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Galerkin Method for Wave Equations with Uncertain Coefficients

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Abstract. Polynomial chaos methods (and generalized polynomial chaos methods) have been extensively applied to analyze PDEs that contain uncertainties. However this approach is rarely applied to hyperbolic systems. In this paper we analyze the properties of the resulting deterministic system of equations obtained by stochastic Galerkin projection. We consider a simple model of a scalar wave equation with random wave speed. We show that when uncertainty causes the change of characteristic directions, the resulting deterministic system of equations is a symmetric hyperbolic system with both positive and negative eigenvalues. A consistent method of imposing the boundary conditions is proposed and its convergence is established. Numerical examples are presented to support the analysis.

AMS subject classifications: 65C20, 65C30

Key words: Generalized polynomial chaos, stochastic PDE, Galerkin method, hyperbolic equation, uncertainty quantification.

1 Introduction

In recent years there is a growing interests in studying efficient numerical methods for solving differential equations with random inputs. The Polynomial Chaos (PC) based methods have received intensive attention. The original PC method was developed by R. Ghanem, cf. [3], and was inspired by the Wiener chaos expansion which uses Hermite polynomials of Gaussian random variables to represent random processes [5]. Later the approach was extended to generalized Polynomial Chaos (gPC) where general orthogonal polynomials are adopted for improved representations of more general random processes [7]. With PC/gPC serving as a complete basis to represent random processes, a

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stochastic Galerkin projection can be used to transform the (stochastic) governing equations to a set of deterministic equations that can be readily discretized via standard numerical techniques. Although such a Galerkin approach is effective in many problems, see, e.g., [2, 6, 8], its application to hyperbolic problems has been limited as of now. We believe that the primary reason is that the properties of the system of equations resulting from a Galerkin projection is not fully understood. (When the uncertainty does not change the direction of the characteristics, the Galerkin system can be shown to be hyperbolic and solved in a straightforward manner [1]).

We discuss in this paper the application of the gPC Galerkin method to the simulations of hyperbolic systems that contain uncertainties. In general these uncertainties may enter through initial conditions, boundary conditions or through uncertainties in the coefficients of the problem. Here we deal with the case that the coefficients are functions of random variables. In particular we use a scalar wave equation as a model and study the situation in which the inflow-outflow conditions change as a function of a random variable. The problem is whether it is possible to impose boundary conditions on the deterministic system, consistent with the boundary conditions of the original equation.

We show, in this paper, that the deterministic system is a symmetric hyperbolic system with positive as well as negative eigenvalues. A consistent and stable method of imposing the boundary conditions is outlined. The boundary conditions are not satisfied exactly at the boundaries but rather to the order of the scheme. Convergence of the scheme is established.

The paper is organized as follows. In Section 2 we present the model problem of a scalar hyperbolic equation where the wave speed is a random variable. A consistent set of boundary conditions are presented for the deterministic system resulted from a gPC Galerkin procedure, and we prove convergence of the scheme. In Section 3 we present numerical results to support the theory.

2 Model problem: Scalar wave equation with uncertainty

A simple scalar equation that illustrates the difficulties in applying the (generalized) Polynomial Chaos to hyperbolic equations is:

$$\frac{\partial u(x,t,y)}{\partial t} = c(y)\frac{\partial u(x,t,y)}{\partial x}, \quad x\in(-1,1), \quad t>0, \tag{2.1}$$

where $c(y)$ is a random transport velocity of a random variable $y\in\Omega$ in a properly defined complete random space with event space Ω and probability distribution function $\rho(y)$. With this the expectation of a given function is

$$\mathbb{E}[f(y)] = \int f(y)\rho(y)dy.$$

At this stage we would like to mention that we can consider (2.1) as a system where c is a symmetric matrix and obtain similar results. For simplicity we stay with the example

above to highlight the fundamental properties. The physical domain is bounded, $(-1,1)$ upon proper scaling, so that we can study the effects of boundary conditions.

The initial condition is given by

$$u(x,0,y)=u_0(x,y). \tag{2.2}$$

The boundary conditions are more complicated as they depend on the sign of the random transport velocity $c(y)$. A well posed set of boundary conditions is given by:

$$\begin{aligned} &u(1,t,y)=u_R(t,y), \qquad c(y)>0,\\ &u(-1,t,y)=u_L(t,y), \qquad c(y)<0. \end{aligned} \tag{2.3}$$

Eqs. (2.1)-(2.3) complete the setup of the problem.

2.1 Polynomial chaos Galerkin approach

Following the standard gPC expansion, we assume that $u(x,t,y)$ is sufficiently smooth in y and has a converging expansion of the form

$$u(x,t,y)=\sum_{k=0}^{\infty}\hat{u}_k(x,t)P_k(y), \tag{2.4}$$

where the polynomials $P_k(y)$ correspond the distribution of the random variable y and satisfy the following orthogonality relation

$$\mathbb{E}[P_kP_l]=\int P_k(y)P_l(y)\rho(y)dy=\delta_{kl}, \quad \forall k,l, \tag{2.5}$$

where δ_{kl} is the Kronecker delta function. Note the polynomials are normalized. The commonly seen correspondences between the polynomials $P_k(y)$ and the distribution of the random variable y include Hermite-Gaussian (the original PC expansion), Legendre-uniform, Laguerre-Gamma, etc., cf. [7, 8]. For simplicity we will discuss in this paper the case of random variable y with beta distribution in $(-1,1)$ (upon proper scaling). In this case the expansion functions P_k are the (normalized) Jacobi polynomials. (Note this includes the special case of Legendre polynomials with uniformly distributed random variable y.) For the converged series (2.4), we also assume that the expansion coefficients decay fast asymptotically, i.e.,

$$\|\hat{u}_j(x,t)\|_1^2\leq\frac{K}{j^{2m}}, \quad j\gg 1, \tag{2.6}$$

where $K,m>0$ are constants and the $\|\cdot\|_1$ norm is defined as

$$\|\hat{u}_j(x,t)\|_1^2=\int_{-1}^{1}\left[\hat{u}_j^2+\left(\frac{\partial\hat{u}_j}{\partial x}\right)^2\right]dx. \tag{2.7}$$

We also use $\|\cdot\|_2$ to denote the standard L_2 norm, i.e.,

$$\|f(x)\|_2^2 = \int_{-1}^{1} f^2(x)dx.$$

By utilizing the expansion (2.4) and employing a Galerkin projection, it is straightforward to verify that the coefficients $\hat{u}_j(x,t)$ satisfy the following infinite system of equations

$$\frac{\partial \hat{u}_j(x,t)}{\partial t} = \sum_{k=0}^{\infty} a_{j,k} \frac{\partial \hat{u}_k(x,t)}{\partial x} \qquad j=0,1,\cdots, \tag{2.8}$$

$$a_{j,k} = \int_{-1}^{1} c(y) P_j(y) P_k(y) \rho(y) dy. \tag{2.9}$$

The equations for the first $(N+1)$ coefficients can be written as

$$\frac{\partial \hat{u}_j(x,t)}{\partial t} = \sum_{k=0}^{N} a_{j,k} \frac{\partial \hat{u}_k(x,t)}{\partial x} + \sum_{k=N+1}^{\infty} a_{j,k} \frac{\partial \hat{u}_k(x,t)}{\partial x}, \quad j=0,\cdots,N. \tag{2.10}$$

In the gPC Galerkin method we seek an approximation to the true solution via a finite-term gPC expansion

$$v(x,t,y) = \sum_{k=0}^{N} \hat{v}_k(x,t) P_k(y) \tag{2.11}$$

and project

$$\frac{\partial v(x,t,y)}{\partial t} - c(y) \frac{\partial v(x,t,y)}{\partial x} = 0$$

onto the subspace spanned by the first $(N+1)$ gPC basis polynomials and obtain the following system

$$\frac{\partial \hat{v}_j(x,t)}{\partial t} = \sum_{k=0}^{N} a_{j,k} \frac{\partial \hat{v}_k(x,t)}{\partial x}, \qquad j=0,\cdots,N, \tag{2.12}$$

where $a_{j,k}$ are defined as in (2.9). If we denote by $\mathbf{A}$ the $(N+1)\times(N+1)$ matrix whose entries are $\{a_{j,k}\}_{0\le j,k\le N}$ and $\mathbf{v}=(\hat{v}_0,\cdots,\hat{v}_N)^T$ a vector of length $(N+1)$, then system (2.12) can be written as

$$\frac{\partial \mathbf{v}(x,t)}{\partial t} = \mathbf{A} \frac{\partial \mathbf{v}(x,t)}{\partial x}. \tag{2.13}$$

Note that from the definition $a_{j,k}=a_{k,j}$, i.e., $\mathbf{A}=\mathbf{A}^T$, the system (2.13) is therefore *symmetric hyperbolic*, this is consistent with the fact that the original equation (2.1) is hyperbolic for each realization of y.

2.2 Eigenvalues of the PC Galerkin equations

A less trivial question is the nature of the inflow-outflow boundary conditions. The boundary conditions for the original scalar equation (2.1) depend on the particular realization of the random variable y (see (2.3)). However upon the Galerkin projection in the random dimension the deterministic system (2.13) is independent of y. In Theorem 2.1 we investigate how the inflow-outflow conditions are reflected in the system (2.13).

Theorem 2.1. *Consider the deterministic system* (2.13) *where the coefficients are defined in* (2.9). *Then if* $c(y)\geq 0$ *(reps.* $c(y)\leq 0$*) for all* y*, then the eigenvalues of* $\mathbf{A}$ *are all non-negative (reps. non-positive); if* $c(y)$ *changes sign, i.e.,* $c(y)>0$ *for some* y *and* $c(y)<0$ *for some other* y*, then* $\mathbf{A}$ *has both positive and negative eigenvalues for sufficiently large* N.

Proof. First let us consider the case of $c(y)\geq 0$. Let $\beta(y)$ be a random variable with an expansion

$$\beta(y)=\sum_{k=0}^{N} b_k P_k(y).$$

Let $\mathbf{b}=(b_0,\cdots,b_N)^T$ be the coefficient vector with length $(N+1)$. Note here $\mathbf{b}$ is an arbitrary vector. Then

$$\begin{aligned}\mathbf{b}^T\mathbf{A}\mathbf{b} &= \sum_{j=0}^{N}\sum_{k=0}^{N} b_j a_{j,k} b_k \\ &= \sum_{j=0}^{N}\sum_{k=0}^{N} b_j\left(\int_{-1}^{1} c(y)P_j(y)P_k(y)\rho(y)dy\right)b_k \\ &= \int_{-1}^{1}\beta^2(y)c(y)\rho(y)dy.\end{aligned}$$

Since $c(y)$ is non-negative

$$\mathbf{b}^T\mathbf{A}\mathbf{b}\geq 0$$

for all $\mathbf{b}$, thus all the eigenvalues of $\mathbf{A}$ are non-negative. The case of $c(y)\leq 0,\forall y$ follows similarly.

A more interesting case is when $c(y)$ changes sign. Let us divide the domain Ω where y belongs into the following non-overlapping open sets: $\Omega^+=\Omega_1\cup\Omega_2$ be defined as the subdomain of y where $c(y)>0$ and Ω_3 is the subdomain of y in which $c(y)\leq 0$. Let us also define $\gamma(y)$ be a smooth function such that

$$\gamma(y)>\delta, \qquad y\in\Omega_1, \tag{2.14}$$

$$0<\gamma(y)\leq\delta, \qquad y\in\Omega_2, \tag{2.15}$$

$$\gamma(y)=0, \qquad y\in\Omega_3. \tag{2.16}$$

Let $\beta_N(y)$ be the best polynomial approximation of degree N to $\sqrt{\gamma(y)}$ such that

$$\max_y|\beta_N^2(y)-\gamma(y)|\leq\epsilon, \tag{2.17}$$

where N is sufficiently large such that

$$\epsilon < \delta \frac{\int_{\Omega_1} c(y)\rho(y)dy}{\int_{\Omega} |c(y)|\rho(y)dy}. \tag{2.18}$$

Then

$$\begin{aligned}
&\int_{\Omega} \beta_N^2(y)c(y)\rho(y)dy \\
&= \int_{\Omega^+} \gamma(y)c(y)\rho(y)dy + \int_{\Omega^+} (\beta_N^2(y)-\gamma(y))c(y)\rho(y)dy + \int_{\Omega_3} \beta_N^2(y)c(y)\rho(y)dy \\
&\geq \int_{\Omega_1} \gamma(y)c(y)\rho(y)dy - \left| \int_{\Omega^+} (\beta_N^2(y)-\gamma(y))c(y)\rho(y)dy \right| - \left| \int_{\Omega_3} \beta_N^2(y)c(y)\rho(y)dy \right|.
\end{aligned}$$

Now

$$\begin{aligned}
&\left| \int_{\Omega_3} \beta_N^2(y)c(y)\rho(y)dy \right| \leq \epsilon \int_{\Omega_3} |c(y)|\rho(y)dy, \\
&\left| \int_{\Omega^+} (\beta_N^2(y)-\gamma(y))c(y)\rho(y)dy \right| \leq \epsilon \int_{\Omega^+} |c(y)|\rho(y)dy,
\end{aligned}$$

and therefore

$$\int_{\Omega} \beta_N^2(y)c(y)\rho(y)dy > 0$$

under the condition (2.18). Thus there exists a polynomial $\beta_N(y)$ with expansion coefficients $\mathbf{b} = (b_0, \cdots, b_N)^T$ such that $\mathbf{b}^T\mathbf{A}\mathbf{b}$ is positive. Similarly, if $c(y)$ is negative in a subinterval there exists a polynomial $\beta_N(y)$ with sufficiently large N such that

$$\int_{-1}^{1} \beta_N^2(y)c(y)\rho(y)dy < 0.$$

Thus the matrix $\mathbf{A}$ has positive and negative eigenvalues. This concludes the proof. □

2.3 Boundary conditions and convergence

We now turn to the issue of imposing the boundary conditions. Since $\mathbf{A}$ is symmetric there is an orthogonal matrix $\mathbf{S}^T = \mathbf{S}^{-1}$ such that

$$\mathbf{S}^T\mathbf{A}\mathbf{S} = \Lambda,$$

where Λ is a diagonal matrix whose entries on the eigenvalues of $\mathbf{A}$, i.e.,

$$\Lambda = \text{diag}(\lambda_0, \cdots, \lambda_{j_+}, \cdots, \lambda_{j_-}, \cdots, \lambda_N).$$

Here the positive eigenvalues occupy indices $j = 0, \cdots, j_+$, the negatives ones $j = j_-, \cdots, N$, and the rest, if exist, are zeros. Obviously, $j_+, j_- \leq N$.

Denote by $\mathbf{q}=(q_0,\cdots,q_N)^T=\mathbf{S}^T\mathbf{v}$, i.e.,

$$q_j(x,t)=\sum_{k=0}^{N}s_{k,j}\hat{v}_k(x,t),$$

where $s_{j,k}$ are the entries for $\mathbf{S}$, then we obtain

$$\frac{\partial\mathbf{q}(x,t)}{\partial t}=\Lambda\frac{\partial\mathbf{q}(x,t)}{\partial x}. \tag{2.19}$$

The boundary conditions of this diagonal system are determined by the sign of the eigenvalues, i.e., we need to specify

$$\begin{aligned} q_j(1,t)&=\sum_{k=0}^{N}s_{k,j}\hat{u}_k(1,t), \qquad j=0,\cdots,j_+,\\ q_j(-1,t)&=\sum_{k=0}^{N}s_{k,j}\hat{u}_k(-1,t), \qquad j=j_-,\cdots,N. \end{aligned} \tag{2.20}$$

Here the coefficients $\hat{u}_k$ at the boundaries are determined by the exact gPC projection of the boundary conditions of u, i.e.,

$$\begin{aligned} u_R(t,y)&=\sum_{j=0}^{\infty}\hat{u}_k(1,t)P_k(y),\\ u_L(t,y)&=\sum_{j=0}^{\infty}\hat{u}_k(-1,t)P_k(y). \end{aligned}$$

Subsequently the boundary conditions for the gPC Galerkin system of Eq. (2.13) are specified as

$$\mathbf{v}(1,t)=\mathbf{S}\mathbf{q}(1,t), \quad \mathbf{v}(-1,t)=\mathbf{S}\mathbf{q}(-1,t). \tag{2.21}$$

Note the above specification of boundary conditions via (2.20) and (2.21) implicitly satisfy the following relation

$$\begin{aligned} \sum_{k=0}^{N}s_{k,j}\hat{v}_k(1,t)&=\sum_{k=0}^{N}s_{k,j}\hat{u}_k(1,t), \qquad j=0,\cdots,j_+,\\ \sum_{k=0}^{N}s_{k,j}\hat{v}_k(-1,t)&=\sum_{k=0}^{N}s_{k,j}\hat{u}_k(-1,t), \qquad j=j_-,\cdots,N. \end{aligned}$$

For vanishing eigenvalues, if they exist, no boundary conditions are required.

Theorem 2.2. *Consider the hyperbolic equation* (2.1) *where y is a random variable with beta distribution in* $(-1,1)$. *Let $u(x,t,y)$ be the solution of* (2.1) *whose exact gPC expansion is* (2.4)

and let $v(x,t,y)$ be the $(N+1)$-term gPC solution (2.11) *solved via the Galerkin system* (2.13) *with boundary conditions given in* (2.21)*. Then for any finite time t*

$$\mathbb{E}\left[\|u-v\|_2^2\right]=\sum_{j=0}^{N}\left(\int_{-1}^{1}(\hat{u}_j(x,t)-\hat{v}_j(x,t))^2dx\right)$$
$$\leq\frac{K}{N^{2m-1}}t. \tag{2.22}$$

Note the linear growth in time.

Proof. Let

$$e_j(x,t)=\hat{u}_j(x,t)-\hat{v}_j(x,t),\quad j=0,\cdots,N.$$

From (2.10) and (2.12) we have

$$\frac{\partial e_j(x,t)}{\partial t}=\sum_{k=0}^{N}a_{j,k}\frac{\partial e_k(x,t)}{\partial x}+\sum_{k=N+1}^{\infty}a_{j,k}\frac{\partial\hat{u}_k(x,t)}{\partial x},\quad j=0,\cdots,N. \tag{2.23}$$

Denote by $\mathbf{e}=(e_0,\cdots,e_N)^T$ and let $\mathbf{d}=\mathbf{S}^T\mathbf{e}$. Then we obtain

$$\frac{\partial\mathbf{d}}{\partial t}=\Lambda\frac{\partial\mathbf{d}}{\partial x}+\mathbf{R}, \tag{2.24}$$

where the residual vector $\mathbf{R}=(R_0,\cdots,R_N)^T$ is

$$R_j(x,t)=\sum_{l=0}^{N}\sum_{k=N+1}^{\infty}s_{l,j}a_{l,k}\frac{\partial\hat{u}_k}{\partial x}. \tag{2.25}$$

By multiplying (2.24) by $\mathbf{d}^T$ and integrating in x one gets

$$\frac{1}{2}\frac{d}{dt}\int_{-1}^{1}\mathbf{d}^T\mathbf{d}dx=\frac{1}{2}\sum_{j=0}^{N}\lambda_j\left(d_j^2(1,t)-d_j^2(-1,t)\right)+\int_{-1}^{1}\mathbf{d}^T\mathbf{R}dx. \tag{2.26}$$

From the boundary conditions (2.21) it follows that if $\lambda_j>0$ then $d_j(1,t)=0$ and if $\lambda_j<0$ then $d_j(-1,t)=0$. Thus the first term in the right hand side of the above equation is negative. This leads to

$$\frac{1}{2}\frac{d}{dt}\int_{-1}^{1}\mathbf{d}^T\mathbf{d}dx\equiv\frac{1}{2}\frac{d}{dt}\|\mathbf{d}\|^2\leq\|\mathbf{d}\|\cdot\|\mathbf{R}\|,$$

where

$$\|\mathbf{R}\|^2=\sum_{j=0}^{N}\int_{-1}^{1}R_j^2(x,t)dx$$

and $\|\mathbf{d}\|$ is defined similarly. Thus

$$\frac{d}{dt}\|\mathbf{d}\| \le \|\mathbf{R}\|, \quad \|\mathbf{d}(x,t)\| \le \max_{t\ge 0}\|\mathbf{R}\|\cdot t.$$

Since the matrix $\mathbf{S}$ is unitary and the elements of the matrix $\mathbf{A}$ are bounded then

$$\|\mathbf{R}\|^2 \le \sum_{j=N+1}^{\infty} \|\hat{u}_j\|_1^2 \tag{2.27}$$

and the proof is established under assumption (2.6). □

3 Numerical results

In this section we present a few numerical examples to support the theoretical results derived above. In all of the following computations, we have used sufficiently fine resolutions in physical space and time domain, such that the spatial and temporal errors are negligible. In all computations, y is a random variable uniformly distributed in $(-1,1)$ and thus $P_k(y)$ are (normalized) Legendre polynomials.

3.1 Periodic problem

We first consider problem (2.1) with a periodic boundary condition in physical space. Subsequently the gPC Galerkin system (2.13) requires periodic boundary conditions that can be trivially implemented. Therefore no errors will be induced by specifying boundary conditions via (2.21). Let us consider

$$\begin{aligned} &u_t(x,t,y) = yu_x(x,t,y), \quad 0<x<2\pi,\ t\ge 0,\\ &u(x,0,y) = \cos(x), \qquad\quad 0<x<2\pi, \end{aligned} \tag{3.1}$$

The exact solution is $u_{\text{ex}} = \cos(x-yt)$. In Fig. 1 we plot the evolution of the mean square solution

$$\mathbb{E}[\|u\|_2^2] = \int\int_{-1}^{1} u^2(x,t,y)\rho(y)dydx$$

and its numerical solution via gPC Galerkin method. We observe that there is a finite time where the numerical solutions lose accuracy, i.e., errors become $\mathcal{O}(1)$. The size of the time domain in which the errors remain small grows almost linearly as the orders of gPC expansion are increased. This observation can be cross-examined by comparing the mean-square errors at different time level, as shown in Fig. 2. We observe that with sufficiently high orders of gPC expansions, exponential error convergence can be achieved. However, as time increases, the critical orders of expansions, beyond which errors start to decay exponentially fast, increase linearly. All of these results support the convergence analysis (2.22) where a linear error growth in time exists.

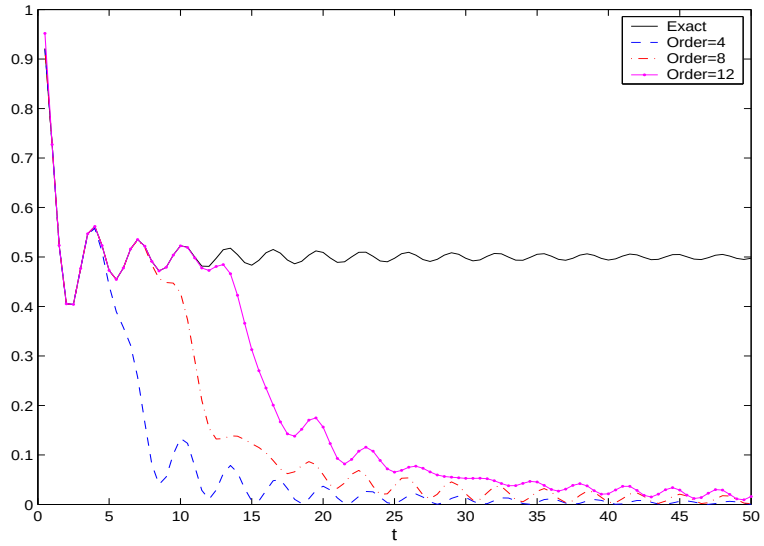

Figure 1: Evolution of errors in mean-square norm over time.

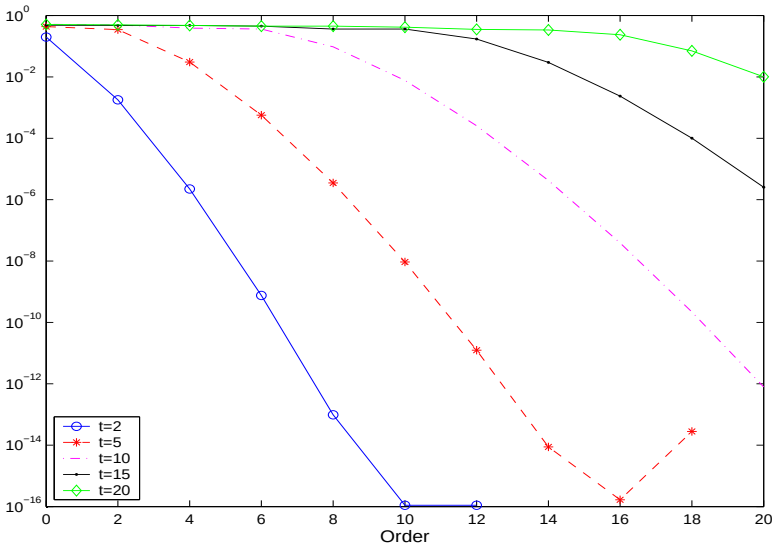

Figure 2: Convergence of mean-square errors with increasing orders.

3.2 Boundary conditions and discontinuity in random space

We now study a wave equation with a random wave speed that changes signs and also contains a discontinuity in the random space

$$\begin{aligned} &u_t = c(y)u_x, && -1\le x\le 1,\ \ t>0,\\ &u(x,0,y) = \sin(\kappa x), && -1\le x\le 1,\ \ y>0,\\ &u(x,0,y) = \sin(2\kappa x), && -1\le x\le 1,\ \ y<0. \end{aligned} \tag{3.2}$$

Here $c(y)=\sigma y$ with $0<\sigma<1$ controlling the variability of the random input and $\kappa>0$ is a real constant. We prescribe boundary conditions as

$$\begin{aligned} u(1,t,y)&=\sin[\kappa(1+c(y)t)], && y>0,\\ u(-1,t,y)&=\sin[2\kappa(-1+c(y)t)], && y<0. \end{aligned} \tag{3.3}$$

The exact solution of (3.2)-(3.3) is $u_e(x,t,y)=\sin[\kappa(x+c(y)t)]$ for $y>0$ and $u_e(x,t,y)=\sin[2\kappa(x+c(y)t)]$ for $y<0$. Note the solution is discontinuous in term of y, although each realization of y is a smooth function in x.

The numerical solutions are solved with $\sigma=0.5$ and $\kappa=1$. The numerical boundary conditions are implemented via the eigenvalue analysis (2.21). For numerical solutions of gPC order N, we examine three error measures: error in mean

$$e_{\text{mean}}(N,t)=\max_x|\mathbb{E}(v)-\mathbb{E}(u_e)|,$$

error in standard deviation (STD)

$$e_{\text{std}}(N,t)=\max_x|\sigma_v-\sigma_{u_e}|,$$

and the mean-square error

$$e_2(N,t)=\max_x(\mathbb{E}[(v-u_e)^2])^{1/2}.$$

Numerical simulations are conducted up to $t=1$, and we define convergence rate as

$$r(N)=[\ln(e(N))-\ln(e(M))]/[\ln(N)-\ln(M)]$$

for expansion orders $N>M\geq 1$, where e is one of the three error measures.

Fig. 3 shows the convergence of the three errors with increasing order of Legendre expansions. In this case, we observe different convergence properties between even and odd orders of expansions, although they appear to have similar asymptotic convergence rate. Note that such different error behaviors between even and odd expansions can be seen in classical spectral methods, cf. [4]. The errors, along with their convergence rates, are tabulated in Tables 1 and 2, for odd and even orders of expansions, respectively. No exponential convergence is achieved, as opposed to that in the earlier examples. Also, the weak error measures (error in mean and error in STD) converge more rapidly than the strong error measure in term of mean-square.

The slower convergence is due to the discontinuity in random space, and is manifested in Fig. 4, where the numerical solution $v(x,t,y)$ is shown at location $x=0.454$ with $N=21$ order of expansion. Fig. 4(a) shows the approximation at $t=0$, i.e., the initial condition, and Fig. 4(b) shows the numerical solution at $t=1$. The Gibbs' oscillations around the discontinuity at $y=0$ are clearly visible.

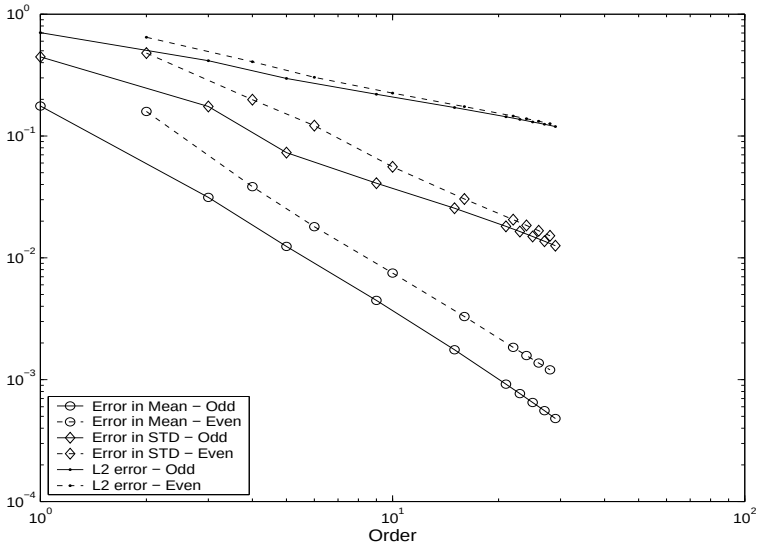

Figure 3: Errors for odd and even orders of Legendre-chaos expansions.

Table 1: Errors for odd-order Legendre-chaos expansions and their convergence rate.

N	e_{mean}	r_{mean}	e_{std}	r_{std}	e_2	r_2
1	0.1761	–	0.4460	–	0.7044	–
3	3.137(-2)	1.57	0.1747	0.85	0.4147	0.48
5	1.241(-2)	1.82	7.300(-2)	1.71	0.2965	0.66
9	4.470(-3)	1.74	4.095(-2)	0.98	0.2202	0.51
15	1.758(-3)	1.83	2.553(-2)	0.92	0.1713	0.49
21	9.188(-4)	1.93	1.815(-2)	1.01	0.1438	0.52
23	7.680(-4)	1.97	1.644(-2)	1.08	0.1368	0.55
25	6.503(-4)	2.00	1.498(-2)	1.12	0.1305	0.56
27	5.564(-4)	2.03	1.370(-2)	1.16	0.1248	0.58
29	4.803(-4)	2.06	1.257(-2)	1.20	0.1195	0.60

Table 2: Errors for even-order Legendre-chaos expansions and their convergence rate.

N	e_{mean}	r_{mean}	e_{std}	r_{std}	e_2	r_2
2	0.1589	–	0.4791	–	0.6457	–
4	3.841(-2)	2.05	0.1985	1.27	0.4065	0.67
6	1.803(-2)	1.87	0.1215	1.21	0.3030	0.72
10	7.503(-3)	1.72	5.588(-2)	1.52	0.2249	0.58
16	3.295(-3)	1.75	3.042(-2)	1.29	0.1741	0.54
22	1.843(-3)	1.82	2.058(-2)	1.23	0.1457	0.56
24	1.577(-3)	1.79	1.846(-2)	1.25	0.1386	0.58
26	1.370(-3)	1.76	1.668(-2)	1.27	0.1321	0.59
28	1.204(-3)	1.74	1.516(-2)	1.29	0.1263	0.61

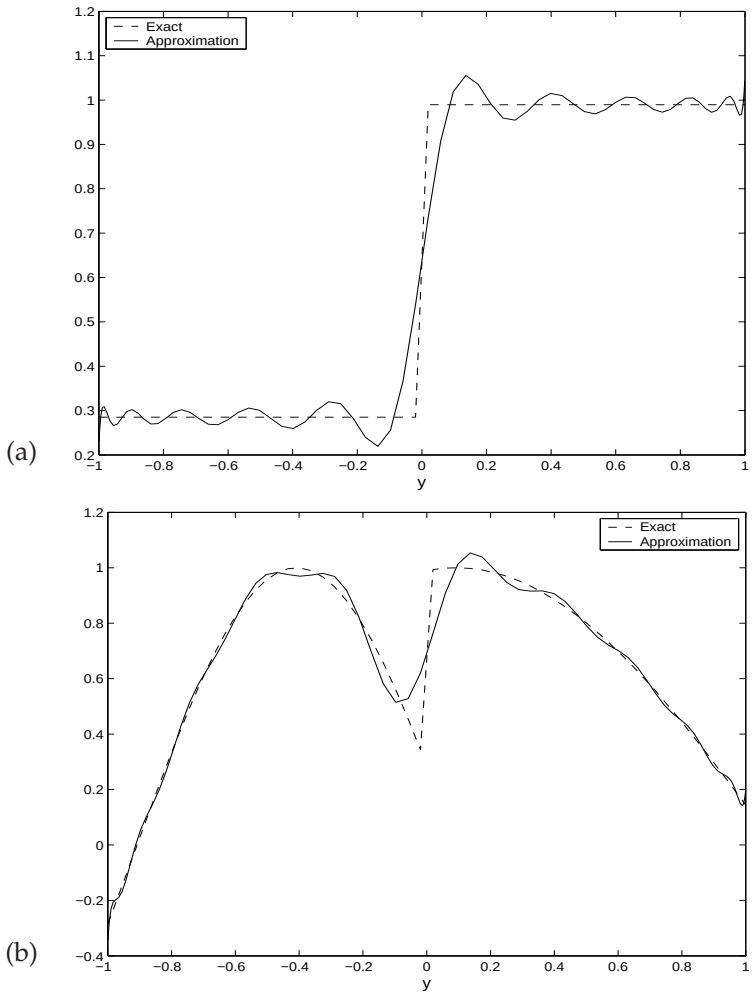

Figure 4: Numerical approximations of $u_N(x,t,y)$ at $x=0.454$ with $N=21$. (a) $t=0$; (b) $t=1$.

4 Summary

The properties of (generalized) Polynomial Chaos method for uncertainty analysis of hyperbolic equations are studied. We show, via a simple model problem of a scalar wave equation with random wave speed, some prominent features of the resulting deterministic system of equations obtained by a Galerkin projection in random space. We proved the existence of both positive and negative eigenvalues when the wave speed changes sign in random space and presented a consistent and stable method for imposing boundary conditions for the deterministic equations. The gPC Galerkin method, with the proper boundary treatment, is shown to be convergent. Furthermore, the error contains a linear growth in time which is independent of the boundary conditions. We remark that although the linear wave equation considered here is rather simple, it possesses one of

the key issues in applying gPC Galerkin method to hyperbolic problems – the proper way to enforce boundary conditions when the characteristic wave changes directions in random space. This issue is addressed here and it opens up the possibility of applying gPC Galerkin method to other hyperbolic problems with uncertainty, e.g., nonlinear wave equations, Maxwell equations, etc.

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References

[1] Q.-Y. Chen, D. Gottlieb and J. S. Hesthaven, Uncertainty analysis for the steady-state flows in a dual throat nozzle, J. Comput. Phys., 204 (2005), 387-398.
[2] R. G. Ghanem, Ingredients for a general purpose stochastic finite element formulation, Comput. Method. Appl. Mech. Engrg., 168 (1999), 19-34.
[3] R. G. Ghanem and P. Spanos, Stochastic Finite Elements: A Spectral Approach, Springer-Verlag, 1991.
[4] D. Gottlieb and S. A. Orszag, Numerical Analysis of Spectral Methods: Theory and Applications, CBMS-NSF, SIAM, Philadelphia, 1977.
[5] N. Wiener, The homogeneous chaos, Am. J. Math., 60 (1938), 897-936.
[6] D. Xiu, Efficient collocational approach for parametric uncertainty analysis, Commun. Comput. Phys., 2 (2007), 293-309.
[7] D. Xiu and G. E. Karniadakis, The Wiener-Askey polynomial chaos for stochastic differential equations, SIAM J. Sci. Comput., 24(2) (2002), 619-644.
[8] D. Xiu and G. E. Karniadakis, Modeling uncertainty in flow simulations via generalized polynomial chaos, J. Comput. Phys., 187 (2003), 137-167.