COMMUNICATIONS IN COMPUTATIONAL PHYSICS
Vol. **6**, No. 2, pp. 396-405

Commun. Comput. Phys.
August 2009

On Approximating Strongly Dispersion-Managed Solitons

Jinglai Li*

Department of Engineering Sciences and Applied Mathematics, Northwestern University, Evanston, IL 60208, USA.

Received 22 July 2008; Accepted (in revised version) 2 October 2008

Available online 15 December 2008

Abstract. We use a generalized scaling invariance of the dispersion-managed nonlinear Schrödinger equation to derive an approximate function for strongly dispersion-managed solitons. We then analyze the regime in which the approximation is valid. Finally, we present a method for extracting the underlying soliton part from a noisy pulse, using the resulting approximate formula.

PACS: 42.65.Tg, 42.65.Sf

Key words: Lasers, fiber optics, soliton, dispersion-management, scaling invariance, Gaussian ansatz.

1 Introduction

The technique of dispersion-management (DM), developed to improve the performance of the fiber-optic transmission lines in 1990s, has become an essential component of modern optical fiber communication systems [1, 2]. Technically speaking, DM is realized by concatenating fiber sections with different chromatic dispersion to build the transmission line. Mathematically, dispersion-managed systems are described by the nonlinear Schrödinger equation (NLS) equation with periodically varying dispersion (see, e.g., [3, 4]):

$$i\frac{\partial u}{\partial t}+\frac{1}{2}D(t/t_a)\frac{\partial^2 u}{\partial x^2}+|u|^2u=0, \tag{1.1}$$

where all quantities are expressed in dimensionless units, and where, t stands for the propagation distance and x stands for time. The function $D(t/t_a)$ represents the local value of fiber dispersion. The quantity t_a appearing in Eq. (1.1) is the characteristic (dimensionless) distance between amplifiers, which we assume to be small compared to the nonlinear distance and the dispersion distance; that is, $t_a \ll 1$. For example, with a typical

*Corresponding author. *Email address:* `jinglai-li@northwestern.edu` (J. Li)

 ©2009 Global-Science Press

amplifier spacing of about 50 km and typical nonlinear distance of about $400\sim1000$ km, it is $t_a=0.05\sim0.125$. Eq. (1.1) contains both large and rapidly varying terms and thus is not useful for studying the long term behavior of solutions. By employing appropriate multiple-scale expansions on Eq. (1.1), one obtains an integro-differential equation, called dispersion-managed nonlinear Schrödinger equation (DMNLS) equation, governing the long-term dynamics of such systems [4,5]:

$$i\frac{\partial u}{\partial t}+\tfrac{1}{2}\bar{d}\frac{\partial^2 u}{\partial x^2}+\int\!\!\int u_{(x+x')}u_{(x+x'')}u^*_{(x+x'+x'')}R_{(x',x'')}\,dx'dx''=0, \tag{1.2}$$

where $\bar{d}$ is the average dispersion, and where the integral kernels $R(x',x'')$ and $r(y)$ are respectively,

$$R(x',x'')=\mathrm{ci}(x'x''/s)/(2\pi|s|),\quad r(y)=\sin(sy)/(4\pi^2 sy).$$

Here the parameter s, called the reduced mapstrength, is defined by

$$s=\frac{1}{4}\int_0^1|\Delta D(\zeta)|d\zeta,$$

where $\Delta D(\cdot)$ is the zero-mean variation [4] in $d(\cdot)$:

$$d(t/t_a)=\bar{d}+\Delta D(t/t_a).$$

By performing an appropriate nondimensionalization, the average dispersion can be normalized to be $\bar{d}=1$ in the abnormal regime [14]. In what follows, we will assume this has been done. Moreover, Eq. (1.2) reduces to the standard NLS equation when $s=0$. The DMNLS equation has been extensively studied in the literature of fiber optics [6–14]. More interestingly, certain types of mode-locked lasers are also dispersion-managed [15], where Eq. (1.1) is again the appropriate model for the pulse dynamics [16, 17]. And it has been suggested recently that Eq. (1.2) describes the asymptotic behavior of the pulses in these mode-locked laser systems as well. Because of the limitation of space, a lot of physical details are left out here for both the fiber-optic communication systems and the lasers, and interested readers are encouraged to consult the cited works and the references therein. In many applications, the lasers are required to produce ultrashort pulses (e.g., fetosecond) [18], which are stable and soliton-like, and hence are usually referred to as the DM solitons (DMS). From a mathematical point of view, the DMS can be associated to either a solution of Eq. (1.1), which is localized in x and periodically varying in t, or a traveling-wave solution of the DMNLS equation (1.2), which preserves its shape during propagation [10].

Considerable efforts have been dedicated to approximations of the periodic solution of Eq. (1.1) [19, 21, 22]. On the other hand, it should also be beneficial to have an approximate function of the DMS as traveling-wave solutions of the DMNLS equation†.

†To avoid confusion, we only refer to the traveling-wave solutions of Eq. (1.2) as the DMS hereafter.

In [20], Ablowitz *et.al.* gave a Gaussian approximation of the DMS, where the amplitude and pulse width parameters have to be determined by matching the pulse with the exact DMS. An explicit approximate function of the DMS is, to the best of our knowledge, yet to be available. The main scope of this work is to derive such a formula. We do so by taking advantage of a generalized scaling invariance of the DMNLS equation. The rest of this paper is organized as follows. In Section 2, we introduce a generalized scaling invariance of the DMNLS equation and apply it to approximating the DMS. In Section 3, we derive an approximate function for the DMS based upon the Gaussian ansatz, and analyze the region where the resulting approximate function is valid. Section 4 provides a numerical example: using the approximate formula to extract the "clean" soliton from a noisy pulse. Finally, Section 5 concludes the work.

2 A generalized scaling invariance

It is known that a specific soliton solution of Eq. (1.2) takes the form of [4,10,14],

$$u(x,t;s)=f(x;s)e^{-\frac{i}{2}t}. \tag{2.1}$$

Note that for the usual NLS equation, $f(x)=\mathrm{sech}(x)$ is the conventional soliton, and by using the scaling invariance of the NLS equation, one obtains a pulse with any desired energy: $u(x,t)=\lambda f(\lambda x,\lambda^2 t)$. Though this invariance is destroyed in Eq. (1.2), a similar scaling property of the DMNLS equation is available: if $u(x,t)$ solves a DMNLS equation with mapstrength s, then $\lambda u(\lambda x,\lambda^2 t)$ solves another DMNLS equation with mapstrength s/a^2. This scaling property is not of direct use, as it changes the equation as well. In [14], however, we have found *a generalized scaling invariance* for the DMNLS equation (1.2), which also involves the transformation of parameter s: namely, if $u(x,t;s)$ solves Eq. (1.2), so does $au(a^2x,at,a^2s)$. Therefore, provided that the solution (2.1) is known, using this generalized scaling invariance, we can generate a family of solutions:

$$u(x,t;s)=\lambda e^{-i\lambda^2 t/2}f(\lambda x;\lambda^2 s), \tag{2.2}$$

where parameter λ represents (or more precisely, is proportional to) the pulse energy. Remarkably, even the exact DMS is not known explicitly, the generalized scaling invariance can also be of help in approximating it. Namely, suppose $f'(x;s)$ is a *good* approximation of $f(x;s)$, in the sense that the relative error is uniformly bounded by a small positive quantity ϵ for $s\in[a,b]$:

$$\frac{\|f(x;s)-f'(x;s)\|_2}{\|f(x;s)\|_2}<\epsilon, \tag{2.3}$$

where $\|\cdot\|_2$ is the L_2-norm, and it is easy to verify that, if λs remains in $[a,b]$,

$$\frac{\|\lambda f(\lambda x;\lambda^2 s)-\lambda f'(\lambda x;\lambda^2 s)\|_2}{\|\lambda f(\lambda x;\lambda^2 s)\|_2}<\epsilon, \tag{2.4}$$

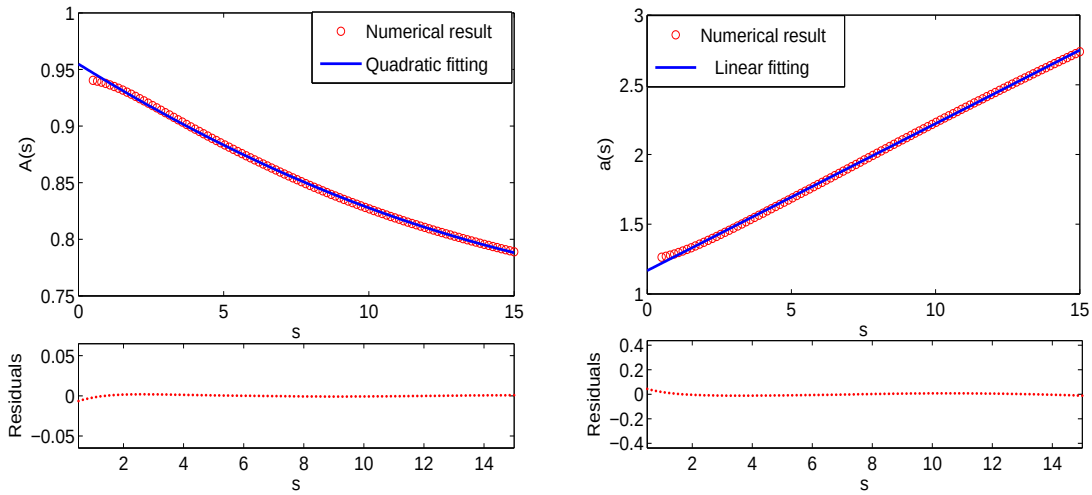

Figure 1: Top: A and a plotted as a function of s, and the polynomial fittings for them. Bottom: the difference between the numerical results and the polynomial fittings.

i.e., $\lambda f'(\lambda x;\lambda^2 s)$ well approximates $\lambda f(\lambda x;\lambda^2 s)$. Hence, to construct a generic approximate formula for the DMS, one only needs to do that for Eq. (2.1), and then scale the obtained approximation to any desired energy by choosing λ appropriately.

3 Gaussian ansatz

Numerical simulations suggest that the strongly dispersion-managed solitons are well approximated by Gaussian pulses [4,5] (while the weakly DMS are more close to hypersecant). Therefore, we here adopt the Gaussian ansatz for Eq. (2.1) (i.e., $\lambda=1$), letting

$$f_{\text{Gauss}}(x)=A(s)\exp\Big(-\frac{x^2}{2a(s)^2}\Big). \tag{3.1}$$

For current applications in fiber optics and mode-locked lasers, it is sufficient to only consider $s\in[0.5,15]$. We take a certain number (e.g., 100) of linearly spaced points s from 0.5 to 15, and numerically solve Eq. (1.2) to obtain the DMS $f(x)$ in Eq. (2.1) for every grid point. Then, for each s, we find the pair of a and A that minimizes $\|f(x)-f_{\text{Gauss}}(x)\|_2$. In Fig. 1, we plot a and A each as a function of s, and by performing a curve fitting, we find that the function $a(s)$ and $A(s)$ are well approximated by a linear and a quadratic functions respectively:

$$a(s)\approx 0.106s+1.16, \tag{3.2a}$$

$$A(s)\approx 0.00032s^2-0.0159s+0.955. \tag{3.2b}$$

Using the generalized scaling invariance, we obtain the approximate function of Eq. (2.2):

$$u_{\text{Gauss}}(x)=\lambda A(\lambda^2 s)\exp\Big(-\frac{(\lambda x)^2}{2a(\lambda^2 s)^2}\Big)\exp\Big(\frac{i}{2}\lambda^2 t\Big). \tag{3.3}$$

In Fig. 2, we demonstrate good agreement between the DMS and its Gaussian approximation (3.3) for two examples: one (on the left) is of energy $E=2.253$, corresponding to $\lambda=1$, and the other (on the right) is of energy $E=4.909$, corresponding to $\lambda=3$, both with $s=4$. The relative errors $\|f-f_{\text{Gauss}}\|_2/\|f\|_2$ are 0.012 and 0.021 respectively. It is natural to ask here: what is the range of s and λ where Eq. (3.3) remains a good approximation of Eq. (2.2)? We answer this question by plotting the relative approximation error against s and λ in Fig. 3. The figure indicates that the region where the approximation appears good (in the sense that the relative error is less than 0.1) covers the practical regime of s and λ largely.

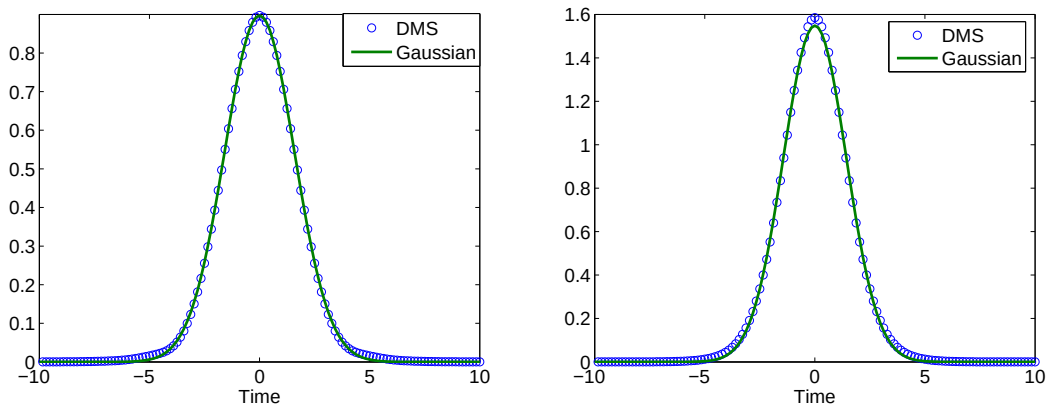

Figure 2: A comparison of the DMS and the Gaussian approximation. Left: $s=4$, $\lambda=1$; Right: $s=4$, $\lambda=2$.

Moreover, applying some invariances [14] to Eq. (3.1) yields the approximate function of the traveling-wave solution of the DMNLS equation:

$$u_{\text{Gauss}}(x,t;s)=\lambda A(\lambda^2 s)\exp\left[-\frac{\lambda^2(x-\omega t-x_o)^2}{2a^2(\lambda^2 s)}+i\Big(\omega x+\frac{(\lambda^2-\omega^2)t}{2}+\phi_o\Big)\right], \tag{3.4}$$

where ω, x_o and ϕ_o are arbitrary real parameters. As an example, in Fig. 4, we compare the actual soliton

$$u(x,t;4)=f(x;4)\exp\left[i\Big(2x-\frac{3}{2}t\Big)\right]$$

(that is, $\lambda=1$, $\omega=2$, $x_o=0$, $\phi_o=0$ and $s=4$), obtained by numerically solving the DMNLS equation, with the approximation obtained from Eq. (3.4), at three locations: $t=0$, $t=10$ and $t=20$. The approximation (3.4) will be used in extracting "clean" solitons from noise in Section 4.

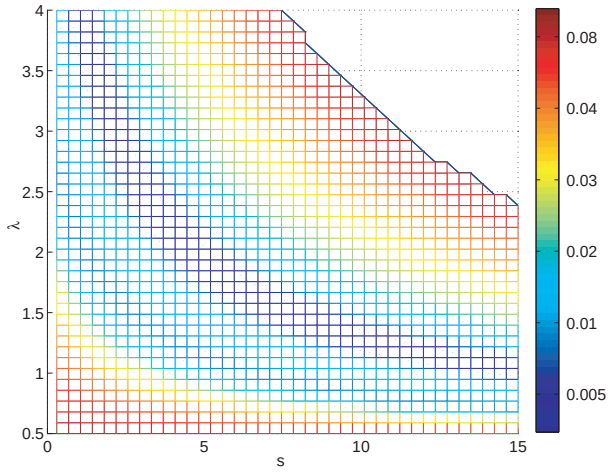

Figure 3: (color online) A pseudo-color plot of the relative error of the approximate function (3.3) against s and λ. The region where the error is larger than 0.1 is left blank. A color bar is also shown at the right.

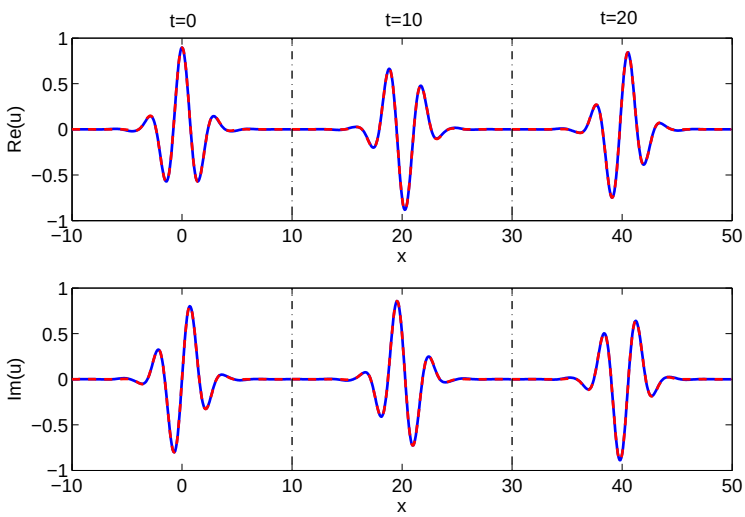

Figure 4: (color online) Solid line (blue) is the traveling wave obtained by solving the DMNLS equation, and dashed line (red) is the Gaussian approximation.

The approximation accuracy of Eq. (2.1) is inherently limited by the Gaussian ansatz which is completely characterized by two parameters, A and a. Thus, if higher accuracy is desired, one must extend the ansatz to allow more "degrees of freedom", i.e., to use an ansatz characterized by more parameters. In principle, increasing the number of parameters allows one to get a better approximation, but on the other hand, doing so complicates the minimization problem and the curve-fitting problem as well. Nevertheless, the generalized scaling invariance can be used along with any ansatz to obtain approximations of the DMS.

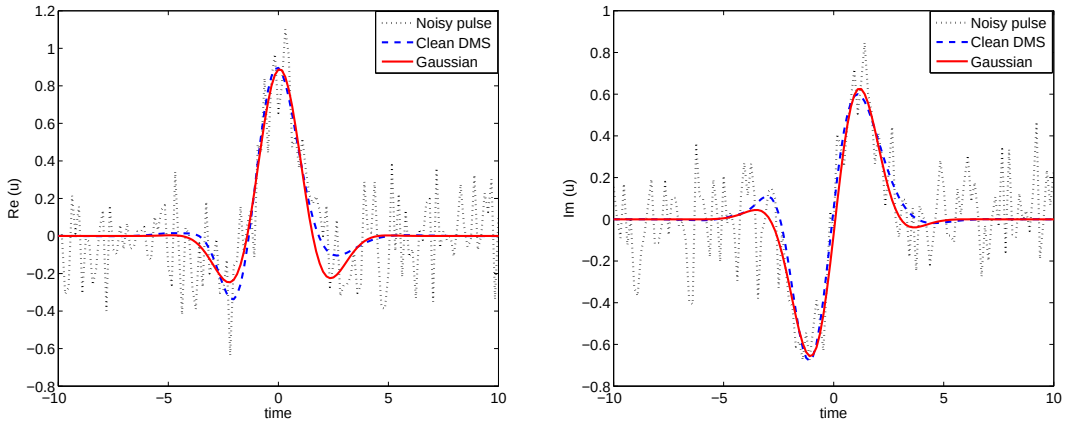

Figure 5: (color online) The noisy pulse (dotted, black), the underlying soliton part (dashed, blue) of it obtained by solving Eq. (4.1), and the Gaussian approximation (solid,red) obtained by the method provided in Section 4.

4 Extracting DMS from noise

The performance of lightwave systems suffers from random noise (e.g, the ASE noise induced by amplifiers). When investigating the impairment of the noise on the system performance, one often needs to extract the underlying soliton part from a noisy signal. For example, in the importance-sampled Monte-Carlo simulations performed in [14, 23, 24], the underlying soliton is needed for detecting the most likely error at each amplifier. In [14], we used a method to extract the underlying DMS from noise, which is consist of the following: first filter the noisy signal and then use the filtered pulse as the initial data to solve the non-local equation,

$$\hat{f}(\omega)=\frac{2}{\lambda^2+\bar{d}\omega^2}\iint \hat{f}(\omega+\omega')\hat{f}(\omega+\omega'')\hat{f}^*(\omega+\omega'+\omega'')r(\omega'\omega'')d\omega' d\omega'', \tag{4.1}$$

where $\hat{f}(\omega)$ is the Fourier transform of $f(x)$. Details of the method can be found in Appendix 3 in [14]. This method, unless modified, can not give the phase and position parameters directly. Another drawback is that it is a bit expensive computationally, especially when the extraction has to be performed many times. For example, when doing a IS-MC simulation of 50,000 soliton trials for a system with 200 amplifiers, one has to extract the DMS, i.e., to solve Eq. (4.1), ten million times. Here we provide an alternative method, extracting the DMS by a combination of the approximate function (2.2) and the perturbation theory for DMS developed in [14]. The new method does not require to solve Eq. (4.1) numerically, and hence improves the efficiency considerably (the trade-off is that the resulting soliton is only an approximation of course). Moreover, the new method is also able to obtain all the four soliton parameters simultaneously. The basic idea of the new method is to separate the noisy signal $u(x,t)$ into a clean soliton

$$u_s(x,t)=u_o(x,t)e^{i\Theta},$$

with

$$u_o(x,t)=\lambda f(\lambda(x-X)),\quad \Theta=\Omega(x-X)+\Phi(t),$$

and a purely dispersive field $\Delta u(x,t)=u(x,t)-u_s(x,t)$, which do not interact with each other at the first order. Then using the soliton perturbation theory (SPT), we turn such a requirement into the following equations:

$$\Delta\lambda=\mathrm{Re}\int e^{-i\Theta}u_o\Delta u\,dx=0, \tag{4.2a}$$

$$\Delta X=\mathrm{Re}\int e^{-i\Theta}\Big(-\frac{1}{\lambda}(x-X)u_o\Big)\Delta u\,dx=0. \tag{4.2b}$$

It should be noted that, in order for Δu to be purely dispersive, we shall also require that $\Delta\Omega=0$ and $\Delta\Phi=0$. However, because the perturbation equations for Ω and Φ involve the derivatives of the soliton with respect to x, which are not well approximated by differentiating Eq. (2.2) with respect to x, it is not feasible to calculate Ω and Φ this way. The extraction proceeds as follows:

Algorithm 4.1:

---

1. Obtain a first approximation to the soliton frequency by computing the mean frequency of the noisy pulse
$$\Omega_o=\int\omega|\hat{u}|^2d\omega\Big/\int|\hat{u}|^2d\omega, \tag{4.3}$$
where as before $\hat{u}$ is the Fourier transform of u.
2. Use a low-pass Gaussian filer centered at Ω_o to filter the noisy pulse (see Appendix 3 in [14] for details) and denote the filtered pulse by u_f.
3. Use Eq. (4.3) with the filtered pulse u_f to compute the pulse frequency Ω.
4. Compute $E_o=\int|u|^2dx$ and $X_o=\int x|u|^2dx/E_o$. Then calculate the corresponding λ_o from
$$E_o=\int[\lambda A(\lambda_o^2 s)]^2e^{-\lambda_o^2x^2/a(\lambda_o^2)}dx.$$
5. Let $n=0$; choose an error tolerance T.
6. Let $u_o(x,t)=\lambda_n f_{\mathrm{Gauss}}(\lambda_n(x-X_n))$, and find Φ_n by minimizing $\|u-u_oe^{i(\Omega x+\Phi_n)}\|_2$.
7. Let $u_s(x,t)=u_o\exp[i(\Omega x+\Phi_n)]$. Compute $\Delta\lambda_n$ and ΔX_n from Eq. (4.2). Let
$$\lambda_{n+1}=\lambda_n+\Delta\lambda_n,\quad X_{n+1}=X_n+\Delta X_n.$$
8. Stop if $\|\Delta\lambda_n+\Delta X_n\|_2<T$; otherwise, let $n=n+1$ and goto step 6.

A few remarks: first, the frequency parameter Ω is calculated using the same way as [14]; second, steps 7 and 8 are actually the fixed-point iteration for solving Eqs. (4.2); third, the pulse phase is obtained by minimizing the L_2-norm of the dispersive field Δu. Finally, as is shown in Fig. 5, the approximation obtained by using the proposed method

is fairly close to the soliton obtained by using the method in [14], while the new method is significantly faster.

5 Conclusions

In conclusion, we have obtained an explicit approximate function for the DMS, based upon the Gaussian ansatz and the generalized scaling invariance of the DMNLS equation. It should be noted that, the approximation is invalid when the DM is weak ($s \ll 1$), because the weakly dispersion-managed solitons are not close to the shape of Gaussian. We emphasize that the approximate function is *explicit*, and the use of it does not require any information of the soliton, such as the pulse energy, width, and so forth. Finally we provide a method to approximately extract the DMS from noise by using the approximate function, which is more efficient than the method used in [14].

Acknowledgments

It is a pleasure to thank William L. Kath and Gino Biondini for many insightful discussions. The author is also grateful to two anonymous referees for their constructive criticism and suggestions and to Xiaoying Han for reading the revised manuscript. The author is, however, solely responsible for all the statements made and opinions expressed in this work.

References

[1] G. P. Agrawal, Nonlinear Fiber Optics, 3rd ed., Academic Press, San Diego, 2003.
[2] L. F. Mollenauer and J. P. Gordon, Solitons in Optical Fibers: Fundamentals and Applications, Academic Press, 2006.
[3] T. S. Yang and W. L. Kath, Analysis of enhanced-power solitons in dispersion-managed optical fibers, Opt. Lett., 22 (1997), 985-987.
[4] M. J. Ablowitz and G. Biondini, Multiple scale dynamics in communication systems with strong dispersion management, Opt. Lett., 23 (1998), 1668-1670.
[5] I. R. Gabitov and S. K. Turitsyn, Averaged pulse dynamics in a cascaded transmission system with passive dispersion compensation, Opt. Lett., 21 (1996), 327-329.
[6] M. J. Ablowitz, G. Biondini and T. Hirooka, On quasi-linear dispersion-managed transmission systems, Opt. Lett., 26 (2001), 459-461.
[7] M. J. Ablowitz, G. Biondini and E. S. Olson, Incomplete collisions of wavelength-division multiplexed dispersion-managed solitons, J. Opt. Soc. Am. B, 18 (2001), 577-583.
[8] G. Biondini and S. Chakravarty, Nonlinear chirp of dispersion-managed return-to-zero pulses, Opt. Lett., 26 (2001), 1761-1763.
[9] P. M. Lushnikov, Dispersion-managed soliton in a strong dispersion map limit, Opt. Lett., 26 (2001), 1535-1537.
[10] D. E. Pelinovsky and V. Zharnitsky, Averaging of dispersion-managed pulses: existence and stability, SIAM J. Appl. Math., 63 (2003), 745-776.

[11] T. I. Lakoba and D. E. Pelinovsky, Persistent oscillations of scalar and vector dispersion-managed solitons, Chaos, 10 (2000), 539-550.
[12] G. Biondini and S. Chakravarty, Nonlinear chirp of dispersion-managed return-to-zero pulses, Opt. Lett., 26 (2001), 1761-1763.
[13] A. Tonello, A. D. Capobianco, G. Nalesso, F. Gringoli and C. De Angelis, Perturbations, internal modes and noise in dispersion-managed soliton transmission, Opt. Commun., 246 (2005), 393-403.
[14] J. Li, E. T. Spiller and G. Biondini, Noise-induced perturbations of dispersion-managed solitons, Phys. Rev. A, 75 (2007), 053818:1-13.
[15] J. Ye and S. T. Cundiff (Eds.), Femtosecond Optical Frequency Comb: Principle, Operation and Applications, Springer, New York, 2004.
[16] Q. Quraishi, S Cundiff, B. Ilan and M. J. Ablowitz, Dynamics of nonlinear and dispersion managed solitons, Phys. Rev. Lett., 94 (2005), 243904.
[17] M. J. Ablowitz, T. P. Horikis and B. Ilan, Solitons in dispersion-managed mode-locked lasers, Phys. Rev. A, 77 (2008), 033814.
[18] J. N. Kutz, Mode-locked soliton lasers, SIAM Rev., 48 (2006), 629-678.
[19] T. I. Lakoba and D. J. Kaup, Hermite-Gaussian expansion for pulse propagation in strongly dispersion managed fibers, Phys. Rev. E, 58 (1998), 6728-6741.
[20] M. J. Ablowtiz, G. Biondini and E. S. Olson, On the evolution and interaction of dispersion-managed solitons, in: A. Hasegawa (Ed.), Massive WDM and TDM soliton transmission systems, 2000, pp. 362-367.
[21] R. K. Jackson, C. K. R. T. Jones and V. Zharnitsky, Dispersion-managed solitons via an averaged variational principle, Phys. D, 190 (2004), 63-77.
[22] P. T. Dinda, K. Nakkeeran and A. B. Moubissi, Optimized Hermite-gaussian ansatz functions for dispersion-managed solitons, Opt. Commun., 187 (2001), 427-433.
[23] R. O. Moore, G. Biondini and W. L. Kath, A method to compute statistics of large, noise-induced perturbations of nonlinear Schrödinger solitons, SIAM J. Appl. Math., 67 (2007), 1418-1439; SIAM Rev., 50 (2008), 523-549.
[24] E. T. Spiller and W. L. Kath, A method for determining most probable errors in nonlinear lightwave systems, SIAM J. Appl. Dyn. Sys., 7 (2008), 868-894.