COMMUNICATIONS IN COMPUTATIONAL PHYSICS
Vol. **5**, No. 5, pp. 942-958

Commun. Comput. Phys.
May 2009

The Recursive Formulation of Particular Solutions for Some Elliptic PDEs with Polynomial Source Functions

J. Ding, H. Y. Tian and C. S. Chen[*]

Department of Mathematics, University of Southern Mississippi, Hattiesburg, MS 39406, USA.

Received 2 April 2008; Accepted (in revised version) 3 September 2008

Available online 14 October 2008

Abstract. In this paper we develop an efficient meshless method for solving inhomogeneous elliptic partial differential equations. We first approximate the source function by Chebyshev polynomials. We then focus on how to find a polynomial particular solution when the source function is a polynomial. Through the principle of the method of undetermined coefficients and a proper arrangement of the terms for the polynomial particular solution to be determined, the coefficients of the particular solution satisfy a triangular system of linear algebraic equations. Explicit recursive formulas for the coefficients of the particular solutions are derived for different types of elliptic PDEs. The method is further incorporated into the method of fundamental solutions for solving inhomogeneous elliptic PDEs. Numerical results show that our approach is efficient and accurate.

AMS subject classifications: 35J05, 35J25, 65D05, 65D15

Key words: The method of fundamental solutions, particular solution, Helmholtz equation, Chebyshev polynomial, Laplace-Helmholtz equation, convection-reaction equation.

1 Introduction

We consider the boundary value problem,

$$\begin{aligned} Lu(x,y) &= f(x,y), \ (x,y)\in\Omega, \\ Bu(x,y) &= g(x,y), \ (x,y)\in\partial\Omega, \end{aligned} \tag{1.1}$$

[*] Corresponding author. *Email addresses:* Jiu.Ding@usm.edu (J. Ding), Haiyan.Tian@usm.edu (H. Y. Tian), CS.Chen@usm.edu (C. S. Chen)

where $\Omega \subset R^2$ is a simply connected domain whose boundary is a simple closed curve $\partial\Omega$, L and B are the differential operators on u over the interior of Ω and the boundary $\partial\Omega$ respectively. We assume that the operator L is of elliptic type. Efficient and accurate solution techniques for the elliptic boundary value problem can easily find applications in diverse problems in mechanics, gravitation, electricity, and magnetism.

In the framework of boundary methods, a widely used approach is to split the solution of Problem (1.1) into a particular solution $u_p(x,y)$ that satisfies

$$Lu_p(x,y) = f(x,y), \ (x,y) \in \Omega, \tag{1.2}$$

and its associated homogeneous solution $u_h(x,y)$ that satisfies

$$\begin{aligned} &Lu_h(x,y) = 0, \ (x,y) \in \Omega, \\ &Bu_h(x,y) = g(x,y) - Bu_p(x,y), \ (x,y) \in \partial\Omega. \end{aligned} \tag{1.3}$$

Once a particular solution u_p is known, the influence on the solution by the inhomogeneous term f has in fact been transferred to the boundary, giving rise to Problem (1.3) involving the homogeneous elliptic equation subject to a new boundary condition. The solution of Problem (1.3) can then be found by standard boundary techniques [15–17]. The solution u of Problem (1.1) is then obtained as $u = u_p + u_h$.

Following this approach we face the challenge of approximating f in such a way that also allows us to find a particular solution. For general differential operators, the method of particular solution (MPS) [17] has been used to overcome the difficulties of evaluating u_p. It allows for the decoupling of the original given problem (1.1) into a particular solution and a homogeneous solution. In the framework of the MPS, a variety of basis functions can be used to approximate the source function. Most commonly, the source function is approximated by a series of radial basis functions (RBFs). For example, Partridge et al. in [25] and Muleshkov et al. in [22] used the RBF approximation for the Laplacian and Helmholtz-type operators, respectively. Despite the important interpolating properties of RBFs, one of their drawbacks is that it is difficult to obtain rapidly convergent RBF interpolants. As a consequence, one often has to use a large number of interpolating points, which could lead to a large, dense and highly ill-conditioned system of equations.

Other classes of approximations, such as trigonometric [1] and polynomial [6, 13, 18] ones, have been considered to overcome the difficulties encountered in the use of RBFs in the MPS. Chen et al in [6] obtained particular solutions in analytical form for the 2-D Poisson equation when the source function f is a homogeneous polynomial. Golberg et al. in [18] implemented the MPS when they used Chebyshev interpolants in their approach. In [18], particular solutions in analytical form for 2-D and 3-D Helmholtz-type equations when the source function is a monomial and for the 3-D Poisson equation when the source function is a homogeneous polynomial are obtained. Symbolic software packages such as Maple and Mathematica can be used for the implementation of the algorithm to get particular solutions. However, after the source function f is approximated by a polynomial $\widetilde{f}$, for example, for a 2-D Helmholtz-type equation, book-keeping of the many monomial terms of the polynomial $\widetilde{f}$ and the particular solutions corresponding to these terms becomes tedious and inefficient. For example, when using the Chebyshev approximation, we get

$$\widetilde{f}=\sum_{k=0}^{m}\sum_{l=0}^{n}a_{kl}T_k(x)T_l(y),$$

where $T_k(x)$ and $T_l(y)$ are Chebyshev polynomials of degree k and l respectively. To get the particular solution $\widetilde{u}_p$ corresponding to $\widetilde{f}$, we will need to expand $\widetilde{f}$ in terms of monomials and derive the particular solution for each monomial term. Then $\widetilde{u}_p$ is obtained by the superposition principle.

Chen et al. [9], by a simple and direct way, avoided the particular solutions corresponding to monomial terms. Instead, they used a finite term geometric series expansion on a differential operator to directly obtain a particular solution Ψ_{kl} corresponding to each $T_k(x)T_l(y)$. In their paper, particular solutions in the form of finite series involving certain differential operators are obtained for Helmholtz, Bi-Helmholtz, Laplace-Helmholtz, and convection-reaction types of equations. Since the particular solutions Ψ_{kl} are found by applying the finite series of a differential operator onto $T_k(x)T_l(y)$, symbolic computer packages such as Mathematica or Maple can be used to serve this purpose. Then a look up table of Ψ_{kl} is generated so that the algorithm can be implemented together with the method of fundamental solutions (MFS) using Fortran or C++ for fast computation. In this way, they are not limited to only symbolic software package as in [18]. Recently, Karageorghis and Kyza derived particular solutions for Poisson's equation and the bi-harmonic equation directly using Chebyshev polynomials as basis functions. In their work, Karageorghis and Kyza [19] made use of special properties of the second derivatives of Chebyshev polynomials. However, they need to solve a matrix system which is computationally intensive when the degrees of the Chebyshev polynomials used is high. Here we derive a particular solution corresponding to $\widetilde{f}$ in a way that can be implemented using only Fortran or C++ for the purpose of high speed scientific computing. In the recursive formulation for obtaining a particular solution we developed in this paper, no matrix inversion is required. As a result, as we shall see in our numerical examples, our algorithm is very efficient.

The Helmholtz and poly-Helmholtz equations often appear in physical problems such as a multilayered aquifer system [10,11], or a multiple porosity system [12].

In this paper, we use Chebyshev polynomials to approximate the source function f for various types of differential operators. The collocation points are distributed in a rectangular region containing the physical domain. Explicit formulas are known for these interpolants. There is no need to solve a system of linear equations in the approximation. Therefore, the difficulty of solving a large and ill-conditioned system due to a large number of interpolation points as encountered in the RBF approach does not exist. We then focus on the derivation of a polynomial particular solution when the source function is a polynomial. In Section 2, we consider a Helmholtz type equation with its source function being a bivariate polynomial of the form

$$f(x,y)=\sum_{i=0}^{m}\sum_{j=0}^{n}d_{ij}x^iy^j. \tag{1.4}$$

In determining a polynomial particular solution, the method of undetermined coefficients is used. A nonsingular upper triangular system of the unknown coefficients is obtained by a proper arrangement of the coefficients. The solution of the system can be easily obtained by an explicit recursive formula. Instead of first getting a particular solution corresponding to each monomial and then using the superposition principle to get the particular solution corresponding to the polynomial, here we get the particular solution corresponding to a polynomial directly.

The structure of this paper is as follows. In Section 2, we give an algorithm for determining the coefficients d_{ij} when expressing a Chebyshev approximation $\widetilde{f}$ in the form of (1.4) in order to use the recursive formulas derived in Section 3 and Section 4. In Section 3, the particular solution for the Helmholtz equation is derived. In Section 4, particular solutions in the form of a polynomial are derived for other equations including the Bi-Helmholtz equation, the Laplace-Helmholtz equation, and the convection-reaction equation. Numerical examples are presented in Section 5. The numerical results are compared to the corresponding analytical solutions and the proposed method is shown to be accurate and efficient.

2 Expansion of Chebyshev polynomials

In this section, we first briefly review the Chebyshev approximation $\widetilde{f}(x,y)$ of a function $f(x,y)$. For the purpose of real implementation, we describe an algorithm on how to expand products of Chebyshev polynomials.

The univariate Chebyshev polynomial interpolation is extended to the bivariate case in a rectangular domain by a tensor product. The Chebyshev interpolant using the Gauss-Lobatto nodes [2,3,21] for the rectangular domain $[a,b]\times[c,d]$ takes the form:

$$\widetilde{f}(x,y)=\sum_{k=0}^{m}\sum_{l=0}^{n}a_{kl}T_k\left(\frac{2x-b-a}{b-a}\right)T_l\left(\frac{2y-d-c}{d-c}\right), \tag{2.1}$$

where

$$a_{kl}=\frac{4}{mn\bar{c}_k^x\bar{c}_l^y}\sum_{p=0}^{m}\sum_{q=0}^{n}\frac{f\left(x_p,y_q\right)}{\bar{c}_p^x\bar{c}_q^y}\cos\left(\frac{\pi pk}{m}\right)\cos\left(\frac{\pi ql}{n}\right) \tag{2.2}$$

and

$$\begin{aligned}&\bar{c}_0^x=\bar{c}_m^x=2,\ \bar{c}_j^x=1,\ 1\le j\le m-1,\\&\bar{c}_0^y=\bar{c}_n^y=2,\ \bar{c}_j^y=1,\ 1\le j\le n-1.\end{aligned} \tag{2.3}$$

Note that m and n are the numbers of Gauss-Lobatto nodes in the x and y direction, respectively. It is well known that the use of Gauss-Lobatto nodes will ensure the spectral convergence for the Chebyshev interpolation.

We assume $a=-1$ and $b=1$ in the Chebyshev expansion $\widetilde{f}(x,y)$ given by (2.1). As we shall see in the next section, the recursive formulation of the particular solution is based on the polynomial basis $\{x^k y^l\}_{k=0,l=0}^{m,n}$ instead of Chebyshev polynomial basis $\{T_k(x)T_l(y)\}_{k=0,l=0}^{m,n}$. How to convert from one basis to the other is not trivial.

The function $\widetilde{f}(x,y)$ in (2.1) can be expressed in the following form

$$\widetilde{f}=\sum_{i=0}^{m}\sum_{j=0}^{n}d_{ij}x^i y^j.$$

Assume that each $T_k(x)T_l(y)$ in (2.1) is expressed as

$$T_k(x)T_l(y)=\sum_{i=0}^{k}\sum_{j=0}^{l}d_{ij}^{(kl)}x^i y^j, \tag{2.4}$$

then

$$d_{ij}=\sum_{k=i}^{m}\sum_{l=j}^{n}a_{kl}d_{ij}^{(kl)}. \tag{2.5}$$

In the following, we give an algorithm to determine the coefficients $d_{ij}^{(kl)}$, $i=0,\cdots,k$, $j=0,\cdots,l$ for $k=0,\cdots,m$, $l=0,\cdots,n$. Since $d_{ij}^{(kl)}$ is the coefficient of $x^i y^j$ in $T_k(x)T_l(y)$ we need to collect the coefficient for x^i contributed by $T_k(x)$ and the coefficient for y^j contributed by $T_l(y)$. Let $t_{k,i}$ and $s_{l,j}$ denote the coefficients of x^i in $T_k(x)$ and y^j in $T_l(y)$, respectively.

For the univariate Chebyshev polynomial $T_k(x)$, we have the following recursive formula

$$\begin{aligned}&T_0(x)=1,\\&T_1(x)=x,\\&T_k(x)=2xT_{k-1}(x)-T_{k-2}(x),\ \ k\geq 2.\end{aligned}$$

Then the $t_{k,i}$ can be generated by the following recursive formula with $t_{k-2,k-1}=t_{k-2,k}=0$,

$$\begin{aligned}&t_{0,0}=1,\\&t_{1,0}=0,\ \ t_{1,1}=1,\\&t_{k,i}=2t_{k-1,i-1}-t_{k-2,i},\ \ k=2,\cdots,m;\ \ i=0,\cdots,k.\end{aligned} \tag{2.6}$$

Similarly, we have the following recursive formula for the $s_{l,j}$ with $s_{l-2,l-1}=s_{l-2,l}=0$,

$$\begin{aligned}&s_{0,0}=1,\\&s_{1,0}=0,\ \ s_{1,1}=1,\\&s_{l,j}=2s_{l-1,j-1}-s_{l-2,j},\ \ l=2,\cdots,n;\ \ j=0,\cdots,l.\end{aligned} \tag{2.7}$$

The above algorithm offers a different option for using symbolic software packages such as Mathematica or Maple to find the coefficients $t_{k,i}$ and $s_{l,j}$. The coefficients $d_{ij}^{(kl)}$ are then found to be

$$d_{ij}^{(kl)}=t_{k,i}s_{l,j}, \; i=0,\cdots,k, \; j=0,\cdots,l, \; \text{for}\; k=0,\cdots,m; \; l=0,\cdots,n. \tag{2.8}$$

We note that when $m=n$ in (2.1), the set $\{t_{k,i}\}$ and $\{s_{l,j}\}$ are the same. The generation of coefficients d_{ij} when the Chebyshev approximation $\widetilde{f}(x,y)$ in (2.1) is expressed in the form of (1.4) and the coefficients for the polynomial particular solutions corresponding to (1.4) can be completely implemented using Fortran.

3 Particular solutions for Helmholtz equation

Problem (1.1) can be solved by first approximating the source function f using its Chebyshev polynomial expansion $\widetilde{f}$. We then determine the coefficients d_{ij} for $\widetilde{f}$ as expressed in (1.4). A polynomial particular solution $\widetilde{u}_p$ corresponding to the source function $\widetilde{f}$ can be obtained by a recursive formula.

In this section, we focus on the derivation of a closed-form particular solution when the source function f is a polynomial. We consider the Helmholtz-type partial differential equation in two variables

$$\Delta u(x,y)+ku(x,y)=f(x,y), (x,y)\in\Omega, \tag{3.1}$$

where Δ is the Laplacian, k a nonzero constant, and Ω is a bounded open region in R^2. Assume that the source function f is analytic in Ω in the sense that f has a power series representation in an open disk $D(0;r)$ that contains Ω. Let

$$f(x,y)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}d_{ij}x^iy^j$$

with uniform convergence on any compact subset of $D(0;r)$. Our purpose is to find a particular solution of equation (3.1) in power series form. Suppose a solution u of (3.1) has the power series representation

$$u(x,y)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}c_{ij}x^iy^j.$$

It follows from (3.1) and the expressions of $f(x,y)$ and $u(x,y)$ that

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left[(i+2)(i+1)c_{i+2,j}+(j+2)(j+1)c_{i,j+2}+kc_{ij}\right]x^iy^j=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}d_{ij}x^iy^j, \tag{3.2}$$

which is equivalent to the following system of linear equations for c_{ij}'s:

$$(i+2)(i+1)c_{i+2,j}+(j+2)(j+1)c_{i,j+2}+kc_{ij}=d_{ij},\ \text{for all } i,j=0,1,\cdots.$$

In the process of obtaining a particular solution of the Helmholtz-type equation (3.1), the source function f can be first approximated by Chebyshev polynomials. From now on we assume that

$$f(x,y)=\sum_{i=0}^{m}\sum_{j=0}^{n}d_{ij}x^iy^j.$$

Then a particular solution u_p can be chosen to be the polynomial

$$u_p(x,y)=\sum_{i=0}^{m}\sum_{j=0}^{n}c_{ij}x^iy^j, \tag{3.3}$$

and system (3.2) is reduced to

$$\begin{aligned}&(i+2)(i+1)c_{i+2,j}+(j+2)(j+1)c_{i,j+2}+kc_{ij}=d_{ij},\ \text{for all } 0\le i\le m,\ 0\le j\le n,\\ &c_{ij}=0,\ \text{for all } i>m \text{ or } j>n.\end{aligned} \tag{3.4}$$

The above linear system of $(m+1)(n+1)$ equations for the $(m+1)(n+1)$ unknowns c_{ij} can be explicitly represented by the recursive formula

$$\begin{aligned}&c_{ij}=\frac{1}{k}\left[d_{ij}-(i+2)(i+1)c_{i+2,j}-(j+2)(j+1)c_{i,j+2}\right],\\ &j=n,n-1,\cdots,1,0;\ i=m,m-1,\cdots,1,0,\end{aligned} \tag{3.5}$$

which is equivalent to solving the corresponding nonsingular upper triangular system

$$A\mathbf{c}=\mathbf{d}$$

of linear equations by back substitution, where

$$\mathbf{c}=(c_{00},c_{10},\cdots,c_{m0},c_{01},c_{11},\cdots,c_{m1},\cdots,c_{0n},c_{1n},\cdots,c_{mn})^T, \tag{3.6}$$

$$\mathbf{d}=(d_{00},d_{10},\cdots,d_{m0},d_{01},d_{11},\cdots,d_{m1},\cdots,d_{0n},d_{1n},\cdots,d_{mn})^T, \tag{3.7}$$

and $A=[a_{rs}]$ is an $(m+1)(n+1)\times(m+1)(n+1)$ *upper triangular* matrix that can be written in block matrix form as

$$A=\begin{bmatrix} B & 0 & 2\cdot1I & 0 & 0 & \cdots & 0\\ 0 & B & 0 & 3\cdot2I & 0 & \cdots & 0\\ 0 & 0 & B & 0 & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ 0 & \cdot & 0 & 0 & B & 0 & n(n-1)I\\ 0 & \cdot & 0 & 0 & 0 & B & 0\\ 0 & \cdot & 0 & 0 & 0 & 0 & B \end{bmatrix},$$

where I is the $(m+1)\times(m+1)$ identity matrix and $B=[b_{ij}]$ with $b_{ij}=0$ except that $b_{ii}=k$ for $1\le i\le m+1$ and $b_{i,i+2}=i(i+1)$ for $1\le i\le m-1$.

Example 3.1. Let us find a particular solution of the equation

$$(\Delta+16)\Psi_{2,2}(x,y)=T_2(x)T_2(y). \tag{3.8}$$

Since

$$T_2(x)T_2(y)=(2x^2-1)(2y^2-1)=1-2x^2-2y^2+4x^2y^2,$$

we have $d_{00}=1, d_{20}=d_{02}=-2, d_{22}=4$, and all other $d_{ij}=0$. Solving (3.5) gives

$$c_{22}=\frac{1}{4},\ c_{12}=0,\ c_{02}=-\frac{5}{32},\ c_{21}=c_{11}=c_{01}=0,\ c_{20}=-\frac{5}{32},\ c_{10}=0,\ c_{00}=\frac{13}{128}.$$

Thus, a particular solution of (3.8) is

$$\Psi_{2,2}(x,y)=\frac{13}{128}-\frac{5}{32}x^2-\frac{5}{32}y^2+\frac{1}{4}x^2y^2.$$

4 Applications of the method to other equations

The derivation of a particular solution using the matrix approach to Helmholtz-type equations shown in the previous section can be easily extended to other types of differential equations such as the bi-Helmholtz equation. In the following subsections we apply our new approach to several inhomogeneous equations which are closely related to Helmholtz-type equations.

4.1 Bi-Helmholtz equation

We consider the following bi-Helmholtz equation

$$(\Delta+k)^2u(x,y)=f(x,y). \tag{4.1}$$

Clearly the bi-Helmholtz equation is equivalent to the pair of Helmholtz-type equations

$$(\Delta+k)u(x,y)=v(x,y),\ (\Delta+k)v(x,y)=f(x,y).$$

It is now obvious that if f is a polynomial, then a particular solution v to the second equation above can be taken to be a polynomial and can be calculated by the method from the previous section. Since v is a polynomial, the same method applied to the first equation above produces a particular solution to the original bi-Helmholtz equation.

Example 4.1. Let us consider the equation

$$(\Delta+k)^2\Psi_{3,4}(x,y)=T_3(x)T_4(y). \tag{4.2}$$

For $k=16$, we have

$$\Psi_{3,4}(x,y)=-\frac{165}{1024}x+\frac{21}{256}x^3+\frac{69}{128}xy^2-\frac{5}{16}x^3y^2-\frac{3}{16}xy^4+\frac{1}{8}x^3y^4.$$

For $k=-16$, we have

$$\Psi_{3,4}(x,y)=\frac{15}{1024}x+\frac{5}{256}x^3+\frac{9}{128}xy^2+\frac{1}{16}x^3y^2+\frac{1}{8}x^3y^4.$$

4.2 Laplace-Helmholtz equation

Consider the following Laplace-Helmholtz equation

$$\Delta(\Delta+k)u(x,y)=f(x,y). \tag{4.3}$$

The above equation can be written as a Helmholtz equation and a Poisson equation:

$$(\Delta+k)u(x,y)=g(x,y),\ \ \Delta g(x,y)=f(x,y). \tag{4.4}$$

A particular solution of the first equation in (4.4) can be obtained via our matrix method if we can find a particular solution of the second equation which may be obtained by the method described in [8]. However, we can still use the idea of our matrix approach to find a particular solution of the second equation if we view the polynomial f as a sum of homogeneous polynomials, as the following shows.

A polynomial is said to be *homogeneous* if each term has the same degree. Consider the Poisson equation

$$\Delta g(x,y)=\sum_{i=0}^{n}d_ix^{n-i}y^i. \tag{4.5}$$

Let

$$g(x,y)=\sum_{i=0}^{n}c_ix^{n+2-i}y^i,$$

where the coefficients c_i are to be determined. Using the above expression and some straightforward calculations, we have

$$\Delta g(x,y)=\sum_{i=0}^{n-2}[c_i(n+2-i)(n+1-i)+c_{i+2}(i+2)(i+1)]x^{n-i}y^i+c_{n-1}3\cdot 2xy^{n-1}+c_n2\cdot 1y^n.$$

Thus, (4.5) is reduced to the algebraic system $A\mathbf{c}=\mathbf{d}$ which can be solved easily by back substitution, where A is an $(n+1)\times(n+1)$ upper triangular matrix with all zero entries except for the main diagonal entries

$$(n+2)(n+1),(n+1)n,n(n-1),(n-1)(n-2),\cdots,4\cdot 3,3\cdot 2,2\cdot 1$$

and the second super-diagonal entries

$$2\cdot 1, 3\cdot 2, 4\cdot 3, \cdots, (n-2)(n-3), (n-1)(n-2), n(n-1),$$

and $\mathbf{c}$ and $\mathbf{d}$ are the $(n+1)$-dimensional vectors of components c_k's and d_k's, respectively. Therefore, a particular solution of the second equation of (4.4) can be calculated by first decomposing the source polynomial as a sum of homogeneous polynomials and applying the above matrix method and the superposition principle.

Example 4.2. Let us consider the equation

$$\Delta(\Delta+k)\Psi_{3,4}(x,y) = T_3(x)T_4(y).$$

For $k=16$, we have

$$\begin{aligned}\Psi_{3,4}(x,y) = &\frac{129}{2048}x - \frac{19}{256}x^3 + \frac{1}{80}x^5 + \frac{1}{210}x^7 - \frac{9}{32}xy^2 + \frac{7}{32}x^3y^2 - \frac{1}{10}x^5y^2 \\ &+ \frac{17}{64}xy^4 - \frac{1}{8}x^3y^4 - \frac{1}{20}xy^6 + \frac{1}{15}x^3y^6 - \frac{1}{140}xy^8.\end{aligned}$$

For $k=-16$, we have

$$\begin{aligned}\Psi_{3,4}(x,y) = &\frac{3}{2048}x + \frac{5}{256}x^3 - \frac{1}{80}x^5 - \frac{1}{210}x^7 - \frac{3}{64}xy^2 + \frac{1}{32}x^3y^2 + \frac{1}{10}x^5y^2 \\ &- \frac{5}{64}xy^4 - \frac{1}{8}x^3y^4 + \frac{1}{20}xy^6 - \frac{1}{15}x^3y^6 + \frac{1}{140}xy^8.\end{aligned}$$

4.3 Convection-reaction equation

Consider the following convection-reaction equation

$$\left(\Delta + a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\right)u(x,y) = f(x,y), \tag{4.6}$$

where a, b, and $c \neq 0$ are constants.

The idea of the matrix method applied to the Helmholtz-type equation can be easily extended to find a particular solution of the convection-reaction equation as follows

Suppose f is the polynomial (1.4) and let u be given by (3.3). Then

$$\begin{aligned}u_x(x,y) &= \sum_{i=1}^{m}\sum_{j=0}^{n} c_{ij} i x^{i-1} y^j = \sum_{i=0}^{m-1}\sum_{j=0}^{n} c_{i+1,j}(i+1)x^i y^j, \\ u_y(x,y) &= \sum_{i=0}^{m}\sum_{j=1}^{n} c_{ij} j x^i y^{j-1} = \sum_{i=0}^{m}\sum_{j=0}^{n-1} c_{i,j+1}(j+1)x^i y^j, \\ u_{xx}(x,y) &= \sum_{i=2}^{m}\sum_{j=0}^{n} c_{ij} i(i-1) x^{i-2} y^j = \sum_{i=0}^{m-2}\sum_{j=0}^{n} c_{i+2,j}(i+2)(i+1)x^i y^j, \\ u_{yy}(x,y) &= \sum_{i=0}^{m}\sum_{j=2}^{n} c_{ij} j(j-1) x^i y^{j-2} = \sum_{i=0}^{m}\sum_{j=0}^{n-2} c_{i,j+2}(j+2)(j+1)x^i y^j.\end{aligned}$$

The following upper triangular system of linear equations for the c_{ij}'s can be obtained from (4.6):

$$\begin{aligned}&(i+2)(i+1)c_{i+2,j}+(j+2)(j+1)c_{i,j+2}+a(i+1)c_{i+1,j}+b(j+1)c_{i,j+1}+cc_{ij}=d_{ij},\\&\text{for all } i=0,1,\cdots,m,\ j=0,1,\cdots,n,\\&c_{ij}=0,\ \text{ for all } i>m \text{ or } j>n.\end{aligned}\tag{4.7}$$

The solution of (4.7) is given recursively by

$$\begin{aligned}&c_{ij}=\frac{1}{c}\left[d_{ij}-(i+2)(i+1)c_{i+2,j}-(j+2)(j+1)c_{i,j+2}-a(i+1)c_{i+1,j}-b(j+1)c_{i,j+1}\right]\\&j=n,n-1,\cdots,1,0;\ i=m,m-1,\cdots,1,0.\end{aligned}\tag{4.8}$$

Example 4.3. Consider the equation

$$\left(\Delta+2\frac{\partial}{\partial x}+2\frac{\partial}{\partial y}-16\right)\Psi_{3,4}(x,y)=T_3(x)T_4(y).\tag{4.9}$$

Based on (4.8), we obtain a particular solution $\Psi_{3,4}(x,y)$ as follows:

$$\begin{aligned}\Psi_{3,4}(x,y)=&-\frac{9531}{65536}-\frac{1995x}{8192}-\frac{573x^2}{2048}-\frac{71x^3}{256}-\frac{621y}{2048}-\frac{369xy}{512}-\frac{39x^2y}{64}-\frac{11x^3y}{32}\\&-\frac{345y^2}{1024}-\frac{171xy^2}{128}-\frac{51x^2y^2}{64}+\frac{x^3y^2}{8}-\frac{9y^3}{64}-\frac{9xy^3}{32}-\frac{3x^2y^3}{4}-x^3y^3\\&-\frac{3y^4}{128}+\frac{9xy^4}{16}-\frac{3x^2y^4}{4}-2x^3y^4.\end{aligned}$$

5 Numerical results

Once we find an approximate particular solution $\widetilde{u}_p$ to Problem (1.2), we then solve the resulting homogeneous problem (1.3) by the method of fundamental solutions.

Let $\{G(\mathbf{x},¸_j)\}_{j=1}^N$ be the fundamental solutions corresponding to the operator L, with each $¸_j$ being a source point located outside the domain Ω. Let

$$\widetilde{u}_h=\sum_{j=1}^N q_j G(\mathbf{x},¸_j).$$

The coefficients q_j, $j=1,\cdots,N$, will be determined by the collocation approach of the MFS. That is, we let $\widetilde{u}_h$ satisfy the boundary condition in (1.3) at a set of N collocation points $\{\mathbf{x}_i\}_{i=1}^N$ on $\partial\Omega$,

$$\sum_{j=1}^N q_j BG(\mathbf{x}_i,¸_j)=g(\mathbf{x}_i)-B\widetilde{u}_p(\mathbf{x}_i),\quad i=1,\cdots,N.\tag{5.1}$$

The solution $\widetilde{u}_h$ can be found after we solve the system of equations (5.1) for the coefficients $\{q_j\}_{j=1}^N$. A numerical solution to Problem (1.1) is then obtained as

$$\widetilde{u}(\mathbf{x})=\widetilde{u}_p(\mathbf{x})+\widetilde{u}_h(\mathbf{x}).$$

In order to assess the accuracy of the numerical solution $\widetilde{u}(\mathbf{x})$, we choose a set of test points $\{\mathbf{t}_k\}_{k=1}^K \subset \overline{\Omega}$. For the examples given below, we let E, Ex, and Ey denote the absolute errors of the computed solution $\widetilde{u}(\mathbf{x})$ and its first order derivatives $\widetilde{u}_x(\mathbf{x})$ and $\widetilde{u}_y(\mathbf{x})$ over the test points $\{\mathbf{t}_k\}$,

$$\begin{aligned} E&=\max_k |u(\mathbf{t}_k)-\widetilde{u}(\mathbf{t}_k)|, \\ Ex&=\max_k |u_x(\mathbf{t}_k)-\widetilde{u}_x(\mathbf{t}_k)|, \\ Ey&=\max_k |u_y(\mathbf{t}_k)-\widetilde{u}_y(\mathbf{t}_k)|. \end{aligned}$$

To increase the accuracy, we can simply increase the number of Chebyshev nodes m and n. For large m and n, the coefficients a_{jk} in (2.2) become large and the round-off error could be an issue. We observe that most of the coefficients a_{kl} in (2.2) are almost zero. We use the same a_{kl} for the evaluation of the corresponding particular solution. Hence, we skip the evaluation of a particular solution when $a_{kl}<\varepsilon$, with ε being a given positive number. To increase the efficiency, we save a_{kl} in the memory to avoid the repeated evaluation. As we shall see, with such refinement we can significantly improve the computational efficiency. This results in the fast evaluation of a particular solution and also avoids the potential difficulty of round-off errors.

We test our proposed numerical algorithm on Helmholtz-type and Bi-Helmholtz-type problems in different domains.

Example 5.1. We consider the modified Helmholtz equation,

$$\begin{aligned} &\left(\Delta-\lambda^2\right)u=f, \quad \text{in } \Omega, \\ &u=g, \quad \text{on } \partial\Omega, \end{aligned} \tag{5.2}$$

where $\lambda=10$, the domain Ω is of shape of Cassini whose boundary $\partial\Omega$ is given by

$$\begin{aligned} &x=\rho\cos\theta, \ \ y=\rho\sin\theta, \\ &\rho=q\left(\cos(3\theta)+\sqrt{4-\sin^2(3\theta)}\right)^{1/3}, \end{aligned} \tag{5.3}$$

with $q=1$, the functions f and g are given such that the exact solution of (5.2) is

$$u^*(x,y)=y\sin(\pi x)+x\cos(\pi y). \tag{5.4}$$

We evenly (in terms of angle) distribute 160 collocation points on the Cassini boundary and 160 source points on the Cassini curve with $q=10$ in (5.3). We use a total of 258 test

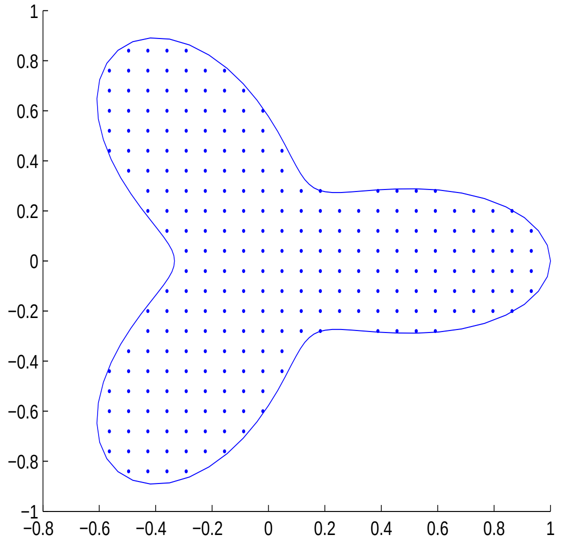

Figure 1: The distribution of the test points on the Cassini domain.

points inside the domain. A plot of these test points is shown in Fig. 1. The computational results are given in Table 1. We observed an increase of one order of accuracy for every increment of Chebyshev node in each axis direction. For $m=n=18$, we can achieve near machine precision. Furthermore, this only requires 0.1698 second of CPU time which is extremely efficient. The fast algorithm is mostly due to the omission of the evaluation of the particular solution when $a_{kl}<\varepsilon=10^{-12}$.

Table 1: The absolute error of the computed solution and its first order derivatives.

$m=n$	E_a	EX_a	EY_a	CPU Time (seconds)
11	5.409E-07	3.270E-06	6.109E-06	0.0815
12	6.118E-08	6.595E-07	3.013E-07	0.0992
13	6.021E-09	3.564E-08	8.660E-08	0.0999
14	5.492E-10	8.525E-09	3.154E-09	0.1093
15	5.652E-11	2.894E-10	8.586E-10	0.1196
16	4.508E-12	6.799E-11	4.235E-11	0.1285
17	3.911E-13	1.788E-12	6.303E-12	0.1387
18	2.756E-14	5.080E-13	1.885E-13	0.1698

Example 5.2. We now consider the Bi-Helmholtz problem,

$$\begin{aligned}&\left(\Delta-\lambda^2\right)^2 u=f, \ \text{ in } \ \Omega,\\ &u=g, \ \text{ on } \ \partial\Omega,\end{aligned} \tag{5.5}$$

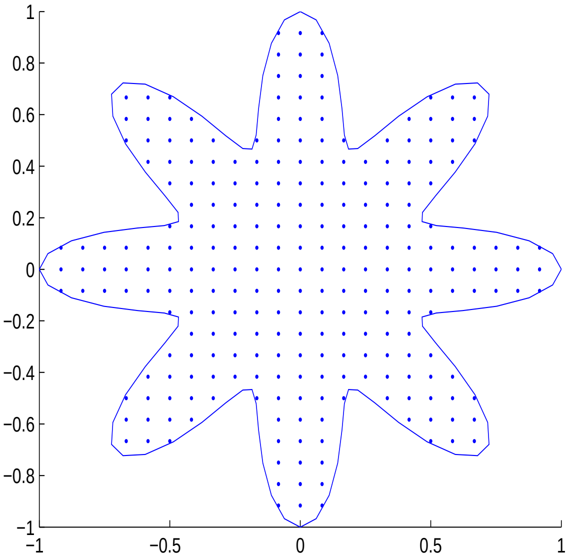

Figure 2: The distribution of the test points on a star-shaped domain.

where $\lambda=15$, the domain Ω is of star shape whose boundary $\partial\Omega$ is determined by the parametric equation,

$$x(t)=\frac{1}{2}q(1+\cos^2(4t))\cos t,\; y(t)=\frac{1}{2}q(1+\cos^2(4t))\sin t, \tag{5.6}$$

with $q=1$, the functions f and g are given such that the exact solution for Problem (5.5) is u^* in (5.4).

We approximate the source function f by its Chebyshev polynomial expansion $\widetilde{f}(x,y)$ in (2.1). We then find particular solutions $\widetilde{u}_{p1}$ and $\widetilde{u}_{p2}$ that satisfy the following equations

$$\begin{aligned}(\Delta-\lambda^2)\,\widetilde{u}_{p1}&=\widetilde{f},\\(\Delta-\lambda^2)\,\widetilde{u}_{p2}&=\widetilde{u}_{p1}.\end{aligned}$$

An approximate particular solution for Problem (5.5) is then given by $\widetilde{u}_{p2}$. We further use the MFS to obtain the approximate solution $\widetilde{u}$ to Problem (5.5) as $u_{p2}+\widetilde{u}_h$. We evenly (in terms of angle) distribute 120 collocation points on the star boundary $\partial\Omega$ and 120 source points on an enlarged star curve for which $q=25$ in (5.6). We use a total of 261 test points distributed inside the domain which is shown in Fig. 2. The computational results are given in Table 2. In this example, we choose $\varepsilon=10^{-9}$. The results are expected not to be as good as the previous example due to the repeated evaluation of the particular solution. Nevertheless, the results are still excellent. Using the efficient algorithm as mentioned in the last example, we obtained better accuracy at much less CPU time as can be seen in Tables 2 and 3.

Table 2: The absolute error of the computed solution and its first order derivatives using efficient algorithm by taking $\varepsilon = 10^{-9}$.

$m=n$	E	Ex	Ey	CPU Time (seconds)
12	1.115E-03	1.191E-02	1.012E-02	0.25
13	2.328E-04	2.085E-03	1.958E-03	0.32
14	1.821E-06	3.300E-05	1.663E-05	0.43
15	3.125E-06	2.845E-05	2.746E-05	0.55
16	6.709E-08	6.420E-07	6.964E-07	0.74
17	1.993E-08	1.837E-07	1.799E-07	0.94
18	6.708E-10	7.000E-09	7.265E-09	1.22
19	3.057E-10	2.970E-09	3.065E-09	1.55
20	1.675E-11	1.184E-10	1.286E-10	1.86

Table 3: The absolute error of the computed solution and its first order derivatives using regular Chebyshev approximation.

$m=n$	E	Ex	Ey	CPU Time (seconds)
12	1.115E-03	1.191E-02	1.012E-02	2.44
13	2.328E-04	2.085E-03	1.958E-03	3.69
14	1.821E-06	3.300E-05	1.663E-05	5.53
15	3.125E-06	2.845E-05	2.746E-05	7.93
16	6.789E-08	6.495E-07	7.032E-07	11.27
17	1.962E-08	1.808E-07	1.768E-07	16.33
18	6.240E-10	6.586E-09	7.036E-09	22.86
19	5.306E-10	4.976E-09	5.499E-09	29.83
20	7.155E-10	5.637E-09	7.245E-09	39.79

6 Conclusions

This paper is focused on the derivation of particular solutions on differential equations in two dimensions through the use of Chebyshev interpolation. Recursive formulas for evaluating the particular solutions for various differential operators have been derived. Due to the exponential convergence of Chebyshev approximations and the MFS, our numerical results are extremely accurate and efficient. Another advantage of the proposed approach is that there is no system of equations to be solved for the evaluation of a particular solution and, hence, there is no ill-conditioning problem. However, this approach is restricted to problems where the source function can be smoothly extended outside the domain [1]. Our approach is not suitable if the source function is given as scattered data. In that case, radial basis function approximations are more appropriate.

Instead of using Chebyshev polynomials as basis functions, other types of orthogonal polynomial basis functions such as Legendre polynomials could be used. The resulting algorithm would be very similar to the one proposed in this paper. No significant advantage would be expected by such a change.

The challenge of our proposed approach is to apply the current algorithm to nonlinear problems and time-dependent problems. These research topics are currently under investigation.

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