end{aligned} \tag{3.51}$$

The proof is complete. □

Now we give the main result in terms of the L^2 -error estimate.

Theorem 3.2. *Let $\mathbf{U}_h = (\mathbf{H}_h, \mathbf{E}_h)^T$ be the solution of (2.20) and $(\mathbf{H}, \mathbf{E})^T$ the exact smooth solution of (2.1)-(2.2). Assume that*

$$\mathbf{H}, \mathbf{E} \in \left(L^2([0, T], H^{k+1}(\Omega)) \right)^3, \quad \nabla \times \mathbf{E}, \nabla \times \mathbf{H} \in \left(H^{r+1}([0, T], L^2(\Omega)) \right)^3.$$

Then

$$\begin{aligned} &\mu \|\mathbf{H}(\mathbf{x}, T) - \mathbf{H}_h(\mathbf{x}, T^-)\|_0 + \epsilon \|\mathbf{E}(\mathbf{x}, T) - \mathbf{E}_h(\mathbf{x}, T^-)\|_0 \\ &\leq C(\Delta t)^{r+1} (\|\nabla \times \mathbf{E}\|_{0, r+1} + \|\nabla \times \mathbf{H}\|_{0, r+1}) \\ &\quad + Ch^{k+\frac{1}{2}} (\|\mathbf{E}\|_{k+1, 0} + \|\mathbf{H}\|_{k+1, 0}) + Ch^{k+1} (\|\mathbf{E}(\mathbf{x}, T)\|_{k+1, \Omega} + \|\mathbf{H}(\mathbf{x}, T)\|_{k+1, \Omega}), \end{aligned}$$

where C is a constant relying on T , but independent of Δt and h .

Proof. By Eq. (3.25), Lemma 3.3 and Lemma 3.4, we have

$$\begin{aligned} & \frac{1}{2} \|\mathbf{Q}^{1/2} \theta(\cdot, t_{j+1}^-)\|_{0,\Omega}^2 - \frac{1}{2} \|\mathbf{Q}^{1/2} \theta(\cdot, t_j^-)\|_{0,\Omega}^2 + \frac{1}{2} \Theta_{I_j, \mathcal{T}_h}(\theta) \\ & \leq \frac{1}{2} \int_{I_j} \|\mathbf{Q}^{1/2} \theta(\cdot, t)\|_{0,\Omega}^2 dt + \frac{1}{2} \Theta_{I_j, \mathcal{T}_h}(\theta) + Ch^{2k+1} \int_{I_j} \|\mathbf{U}(\cdot, t)\|_{k+1,\Omega}^2 dt \\ & \quad + C(\Delta t)^{2r+2} \int_{\Omega} \left(\|\nabla \times \mathbf{E}(\mathbf{x}, \cdot)\|_{r+1, I_j}^2 + \|\nabla \times \mathbf{H}(\mathbf{x}, \cdot)\|_{r+1, I_j}^2 \right) dx, \end{aligned} \tag{3.52}$$

for $j = 1, 2, \dots, n-1$, and

$$\begin{aligned} & \frac{1}{2} \|\mathbf{Q}^{1/2} \theta(\cdot, t_1^-)\|_{0,\Omega}^2 + \frac{1}{2} \|\mathbf{Q}^{1/2} \theta(\cdot, 0^+)\|_{0,\Omega}^2 + \frac{1}{2} \Theta_{I_0, \mathcal{T}_h}(\theta) \\ & \leq \frac{1}{2} \int_{I_0} \|\mathbf{Q}^{1/2} \theta(\cdot, t)\|_{0,\Omega}^2 dt + \frac{1}{2} \Theta_{I_0, \mathcal{T}_h}(\theta) + Ch^{2k+1} \int_{I_0} \|\mathbf{U}(\cdot, t)\|_{k+1,\Omega}^2 dt \\ & \quad + C(\Delta t)^{2r+2} \int_{\Omega} \left(\|\nabla \times \mathbf{E}(\mathbf{x}, \cdot)\|_{r+1, I_0}^2 + \|\nabla \times \mathbf{H}(\mathbf{x}, \cdot)\|_{r+1, I_0}^2 \right) dx. \end{aligned} \tag{3.53}$$

Thus we obtain

$$\begin{aligned} & \|\mathbf{Q}^{1/2} \theta(\cdot, t_{j+1}^-)\|_{0,\Omega}^2 \\ & \leq e^{\Delta t} \|\mathbf{Q}^{1/2} \theta(\cdot, t_j^-)\|_{0,\Omega}^2 + Ce^{\Delta t} h^{2k+1} \int_{I_j} \|\mathbf{U}(\cdot, t)\|_{k+1,\Omega}^2 dt \\ & \quad + Ce^{\Delta t} (\Delta t)^{2r+2} \int_{\Omega} \left(\|\nabla \times \mathbf{E}(\mathbf{x}, \cdot)\|_{r+1, I_j}^2 + \|\nabla \times \mathbf{H}(\mathbf{x}, \cdot)\|_{r+1, I_j}^2 \right) dx, \end{aligned} \tag{3.54}$$

for $j = 1, 2, \dots, n-1$, and

$$\begin{aligned} & \|\mathbf{Q}^{1/2} \theta(\cdot, t_1^-)\|_{0,\Omega}^2 \\ & \leq Ce^{\Delta t} h^{2k+1} \int_{I_0} \|\mathbf{U}(\cdot, t)\|_{k+1,\Omega}^2 dt \\ & \quad + Ce^{\Delta t} (\Delta t)^{2r+2} \int_{\Omega} \left(\|\nabla \times \mathbf{E}(\mathbf{x}, \cdot)\|_{r+1, I_0}^2 + \|\nabla \times \mathbf{H}(\mathbf{x}, \cdot)\|_{r+1, I_0}^2 \right) dx, \end{aligned} \tag{3.55}$$

by using Gronwall's inequality. Hence we obtain

$$\begin{aligned} & \|\mathbf{Q}^{1/2} \theta(\mathbf{x}, T^-)\|_0^2 \\ & \leq Ce^T (\Delta t)^{2r+2} \left(\|\nabla \times \mathbf{E}\|_{0,r+1}^2 + \|\nabla \times \mathbf{H}\|_{0,r+1}^2 \right) \\ & \quad + Ce^T h^{2k+1} \left(\|\mathbf{E}\|_{k+1,0}^2 + \|\mathbf{H}\|_{k+1,0}^2 \right). \end{aligned} \tag{3.56}$$

On the other hand,

$$\|\mathbf{Q}^{1/2} \mathbf{R}(\cdot, T^-)\|_{0,\Omega}^2 = \|\mathbf{Q}^{1/2} \eta(\cdot, T^-)\|_{0,\Omega}^2 \leq Ch^{2k+2} \|\mathbf{U}(\cdot, T)\|_{0,\Omega}^2. \tag{3.57}$$

By using a triangular inequality, we have

$$\begin{aligned} \|\mathbf{Q}^{1/2}\mathbf{e}(\cdot, T^-)\|_{0,\Omega}^2 &\leq C\left(\|\mathbf{Q}^{1/2}\mathbf{R}(\cdot, T^-)\|_{0,\Omega}^2 + \|\mathbf{Q}^{1/2}\theta(\cdot, T^-)\|_{0,\Omega}^2\right) \\ &\leq Ce^T(\Delta t)^{2r+2}\left(\|\nabla \times \mathbf{E}\|_{0,k+1}^2 + \|\nabla \times \mathbf{H}\|_{0,k+1}^2\right) \\ &\quad + Ce^Th^{2k+1}\left(\|\mathbf{E}\|_{k+1,0}^2 + \|\mathbf{H}\|_{k+1,0}^2\right) + Ch^{2k+2}\|\mathbf{U}(\cdot, T)\|_{0,\Omega}^2, \end{aligned} \quad (3.58)$$

which completes the proof. \square

Remark 3.1. It is noted that only semi-norms are used in Lemmas 3.1 and 3.2. Hence all norms in the right hand side of the error estimate in Theorem 3.2 can be replaced by the corresponding weaker semi-norms.

4 Numerical results

In this section, some numerical examples are given to justify our theoretical prediction. The uniform Cartesian mesh is used in all numerical examples. According to the theoretical analysis above, we know that our numerical scheme is stable without any restriction on the time step size Δt . Actually we obtain accurate numerical solutions even when Δt is larger than h . Moreover, in this section, the L^2 -errors are computed in the following way,

$$\|\mathbf{u} - \mathbf{u}_h\|_0 = \left(\sum_{K \in \mathcal{T}_h} \int_K |\mathbf{u} - \mathbf{u}_h|^2 d\Omega \right)^{\frac{1}{2}}.$$

4.1 2-D numerical example

The similar error estimate for 2-D Maxwell equations can be obtained in the same way as we have done for 3-D case, by introducing the scalar and vector curl operators

$$\mathit{curl}\mathbf{E} = \frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y}, \quad \nabla \times E = \left(\frac{\partial E}{\partial y}, -\frac{\partial E}{\partial x} \right)^T.$$

To justify our theoretical analysis, we first give a 2-D numerical example. Consider the following 2-D model problem

$$\begin{aligned} \frac{\partial H_x}{\partial t} + \frac{\partial E_z}{\partial y} &= R_1, \\ \frac{\partial H_y}{\partial t} - \frac{\partial E_z}{\partial x} &= R_2, \\ \frac{\partial E_z}{\partial t} - \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) &= R_3, \end{aligned}$$

Table 1: The convergence rate of L^2 error for $r=k=1$ (2-D case).

time	time step	mesh	$\ \mathbf{E}(\cdot, T) - \mathbf{E}_h(\cdot, T^-)\ _0$	order	$\ \mathbf{H}(\cdot, T) - \mathbf{H}_h(\cdot, T^-)\ _0$	order
T=0.5	N=2	4 × 4	1.1050e-3		2.4184e-3	
	N=4	8 × 8	2.6692e-4	2.0496	6.7874e-4	1.8331
	N=8	16 × 16	6.4997e-5	2.0380	1.8910e-4	1.8437
	N=16	32 × 32	1.6012e-5	2.0212	5.2727e-5	1.8425
T=5	N=20	4 × 4	1.6172e-2		9.3017e-2	
	N=40	8 × 8	3.4368e-3	2.2344	3.2580e-2	1.5135
	N=80	16 × 16	7.1898e-4	2.2570	9.4835e-3	1.7805
	N=16	32 × 32	1.6111e-4	2.1579	2.6381e-3	1.8459
T=50	N=200	4 × 4	1.1388e-1		7.8949e-1	
	N=400	8 × 8	2.2959e-2	2.3104	3.5336e-1	1.1598
	N=800	16 × 16	6.5688e-3	1.8054	1.1757e-1	1.5876
	N=1600	32 × 32	1.5181e-3	2.1134	3.3316e-2	1.8192

Table 2: The convergence rate of L^2 error for $r=k=2$ (2-D case).

time	time step	mesh	$\ \mathbf{E}(\cdot, T) - \mathbf{E}_h(\cdot, T^-)\ _0$	order	$\ \mathbf{H}(\cdot, T) - \mathbf{H}_h(\cdot, T^-)\ _0$	order
T=1	N=4	4 × 4	2.7357e-4		8.7101e-4	
	N=8	8 × 8	3.3301e-5	3.0383	1.1044e-4	2.9794
	N=16	16 × 16	4.1122e-6	3.0176	1.3904e-5	2.9897
T=10	N=40	4 × 4	2.6332e-3		7.6589e-3	
	N=80	8 × 8	3.2889e-4	3.0011	9.4570e-4	3.0177
	N=160	16 × 16	4.1015e-5	3.0034	1.1561e-4	3.0321
T=100	N=400	4 × 4	2.4882e-2		8.0735e-2	
	N=800	8 × 8	3.1777e-3	2.9690	1.0267e-2	2.9752
	N=1600	16 × 16	3.9790e-4	2.9975	1.2652e-3	3.0206

in $[0,1]^2$, where $R_i, i=1,2,3$ are chosen such that the exact solution is

$$\begin{pmatrix} H_x \\ H_y \\ E_z \end{pmatrix} = 100 \begin{pmatrix} x(1-x)(1-2y)t\sin t \\ -y(1-y)(1-2x)t\sin t \\ xy(1-x)(1-y)t\sin(t+x+y) \end{pmatrix}.$$

First we choose the time step size Δt equal to the spatial mesh size h and the polynomials of the same degree for both temporal and spatial variables, i.e. $r=k$. The results and their corresponding convergence order are shown in Tables 1 and 2 for $r=k=1$ and $r=k=2$ respectively. It is observed that the convergence rate of both \mathbf{E}_h and \mathbf{H}_h in L^2 -norm is $\mathcal{O}(h^{k+1})$, which is better than the theoretical prediction.

As mentioned above, this space-time DG scheme is unconditionally stable. Here we use a numerical experiment to verify this claim. We take the time step size Δt larger than the spatial mesh size h and $r=1, k=2$. The L^2 -errors are listed in Table 3. It is noticed that, even when the time step size $\Delta t=0.2,0.3$, which is larger than the spatial mesh size $h=0.125$, the relative errors do not increase when $T=20,30$.

Table 3: The unconditional stability of the space-time DG method for $r=1, k=2, 8 \times 8$ mesh (2-D case).

Δt		$\ E(\cdot, T) - E_h(\cdot, T^-)\ _0$	Relative error in percentage	$\ H(\cdot, T) - H_h(\cdot, T^-)\ _0$	Relative error in percentage
0.2	T=0.2	9.9558e-5	1.6499	9.8708e-5	1.6665
	T=2	1.1146e-4	0.5758	2.8505e-4	0.1051
	T=20	1.5273e-3	0.2792	2.3957e-3	0.0880
0.3	T=0.3	4.9772e-4	5.3331	3.3515e-4	2.5360
	T=3	7.6311e-4	1.0170	5.6473e-4	0.8948
	T=30	7.2032e-3	1.5779	4.7357e-3	0.1072

Table 4: The ultra-convergence of order $2r+1$ in $t, h=(\Delta t)^2, r=k, T=1$ (2-D case).

k	Δt	mesh	$\ E(\cdot, T) - E_h(\cdot, T^-)\ _0$	order	$\ H(\cdot, T) - H_h(\cdot, T^-)\ _0$	order
1	$\frac{1}{2}$	4×4	3.0011e-3		1.0045e-2	
	$\frac{1}{4}$	16×16	1.4082e-4	4.4136	9.8193e-4	3.3547
	$\frac{1}{8}$	64×64	1.1089e-5	3.6667	8.8256e-5	3.4759
2	$\frac{1}{2}$	4×4	1.3478e-4		2.8061e-4	
	$\frac{1}{4}$	16×16	4.3510e-6	4.9531	1.4025e-5	4.3225
	$\frac{1}{8}$	64×64	8.7343e-8	5.6385	2.2980e-7	5.9315

Table 5: The ultra-convergence of order $2r+1$ in $t, h=\Delta t, r=1, k=2$ (2-D case).

time	time step	mesh	$\ E(\cdot, T) - E_h(\cdot, T^-)\ _0$	order	$\ H(\cdot, T) - H_h(\cdot, T^-)\ _0$	order
T=1	N=4	4×4	2.7227e-4		9.1652e-4	
	N=8	8×8	3.3998e-5	3.0015	1.1903e-4	2.9448
	N=16	16×16	4.4534e-6	2.9325	1.8143e-5	2.7138
T=10	N=40	4×4	2.7070e-3		7.6651e-3	
	N=80	8×8	3.3387e-4	3.0193	9.2605e-4	3.0491
	N=160	16×16	4.1596e-5	3.0048	1.2621e-4	2.8753
T=100	N=400	4×4	2.5961e-2		8.0891e-2	
	N=800	8×8	3.3121e-3	2.9705	1.0226e-2	2.9837
	N=1600	16×16	4.1525e-4	2.9957	1.2654e-3	3.0146

Besides the unconditional stability, our approach has another important advantage over many existing numerical schemes, i.e., the implementation of the DG method in time-discretization leads to an ultra-convergence of order $2r+1$ in time step for the numerical fluxes w.r.t. t at the grid points. Numerical results in Tables 4 and 5 show this phenomenon from two different ways. On the one hand, we set $r=k$ and choose $h=(\Delta t)^2$ in Table 4. Then the ultra-convergence rate of $\mathcal{O}((\Delta t)^3)$ and $\mathcal{O}((\Delta t)^5)$ for $k=1$ and $k=2$ are observed numerically. On the other hand, we let $\Delta t=h$ but choose $r=1$ and $k=2$, a convergence rate of order $\mathcal{O}((\Delta t)^3)$ of the L^2 -error is observed in Table 5.

4.2 3-D numerical example

We consider the following 3D Maxwell's equation

Table 6: The convergence rate of L^2 error for $r=k=1$ (3-D case).

time	time step	mesh	$\ \mathbf{E}(\cdot, T) - \mathbf{E}_h(\cdot, T^-)\ _0$	order	$\ \mathbf{H}(\cdot, T) - \mathbf{H}_h(\cdot, T^-)\ _0$	order
T=1	N=2	$2 \times 2 \times 2$	4.4143e-2		5.2740e-2	
	N=4	$4 \times 4 \times 4$	1.1601e-2	1.9279	1.4352e-2	1.8776
	N=8	$8 \times 8 \times 8$	2.9482e-3	1.9763	3.9000e-3	1.8797
T=10	N=20	$2 \times 2 \times 2$	5.6905e-1		6.2508e-1	
	N=40	$4 \times 4 \times 4$	1.4497e-1	1.9728	1.8246e-1	1.7765
	N=80	$8 \times 8 \times 8$	3.7722e-2	1.9423	5.3541e-2	1.7689
T=100	N=200	$2 \times 2 \times 2$	5.2164		5.9581	
	N=400	$4 \times 4 \times 4$	1.3730	1.9257	1.6040	1.8932
	N=800	$8 \times 8 \times 8$	3.5143e-1	1.9660	4.3788e-1	1.8731

Table 7: The convergence rate of L^2 error for $r=k=2$ (3-D case).

time	time step	mesh	$\ \mathbf{E}(\cdot, T) - \mathbf{E}_h(\cdot, T^-)\ _0$	order	$\ \mathbf{H}(\cdot, T) - \mathbf{H}_h(\cdot, T^-)\ _0$	order
T=1	N=2	$2 \times 2 \times 2$	7.3451e-3		7.6816e-3	
	N=4	$4 \times 4 \times 4$	9.8885e-4	2.8974	1.0552e-3	2.8639
	N=8	$8 \times 8 \times 8$	1.2276e-4	3.0099	1.3771e-4	2.9378
T=10	N=20	$2 \times 2 \times 2$	8.3531e-2		6.7427e-2	
	N=40	$4 \times 4 \times 4$	1.0484e-2	2.9941	1.0641e-2	2.6637
	N=80	$8 \times 8 \times 8$	1.2073e-3	3.1183	1.5122e-3	2.8149
T=100	N=200	$2 \times 2 \times 2$	7.5887e-1		7.9374e-1	
	N=400	$4 \times 4 \times 4$	1.1413e-1	2.7332	1.2132e-1	2.7098
	N=800	$8 \times 8 \times 8$	1.3973e-2	3.0300	1.7080e-2	2.8284

$$\frac{\partial \mathbf{H}}{\partial t} + \nabla \times \mathbf{E} = \mathbf{R}_1, \quad \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{H} = \mathbf{R}_2,$$

in $\Omega = [0, 1]^3$. Here $\mathbf{R}_1, \mathbf{R}_2$ are chosen such that the exact solution is

$$\mathbf{E} = \begin{pmatrix} (y-y^2)(z-z^2) \\ (x-x^2)(z-z^2) \\ (x-x^2)(y-y^2) \end{pmatrix} t \cos(t+x+y+z), \quad \mathbf{H} = \begin{pmatrix} y-z \\ z-x \\ x-y \end{pmatrix} t \cos(t+x+y+z).$$

Like the 2-D case we firstly choose the time step size Δt equal to the spatial mesh size h and the polynomials of the same degree for both temporal and spatial variables, i.e. $r=k$. The L^2 -errors and their corresponding convergence order are shown in Tables 6 and 7 for $r=k=1$ and $r=k=2$ respectively. It is observed that the convergence rate of both \mathbf{E}_h and \mathbf{H}_h in L^2 -norm is $\mathcal{O}(h^{k+1})$, which is better than the theoretical prediction also.

To show the unconditional stability, we take the time step size Δt larger than the spatial mesh size h and $r=k=1$. The L^2 -errors are listed in Table 8. It is noticed that the relative errors do not increase when $T = 20, 30$ even when the time step size $\Delta t = 0.2, 0.3$ is larger than the spatial mesh size $h = 0.125$.

Table 8: The unconditional stability of the space-time DG method for $r=k=1$, $8 \times 8 \times 8$ mesh (3-D case).

Δt		$\ E(\cdot, T) - E_h(\cdot, T^-)\ _0$	Relative error in percentage	$\ H(\cdot, T) - H_h(\cdot, T^-)\ _0$	Relative error in percentage
0.2	T=0.2	7.7699e-4	17.5035	9.1458e-4	1.7056
	T=2	5.5194e-3	5.4308	7.5872e-3	0.6083
	T=20	6.5066e-2	6.7243	1.0559e-1	0.8895
0.3	T=0.3	2.3168e-3	32.1315	2.0479e-3	2.3443
	T=3	1.2185e-2	17.1600	1.4581e-2	1.6959
	T=30	9.3355e-2	5.8148	1.4147e-1	0.7178

Table 9: The ultra-convergence of order $2r+1$ in t , $h = (\Delta t)^2$, $k=r$, $T=1$ (3-D case).

r	Δt	mesh	$\ E(\cdot, T) - E_h(\cdot, T^-)\ _0$	order	$\ H(\cdot, T) - H_h(\cdot, T^-)\ _0$	order
1	$\frac{1}{2}$	$4 \times 4 \times 4$	1.3154e-2		1.7351e-2	
	$\frac{1}{4}$	$16 \times 16 \times 16$	1.1674e-3	3.4941	1.9320e-3	3.1669
2	$\frac{1}{2}$	$4 \times 4 \times 4$	1.0361e-3		1.1264e-3	
	$\frac{1}{4}$	$16 \times 16 \times 16$	1.9060e-5	5.7645	2.2478e-5	5.6471

Table 10: The ultra-convergence of order $2r+1$ in t , $h = \Delta t$, $r=1$, $k=2$ (3-D case).

time	time step	mesh	$\ E(\cdot, T) - E_h(\cdot, T^-)\ _0$	order	$\ H(\cdot, T) - H_h(\cdot, T^-)\ _0$	order
T=1	N=2	$2 \times 2 \times 2$	9.3503e-3		1.2122e-2	
	N=4	$4 \times 4 \times 4$	1.3368e-3	2.8062	1.9090e-3	2.6667
	N=8	$8 \times 8 \times 8$	1.8422e-4	2.8593	2.5069e-4	2.9288
T=10	N=20	$2 \times 2 \times 2$	8.8436e-2		6.6557e-2	
	N=40	$4 \times 4 \times 4$	1.1200e-2	2.9811	1.0717e-2	2.6347
	N=80	$8 \times 8 \times 8$	1.3099e-3	3.0960	1.5308e-3	2.8075
T=100	N=200	$2 \times 2 \times 2$	8.3760e-1		7.9378e-1	
	N=400	$4 \times 4 \times 4$	1.2389e-1	2.7572	1.2196e-1	2.7023
	N=800	$8 \times 8 \times 8$	1.5200e-2	3.0269	1.7169e-2	2.8285

Numerical results in Tables 9 and 10 show the ultra-convergence of our method numerically. In Table 9 we set $r=k$ and choose $h = (\Delta t)^2$. Then the ultra-convergence rate of $\mathcal{O}((\Delta t)^3)$ and $\mathcal{O}((\Delta t)^5)$ for $k=1$ and $k=2$ are observed numerically. In Table 10, we choose the time step size Δt equal to the spatial mesh size h , and $r=1$, $k=2$. An ultra-convergence of order $\mathcal{O}((\Delta t)^3)$ is observed numerically.

Remark 4.1. Although the theoretical error bound is $\mathcal{O}(h^{k+1/2} + (\Delta t)^{r+1})$, our numerical tests indicate that the actual error seems to be $\mathcal{O}(h^{k+1} + (\Delta t)^{2r+1})$. Therefore, we propose the following two strategies in practice to optimize the scheme: 1) Use the same polynomial space $(P^k)^3 \times P^k$ and different element scale $h = (\Delta t)^{2 - \frac{1}{k+1}}$; 2) Use the same element scale $\Delta t = h$ and different polynomial spaces $(P^k)^3 \times P^r$ with $2r=k$. The first strategy suggests to use the larger time steps while the second one recommends to use higher order polynomial spaces for spatial discretization.

5 Concluding remarks

A space-time DG method is proposed to solve time-dependent Maxwell's equation in homogeneous media. The L^2 -stability is proved. Based on a technical operator decomposition, the convergence rate of $\mathcal{O}(h^{k+\frac{1}{2}} + (\Delta t)^{r+1})$ in the L^2 -norm is established under the standard Galerkin finite element framework.

The proposed space-time DG method is essentially an implicit scheme. The main advantages of it over the traditional explicit time step approaches are 1) its unconditionally stable property, and 2) its ultra-convergence in time steps. These favorable properties make it possible to compute long time behavior of time-dependent Maxwell's equations and offset the disadvantage of the computational cost of the implicit method. As explained in Remark 4.1, it is advised to use $h \approx (\Delta t)^2$ in our scheme instead of $\Delta t = \mathcal{O}(h)$ in most explicit methods. A systematic study of comparison of the proposed space-time DG method with the explicit time step approaches (spatial semi-discretization by DG) would be a separate work. Other future works include the rigorous justification of the ultra-convergence in time steps, the h -version and hp -version space-time DG methods for Maxwell's equations in dispersive media and meta-materials, and their corresponding theoretical analysis and applications.

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