Analytical Solution for Waves Propagation in Heterogeneous Acoustic/Porous Media. Part II: The 3D Case

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Abstract. We are interested in the modeling of wave propagation in an infinite bilayered acoustic/poroelastic media. We consider the biphasic Biot's model in the poroelastic layer. The first part was devoted to the calculation of analytical solution in two dimensions, thanks to Cagniard de Hoop method. In the first part (Diaz and Ezziani, Commun. Comput. Phys., Vol. 7, pp. 171-194) solution to the two-dimensional problem is considered. In this second part we consider the 3D case.

1 Introduction

The computation of analytical solutions for wave propagation problems is of high importance for the validation of numerical computational codes or for a better understanding of the reflexion/transmission properties of the media. Cagniard-de Hoop method [1, 2] is a useful tool to obtain such solutions and permits to compute each type of waves (P wave, S wave, head wave···) independently. Although it was originally dedicated to the solution of elastodynamic wave propagation, it can be applied to any transient wave propagation problem in stratified media. However, as far as we know, few works have

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been dedicated to the application of this method to poroelastic medium. In [3] the analytical solution of poroelastic wave propagation in an homogeneous 2D medium is provided and in [4] the authors compute the analytical expression of the reflected wave at the interface between an acoustic and a poroelastic layer in two dimension but they do not explicit the expression of the transmitted waves. The coupling between acoustic and poroelastic media is of high interest for the simulation of wave propagation for seismics problem in sea bottom or for ultrasound wave propagation in biological tissues, when the human skin can regarded as a fluid and the bones as a porous medium.

In order to validate computational codes of wave propagation in poroelastic media, we have implemented the codes Gar6more 2D [5] and Gar6more 3D [6] which provide the complete solution (reflected and transmitted waves) of the propagation of wave in stratified 2D or 3D media composed of acoustic/acoustic, acoustic/elastic, acoustic/poroelastic or poroelastic/poroelastic. The 2D code and the 3D code are freely downloadable at

                    http://www.spice-rtn.org/library/software/Gar6more2D
and
                    http://www.spice-rtn.org/library/software/Gar6more3D.
In previous studies [7–9] we have presented the 2D acoustic/poroelastic and poroelastic/poroelastic cases and we focus here on the 3D acoustic/poroelastic case, the 3D poroelastic/poroelastic case will be the object of forthcoming papers.

The paper is organized as follows. We first present the model problem we want to solve and derive the Green problem from it (Section 1). Then we present the analytical solution of wave propagation in a stratified 3D medium composed of an acoustic and a poroelastic layer (Section 2) and we detail the computation of the solution (Section 3). Finally we illustrate our results through numerical applications (Section 4).

2   The model problem

We consider an infinite three-dimensional medium ($\Omega = \boldsymbol{R}^3$) composed of an homogeneous acoustic layer $\Omega^+ = \boldsymbol{R}^2 \times]0, +\infty[$ and an homogeneous poroelastic layer $\Omega^- = \boldsymbol{R}^2 \times]-\infty, 0[$ separated by an horizontal interface Γ (see Fig. 1). We first describe the equa-

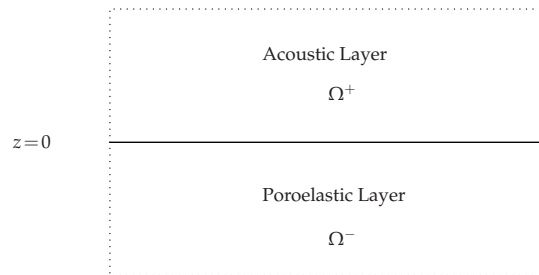Figure 1: Configuration of the study.

tions in the two layers (Section 2.1 and Section 2.2) and the transmission conditions on the interface Γ (Section 2.3). Then we present the Green problem from which we compute the analytical solution (Section 2.4).

The main parts of this section are similar to those of [9]. However, for completeness we will provide a detail description of formulas and notations.

2.1 The equation of acoustics

In the acoustic layer we consider the second order formulation of the wave equation with a point source in space, a regular source function f in time and zero initial conditions:

$$\begin{cases} \ddot{P}^+ - V^{+2}\Delta P^+ = \delta_x \delta_y \delta_{z-h} f(t), & \text{in } \Omega^+ \times]0,T], \\ \ddot{\boldsymbol{U}}^+ = -\dfrac{1}{\rho^+}\nabla P^+, & \text{in } \Omega^+ \times]0,T], \\ P^+(x,y,0) = 0, \quad \dot{P}^+(x,y,0) = 0, & \text{in } \Omega^+, \\ \boldsymbol{U}^+(x,y,0) = 0, \quad \dot{\boldsymbol{U}}^+(x,y,0) = 0, & \text{in } \Omega^+, \end{cases} \tag{2.1}$$

where P^+ is the pressure; $\boldsymbol{U}^+$ is the displacement field; V^+ is the celerity of the wave; and ρ^+ is the density of the fluid.

2.2 Biot's model

In the second layer we consider the second order formulation of the poroelastic equations [10–12]

$$\begin{cases} \rho^- \ddot{\boldsymbol{U}}_s^- + \rho_f^- \ddot{\boldsymbol{W}}^- - \boldsymbol{\nabla} \cdot \Sigma^- = 0, & \text{in } \Omega^- \times]0,T], \\ \rho_f^- \ddot{\boldsymbol{U}}_s^- + \rho_w^- \ddot{\boldsymbol{W}}^- + \dfrac{1}{\mathcal{K}^-}\dot{\boldsymbol{W}}^- + \nabla P^- = 0, & \text{in } \Omega^- \times]0,T], \\ \Sigma^- = \lambda^- \boldsymbol{\nabla} \cdot \boldsymbol{U}_s^- \boldsymbol{I}_3 + 2\mu^- \varepsilon(\boldsymbol{U}_s^-) - \beta^- P^- \boldsymbol{I}_3, & \text{in } \Omega^- \times]0,T], \\ \dfrac{1}{m^-}P^- + \beta^- \boldsymbol{\nabla} \cdot \boldsymbol{U}_s^- + \nabla \cdot \boldsymbol{W}^- = 0, & \text{in } \Omega^- \times]0,T], \\ \boldsymbol{U}_s^-(x,0) = 0, \quad \boldsymbol{W}^-(x,0) = 0, & \text{in } \Omega^-, \\ \dot{\boldsymbol{U}}_s^-(x,0) = 0, \quad \dot{\boldsymbol{W}}^-(x,0) = 0, & \text{in } \Omega^-, \end{cases} \tag{2.2}$$

where $\boldsymbol{I}_3$ is the usual identity matrix of $\mathcal{M}_3(\text{IR})$,

$$(\boldsymbol{\nabla} \cdot \Sigma^-)_i = \sum_{j=1}^{3} \frac{\partial \Sigma_{ij}^-}{\partial x_j} \quad \forall i = 1,3,$$

and $\varepsilon(\boldsymbol{U}_s^-)$ is the solid strain tensor defined by:

$$\varepsilon_{ij}(\boldsymbol{U}) = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right).$$

In (2.2), the unknowns are: U_s^- the displacement field of solid particles; $W^- = \phi^-(U_f^- - U_s^-)$, the relative displacement, U_f^- being the displacement field of fluid particle and ϕ^- the porosity; P^-, the fluid pressure; and Σ^-, the solid stress tensor. The parameters describing the physical properties of the medium are given by: $\rho^- = \phi^- \rho_f^- + (1-\phi^-)\rho_s^-$ is the overall density of the saturated medium, with ρ_s^- the density of the solid and ρ_f^- the density of the fluid; $\rho_w^- = a^- \rho_f^- / \phi^-$, where a^- the tortuosity of the solid matrix; $\mathcal{K}^- = \kappa^-/\eta^-$, κ^- is the permeability of the solid matrix and η is the viscosity of the fluid; m^- and β^- are positive physical coefficients:

$$\beta^- = 1 - K_b^- / K_s^-, \quad m^- = \left[\phi^- / K_f^- + (\beta^- - \phi^-)/K_s^-\right]^{-1},$$

where K_s^- is the bulk modulus of the solid, K_f^- is the bulk modulus of the fluid and K_b^- is the frame bulk modulus; and μ^- is the frame shear modulus, and $\lambda^- = K_b^- - 2\mu^-/3$ is the Lamé constant.

2.3 Transmission conditions

Let n be the unitary normal vector of Γ outwardly directed to. The transmission conditions on the interface between the acoustic and porous medium are [13] :

$$\begin{cases} W^- \cdot n = (U^+ - U_s^-) \cdot n, \\ P^- = P^+, \\ \Sigma^- n = -P^+ n. \end{cases} \tag{2.3}$$

2.4 The Green problem

We won't compute directly the solution to (2.1)-(2.3) but the solution to the following Green problem:

$$\ddot{p}^+ - V^{+2} \Delta p^+ = \delta_x \delta_y \delta_{z-h} \delta_t, \qquad \text{in } \Omega^+ \times]0,T], \tag{2.4a}$$

$$\ddot{u}^+ = -\frac{1}{\rho^+} \nabla p^+, \qquad \text{in } \Omega^+ \times]0,T], \tag{2.4b}$$

$$\rho^- \ddot{u}_s^- + \rho_f^- \ddot{w}^- - \boldsymbol{\nabla} \cdot \sigma^- = 0, \qquad \text{in } \Omega^- \times]0,T], \tag{2.5a}$$

$$\rho_f^- \ddot{u}_s^- + \rho_w^- \ddot{w}^- + \frac{1}{\mathcal{K}^-} \dot{w}^- + \nabla p^- = 0, \qquad \text{in } \Omega^- \times]0,T], \tag{2.5b}$$

$$\sigma^- = \lambda^- \nabla \cdot u_s^- I_3 + 2\mu^- \varepsilon(u_s^-) - \beta^- p^- I_3, \quad \text{in } \Omega^- \times]0,T], \tag{2.5c}$$

$$\frac{1}{m^-} p^- + \beta^- \nabla \cdot u_s^- + \nabla \cdot w^- = 0, \qquad \text{in } \Omega^- \times]0,T], \tag{2.5d}$$

and

$$w^- \cdot n = (u^+ - u_s^-) \cdot n, \quad p^- = p^+, \quad \sigma^- n = -p^+ n, \quad \text{on } \Gamma. \tag{2.6}$$

The solution to (2.1)-(2.3) is then computed from the solution of the Green Problem thanks to a convolution by the source function. For instance we have:

$$P^+(x,y,t) = p^+(x,y,.) * f(.) = \int_0^t p^+(x,y,\tau)f(t-\tau)d\tau$$

(we have similar relations for the other unknowns). We also suppose that the poroelastic medium is non dissipative, i.e the viscosity $\eta^- = 0$. Using Eqs. (2.5c), (2.5d) we can eliminate σ^- and p^- in (2.5) and we obtain the equivalent system:

$$\begin{cases} \rho^- \ddot{u}_s^- + \rho_f^- \ddot{w}^- - \alpha^- \nabla(\nabla \cdot u_s^-) + \mu^- \nabla \times (\nabla \times u_s^-) - m^- \beta^- \nabla(\nabla \cdot w^-) = 0, & z < 0, \\ \rho_f^- \ddot{u}_s^- + \rho_w^- \ddot{w}^- - m^- \beta^- \nabla(\nabla \cdot u_s^-) - m^- \nabla(\nabla \cdot w^-) = 0, & z < 0, \end{cases} \quad (2.7)$$

with $\alpha^- = \lambda^- + 2\mu^- + m^- \beta^{-2}$.

Using Eq. (2.4) the transmission conditions (2.6) on $z = 0$ are rewritten as:

$$\ddot{u}_{sz}^- + \ddot{w}_z^- = -\frac{1}{\rho^+}\partial_z p^+, \quad (2.8a)$$

$$-m^- \beta^- \nabla \cdot u_s^- - m^- \nabla \cdot w^- = p^+, \quad (2.8b)$$

$$\partial_z u_{sx}^- + \partial_x u_{sz}^- = 0, \quad (2.8c)$$

$$\partial_z u_{sy}^- + \partial_y u_{sz}^- = 0, \quad (2.8d)$$

$$(\lambda^- + m^- \beta^{-2})\nabla \cdot u_s^- + 2\mu^- \partial_z u_{sz}^- + m^- \beta^- \nabla \cdot w = -p^+. \quad (2.8e)$$

We split the displacement fields u_s^- and w^- into irrotationnal and isovolumic fields (P-wave and S-wave):

$$u_s^- = \nabla \Theta_u^- + \nabla \times \mathbf{\Psi}_u^-; \quad w^- = \nabla \Theta_w^- + \nabla \times \mathbf{\Psi}_w^-. \quad (2.9)$$

The vectors $\mathbf{\Psi}_u^-$ and $\mathbf{\Psi}_w^-$ are not uniquely defined since:

$$\nabla \times (\mathbf{\Psi}_\ell^- + \nabla C) = \nabla \times \mathbf{\Psi}_\ell^-, \quad \forall \ell \in \{u,w\},$$

for all scalar field C. To define a unique $\mathbf{\Psi}_\ell^-$ we impose the gauge condition:

$$\nabla \cdot \mathbf{\Psi}_\ell^- = 0.$$

The vectorial space of $\mathbf{\Psi}_\ell^-$ verifying this last condition is written as:

$$\mathbf{\Psi}_\ell^- = \begin{bmatrix} \partial_y \\ -\partial_x \\ 0 \end{bmatrix} \Psi_{\ell,1}^- + \begin{bmatrix} \partial_{xz}^2 \\ \partial_{yz}^2 \\ -\partial_{xx}^2 - \partial_{yy}^2 \end{bmatrix} \Psi_{\ell,2}^-,$$

where $\Psi_{\ell,1}^-$ and $\Psi_{\ell,2}^-$ are two scalar fields. The displacement fields $\boldsymbol{u}_s^-$ and $\boldsymbol{w}^-$ are written in the form:

$$
\boldsymbol{u}_s^- = \nabla\Theta_u^- + \begin{bmatrix} \partial_{xz}^2 \\ \partial_{yz}^2 \\ -\partial_{xx}^2-\partial_{yy}^2 \end{bmatrix} \Psi_{u,1}^- - \begin{bmatrix} \partial_y \\ -\partial_x \\ 0 \end{bmatrix} \Delta\Psi_{u,2}^-,
$$

$$
\boldsymbol{w}^- = \nabla\Theta_w^- + \begin{bmatrix} \partial_{xz}^2 \\ \partial_{yz}^2 \\ -\partial_{xx}^2-\partial_{yy}^2 \end{bmatrix} \Psi_{w,1}^- - \begin{bmatrix} \partial_y \\ -\partial_x \\ 0 \end{bmatrix} \Delta\Psi_{w,2}^-.
$$

(2.10)

Using this last change of variables, we can then rewrite the system (2.7) in the following form:

$$
\begin{cases}
A\ddot{\Theta}^- - B\Delta\Theta^- = 0, & z<0, \\
\ddot{\Psi}_{u,1}^- - V_S^{-2}\Delta\Psi_{u,1}^- = 0, & z<0, \\
\ddot{\Psi}_{u,2}^- - V_S^{-2}\Delta\Psi_{u,2}^- = 0, & z<0, \\
\ddot{\Psi}_w^- = -\dfrac{\rho_f^-}{\rho_w^-}\ddot{\Psi}_u^-, & z<0,
\end{cases}
$$

(2.11)

where $\Theta^- = (\Theta_u^-,\Theta_w^-)^t$, A and B are 2×2 symmetric matrices:

$$
A = \begin{pmatrix} \rho^- & \rho_f^- \\ \rho_f^- & \rho_w^- \end{pmatrix}; \quad B = \begin{pmatrix} \lambda^-+2\mu^-+m^-(\beta^-)^2 & m^-\beta^- \\ m^-\beta^- & m^- \end{pmatrix}
$$

and

$$
V_S^- = \sqrt{\frac{\mu\rho_w^-}{\rho^-\rho_w^- - \rho_f^{-2}}}
$$

is the S-wave velocity.

We multiply the first equation of the system (2.11) by the inverse of A. The matrix $A^{-1}B$ is diagonalizable: $A^{-1}B = \mathcal{P}D\mathcal{P}^{-1}$, where $\mathcal{P}$ is the change-of-coordinate matrix, $D = diag(V_{Pf}^{-2},V_{Ps}^{-2})$ is the diagonal matrix similar to $A^{-1}B$, V_{Pf}^- and V_{Ps}^- are respectively the fast P-wave velocity and the slow P-wave velocity ($V_{Ps} < V_{Pf}$).

Using the change of variables

$$
\Phi^- = (\Phi_{Pf}^-,\Phi_{Ps}^-)^t = \mathcal{P}^{-1}\Theta^-,
$$

(2.12)

we obtain the uncoupled system on fast P-waves, slow P-waves and S-waves:

$$
\begin{cases}
\ddot{\Phi}^- - D\Delta\Phi^- = 0, & z<0, \\
\ddot{\Psi}_{u,i}^- - V_S^{-2}\Delta\Psi_{u,i}^- = 0, \ i=1,2, & z<0, \\
\Psi_w^- = -\dfrac{\rho_f^-}{\rho_w^-}\Psi_u^-. & z<0.
\end{cases}
$$

(2.13)

Using the transmission conditions (2.8c)-(2.8d), we obtain:

$$2\partial^2_{xz}\Theta^-_u+\partial_x(\partial^2_{zz}-\Delta_\perp)\Psi^-_{u,1}-\partial^2_{yz}\Delta\Psi^-_{u,2}=0, \qquad \text{on } \Gamma, \qquad (2.14a)$$

$$2\partial^2_{yz}\Theta^-_u+\partial_y(\partial^2_{zz}-\Delta_\perp)\Psi^-_{u,1}+\partial^2_{xz}\Delta\Psi^-_{u,2}=0, \qquad \text{on } \Gamma, \qquad (2.14b)$$

with $\Delta_\perp=\partial^2_{xx}+\partial^2_{yy}$. Applying the derivative ∂_y to Eq. (2.14a), ∂_x to Eq. (2.14b) and subtracting the first obtained equation from the second one, we get:

$$(\partial_z\Delta_\perp)\Delta\Psi^-_{u,2}=0, \quad \text{on } \Gamma. \qquad (2.15)$$

Moreover, using that $\Psi^-_{u,2}$ satisfies the wave equation:

$$\ddot{\Psi}^-_{u,2}-V_S^{-2}\Delta\Psi^-_{u,2}=0, \quad z<0,$$

and that $\boldsymbol{u}^-_s$ and $\boldsymbol{w}^-$ satisfy, at $t=0$, $\boldsymbol{u}^-_s=\dot{\boldsymbol{u}}^-_s=\boldsymbol{w}^-=\dot{\boldsymbol{w}}^-=0$, we obtain:

$$\Psi^-_{u,2}=0, z\le 0,$$

and from (2.14) we deduce the transmission condition equivalent to (2.8c) and (2.8d):

$$2\partial^2_z\Theta^-_u+(\partial^2_{zz}-\Delta_\perp)\Psi^-_{u,1}=0, \quad \text{on } \Gamma. \qquad (2.16)$$

Finally, we obtain the Green problem equivalent to (2.4), (2.5), (2.6):

$$\begin{cases} \ddot{p}^+-V^{+2}\Delta p^+=\delta_x\delta_y\delta_{z-h}\delta_t, & z>0, \\ \ddot{\Phi}^-_i-V_i^{-2}\Delta\Phi^-_i=0, & i\in\{Pf,Ps,S\}, \quad z<0, \\ \mathcal{B}(p^+,\Phi^-_{Pf},\Phi^-_{Ps},\Phi^-_S)=0, & z=0, \end{cases} \qquad (2.17)$$

where we have set $\Phi^-_S=\Psi^-_{u,1}$ in order to have similar notations for the Pf, Ps and S waves. The operator $\mathcal{B}$ represents the transmission conditions on Γ:

$$\mathcal{B}\begin{pmatrix} p^+ \\ \Phi^-_{Pf} \\ \Phi^-_{Ps} \\ \Phi^-_S \end{pmatrix} = \begin{bmatrix} \dfrac{1}{\rho^+}\partial_z & (\mathcal{P}_{11}+\mathcal{P}_{21})\partial^3_{ztt} & (\mathcal{P}_{12}+\mathcal{P}_{22})\partial^3_{ztt} & \left(\dfrac{\rho^-_f}{\rho^-_w}-1\right)\partial^2_{tt}\Delta_\perp \\ 1 & \dfrac{m^-(\beta^-\mathcal{P}_{11}+\mathcal{P}_{21})}{V_{Pf}^{-2}}\partial^2_{tt} & \dfrac{m^-(\beta^-\mathcal{P}_{12}+\mathcal{P}_{22})}{V_{Ps}^{-2}}\partial^2_{tt} & 0 \\ 0 & 2\mathcal{P}_{11}\partial_z & 2\mathcal{P}_{12}\partial_z & \partial^2_{zz}-\Delta_\perp \\ 1 & \mathcal{B}_{42} & \mathcal{B}_{43} & -2\mu^-\partial_z\Delta_\perp \end{bmatrix}\begin{bmatrix} p^+ \\ \Phi^-_{Pf} \\ \Phi^-_{Ps} \\ \Phi^-_S \end{bmatrix},$$

where $\mathcal{P}_{ij}$, $i,j=1,2$ are the components of the change-of-coordinates matrix $\mathcal{P}$, $\mathcal{B}_{42}$ and $\mathcal{B}_{43}$ are given by:

$$\mathcal{B}_{42}=\frac{(\lambda^-+m^-\beta^{-2})\mathcal{P}_{11}+m^-\beta^-\mathcal{P}_{21}}{V_{Pf}^{-2}}\partial^2_{tt}+2\mu^-\mathcal{P}_{11}\partial^2_{zz},$$

$$\mathcal{B}_{43}=\frac{(\lambda^-+m^-\beta^{-2})\mathcal{P}_{12}+m^-\beta^-\mathcal{P}_{22}}{V_{Ps}^{-2}}\partial^2_{tt}+2\mu^-\mathcal{P}_{12}\partial^2_{zz}.$$

To obtain this operator we have used the transmission conditions (2.8a), (2.8b), (2.16), (2.8e), the change of variables (2.9) and the uncoupled system (2.13).

Moreover, we can determine the solid displacement $\boldsymbol{u}_s^-$ by using the change of variables (2.9) and the fluid displacement $\boldsymbol{u}^+$ by using (2.4).

3   Expression of the analytical solution

Since the problem is invariant by a rotation around the z-axis, we will only consider the case $y = 0$ and $x > 0$, so that the y-component of all the displacements are zero. The solution for $y \neq 0$ or $x \leq 0$ is deduced from the solution for $y = 0$ by the relations

$$p(x,y,z,t) = p(\sqrt{x^2+y^2},0,z,t), \tag{3.1}$$

$$u_{sx}(x,y,z,t) = \frac{x}{\sqrt{x^2+y^2}} u_{sx}(\sqrt{x^2+y^2},0,z,t), \tag{3.2}$$

$$u_{sy}(x,y,z,t) = \frac{y}{\sqrt{x^2+y^2}} u_{sx}(\sqrt{x^2+y^2},0,z,t), \tag{3.3}$$

$$u_{sz}(x,y,z,t) = u_{sz}(\sqrt{x^2+y^2},0,z,t). \tag{3.4}$$

To state our results, we need the following notations and definitions:

1. **Definition of the complex square root**. For $q_x \in \mathbb{C} \backslash \mathrm{I\!R}^-$, we use the following definition of the square root $g(q_x) = q_x^{1/2}$:

$$g(q_x)^2 = q_x \quad \text{and} \quad \Re e[g(q_x)] > 0.$$

The branch cut of $g(q_x)$ in the complex plane will thus be the half-line defined by $\{q_x \in \mathrm{I\!R}^-\}$ (see Fig. 2). In the following, we'll use the abuse of notation $g(q_x) = i\sqrt{-q_x}$ for $q_x \in \mathrm{I\!R}^-$.
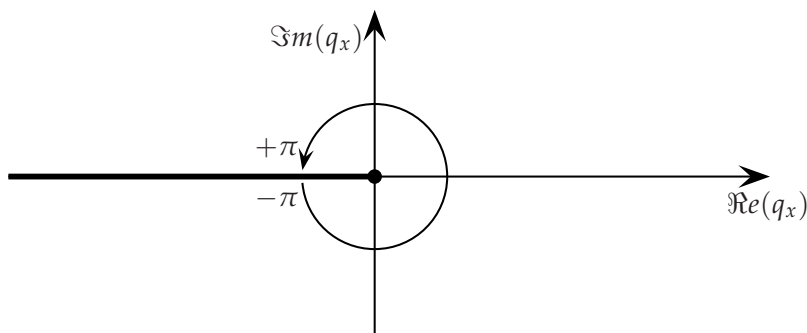

Figure 2: Definition of the function $x \mapsto (x)^{1/2}$.

2. **Definition of the fictitious velocities**. For a given $q \in \mathbb{R}$, we define the fictitious velocities $\mathcal{V}^+(q)$ and $\mathcal{V}_i^-(q)$ for $i \in \{Pf, Ps, S\}$ by

$$\mathcal{V}^+ := \mathcal{V}^+(q) = V^+ \sqrt{\frac{1}{1 + V^{+2} q^2}} \quad \text{and} \quad \mathcal{V}_i^- := \mathcal{V}_i^-(q) = V_i^- \sqrt{\frac{1}{1 + V_i^{-2} q^2}}.$$

These fictitious velocities will be helpful to turn the 3D-problem into the sum of 2D-problems indexed by the variable q. Note that $\mathcal{V}^+(0)$ and $\mathcal{V}_i^-(0)$ correspond to the real velocities V^+ and V_i^-.

3. **Definition of the functions κ^+ and κ_i^-.** For $i \in \{Pf, Ps, S\}$ and $(q_x, q_y) \in \mathbb{C} \times \mathbb{R}$, we define the functions

$$\kappa^+ := \kappa^+(q_x, q_y) = \left(\frac{1}{V^{+2}} + q_x^2 + q_y^2 \right)^{1/2} = \left(\frac{1}{\mathcal{V}^{+2}(q_y)} + q_x^2 \right)^{1/2},$$

$$\kappa_i^- := \kappa_i^-(q_x, q_y) = \left(\frac{1}{V_i^{-2}} + q_x^2 + q_y^2 \right)^{1/2} = \left(\frac{1}{\mathcal{V}_i^{-2}(q_y)} + q_x^2 \right)^{1/2}.$$

4. **Definition of the reflection and transmission coefficients.** For a given $(q_x, q_y) \in \mathbb{C} \times \mathbb{R}$, we denote by $\mathcal{R}(q_x, q_y)$, $\mathcal{T}_{Pf}(q_x, q_y)$, $\mathcal{T}_{Ps}(q_x, q_y)$ and $\mathcal{T}_S(q_x, q_y)$ the solution of the linear system

$$\mathcal{A}(q_x, q_y) \begin{bmatrix} \mathcal{R}(q_x, q_y) \\ \mathcal{T}_{Pf}(q_x, q_y) \\ \mathcal{T}_{Ps}(q_x, q_y) \\ \mathcal{T}_S(q_x, q_y) \end{bmatrix} = -\frac{1}{2\kappa^+(q_x, q_y) V^{+2}} \begin{bmatrix} \dfrac{\kappa^+(q_x, q_y)}{\rho^+} \\ 1 \\ 0 \\ 1 \end{bmatrix}, \tag{3.5}$$

where the matrix $\mathcal{A}(q_x, q_y)$ is defined by:

$$\mathcal{A}(q_x, q_y) =$$
$$\begin{bmatrix} -\dfrac{\kappa^+(q_x,q_y)}{\rho^+} & (\mathcal{P}_{11}+\mathcal{P}_{21})\kappa_{Pf}^-(q_x,q_y) & (\mathcal{P}_{12}+\mathcal{P}_{22})\kappa_{Ps}^-(q_x,q_y) & \left(1-\dfrac{\rho_f^-}{\rho_w^-}\right)(q_x^2+q_y^2) \\ 1 & \dfrac{m^-}{V_{Pf}^{-2}}(\beta^- \mathcal{P}_{11}+\mathcal{P}_{21}) & \dfrac{m^-}{V_{Ps}^{-2}}(\beta^- \mathcal{P}_{12}+\mathcal{P}_{22}) & 0 \\ 0 & 2\mathcal{P}_{11}\kappa_{Pf}^-(q_x,q_y) & 2\mathcal{P}_{12}\kappa_{Ps}^-(q_x,q_y) & \kappa_S^{-2}(q_x,q_y)+q_x^2+q_y^2 \\ 1 & \mathcal{A}_{4,2}(q_x,q_y) & \mathcal{A}_{4,3}(q_x,q_y) & 2\mu^-(q_x^2+q_y^2)\kappa_S^-(q_x,q_y) \end{bmatrix},$$

with

$$A_{4,2}(q_x,q_y) = \frac{\left(\lambda^- + m^- \beta^{-2}\right)\mathcal{P}_{11} + m^- \beta^- \mathcal{P}_{21}}{V_{Pf}^{-2}} + 2\mu^- \kappa_{Pf}^{-2}(q_x,q_y)\mathcal{P}_{11},$$

$$A_{4,3}(q_x,q_y) = \frac{\left(\lambda^- + m^- \beta^{-2}\right)\mathcal{P}_{12} + m^- \beta^- \mathcal{P}_{22}}{V_{Ps}^{-2}} + 2\mu^- \kappa_{Ps}^{-2}(q_x,q_y)\mathcal{P}_{12}.$$

We also denote by $V_{\max}$ the greatest velocity in the two media:

$$V_{\max} = \max(V^+, V_{Pf}^-, V_{Ps}^-, V_S^-).$$

We can now present the expression of the solution to the Green Problem:

Theorem 3.1. The pressure and the displacement in the top medium are given by

$$p^+(x,0,z,t) = p_{\text{inc}}^+(x,z,t) + \frac{d\zeta_{\text{ref}}^+}{dt}(x,z,t), \tag{3.6}$$

$$\boldsymbol{u}^+(x,0,z,t) = \boldsymbol{u}_{\text{inc}}^+(x,z,\tau)d\tau + \boldsymbol{u}_{\text{ref}}^+(x,y,\tau)d\tau, \tag{3.7}$$

and the displacement in the bottom medium is given by

$$\boldsymbol{u}_s^-(x,0,z,t) = \boldsymbol{u}_{Pf}^-(x,z,t) + \boldsymbol{u}_{Ps}^-(x,z,t) + \boldsymbol{u}_S^-(x,z,t),$$

where

- p_{inc}^+ and u_{inc}^+ are respectively the pressure and the displacement of the incident wave and satisfy:

$$\begin{cases} p_{\text{inc}}^+(x,z,t) := \dfrac{\delta(t-t_0)}{4\pi V^{+2} r}, \\[2mm] u_{\text{inc},x}(x,z,t) := \dfrac{xtH(t-t_0)}{4\pi V^{+2} r^3 \rho^+}, \\[2mm] u_{\text{inc},z}(x,z,t) := \dfrac{(z-h)tH(t-t_0)}{4\pi V^{+2} r^3 \rho^+}, \end{cases}$$

where δ and H respectively denote the usual Dirac and Heaviside distributions. Moreover we set $r = (x^2 + (z-h)^2)^{1/2}$ and $t_0 = r/V^+$ denotes the time arrival of the incident wave at point $(x,0,z)$.

- ζ_{ref}^+ and u_{ref}^+ are respectively the primitive of the pressure with respect to the time

and the displacement of the reflected wave and satisfy :

$$
\begin{cases}
\xi_{\text{ref}}^+(x,z,t) = -\int_0^{q_1(t)} \frac{\Im m\left[\kappa^+(v(t,q))\mathcal{R}(v(t,q))\right]}{\pi^2 r\sqrt{q^2+q_0^2(t)}}\,dq, \\[3mm]
u_{\text{ref},x}^+(x,z,t) = -\int_0^{q_1(t)} \frac{\Im m\left[iv(t,q)\kappa^+(v(t,q))\mathcal{R}(v(t,q))\right]}{\pi^2 r\rho^+\sqrt{q^2+q_0^2(t)}}\,dq, \\[3mm]
u_{\text{ref},z}^+(x,z,t) = -\int_0^{q_1(t)} \frac{\Im m\left[\kappa^{+2}(v(t,q))\mathcal{R}(v(t,q))\right]}{\pi^2 r\rho^+\sqrt{q^2+q_0^2(t)}}\,dq,
\end{cases}
$$

if $t_{h_1} < t \le t_0$ and $\frac{x}{r} > \frac{V^+}{V_{\max}}$,

$$
\begin{cases}
\xi_{\text{ref}}^+(x,z,t) = -\int_{q_0(t)}^{q_1(t)} \frac{\Im m\left[\kappa^+(v(t,q))\mathcal{R}(v(t,q))\right]}{\pi^2 r\sqrt{q^2-q_0^2(t)}}\,dq + \int_0^{q_0(t)} \frac{\Re e\left[\kappa^+(\gamma(t,q))\mathcal{R}(\gamma(t,q))\right]}{\pi^2 r\sqrt{q_0^2(t)-q^2}}\,dq, \\[3mm]
u_{\text{ref},x}^+(x,z,t) = -\int_{q_0(t)}^{q_1(t)} \frac{\Im m\left[(iv(t,q)\kappa^+(v(t,q))\mathcal{R}(v(t,q))\right]}{\pi^2 r\rho^+\sqrt{q^2-q_0^2(t)}}\,dq \\[3mm]
\qquad\qquad + \int_0^{q_0(t)} \frac{\Re e\left[i\gamma(t,q)\kappa^+(\gamma(t,q))\mathcal{R}(\gamma(t,q))\right]}{\pi^2 r\rho^+\sqrt{q_0^2(t)-q^2}}\,dq, \\[3mm]
u_{\text{ref},y}^+(x,z,t) = -\int_{q_0(t)}^{q_1(t)} \frac{\Im m\left[\kappa^{+2}(v(t,q))\mathcal{R}(v(t,q))\right]}{\pi^2 r\rho^+\sqrt{q^2-q_0^2(t)}}\,dq + \int_0^{q_0(t)} \frac{\Re e\left[\kappa^{+2}(\gamma(t))\mathcal{R}(\gamma(t))\right]}{\pi^2 r\rho^+\sqrt{q_0^2(t)-q^2}}\,dq,
\end{cases}
$$

if $t_0 < t \le t_{h_2}$ and $\frac{x}{r} > \frac{V^+}{V_{\max}}$,

$$
\begin{cases}
\xi_{\text{ref}}^+(x,z,t) = \int_0^{q_0(t)} \frac{\Re e\left[\kappa^+(\gamma(t,q))\mathcal{R}(\gamma(t,q))\right]}{\pi^2 r\sqrt{q_0^2(t)-q^2}}\,dq, \\[3mm]
u_{\text{ref},x}^+(x,z,t) = \int_0^{q_0(t)} \frac{\Re e\left[i\gamma(t,q)\kappa^+(\gamma(t,q))\mathcal{R}(\gamma(t,q))\right]}{2\pi^2 r\rho^+\sqrt{q_0^2(t)-q^2}}\,dq, \\[3mm]
u_{\text{ref},y}^+(x,z,t) = \int_0^{q_0(t)} \frac{\Re e\left[\kappa^{+2}(\gamma(t))\mathcal{R}(\gamma(t))\right]}{\pi^2 r\rho^+\sqrt{q_0^2(t)-q^2}}\,dq,
\end{cases}
$$

if $t_{h_2} < t$ and $\frac{x}{r} > \frac{V^+}{V_{\max}}$ or if $t_0 < t$ and $\frac{x}{r} \le \frac{V^+}{V_{\max}}$ and

$$
\xi_{\text{ref}}(x,y,t) = 0 \quad \text{and} \quad \boldsymbol{u}_{\text{ref}}(x,y,t) = 0 \text{ else.}
$$

We set here $r = (x^2 + (z+h)^2)^{1/2}$ and $t_0 = r/V^+$ denotes the arrival time of the reflected body wave at point $(x,0,z)$,

$$t_{h_1} = (z+h)\sqrt{\frac{1}{V^{+2}} - \frac{1}{V_{\max}^2}} + \frac{|x|}{V_{\max}} \tag{3.8}$$

denotes the arrival time of the reflected head-wave at point $(x,0,z)$ and

$$t_{h_2} = \frac{r}{z+h}\sqrt{\frac{1}{V^{+2}} - \frac{1}{V_{\max}^2}} \tag{3.9}$$

denotes the time after which there is no longer head wave at point $(x,0,z)$, (contrary to the 2D case, this time does not coincide with the arrival time of the body wave). We also define the functions γ, v, q_0 and q_1 by

$$\gamma \colon \left\{ t \in \mathrm{IR} \,|\, t > t_0 \right\} \times \mathrm{IR} \mapsto \mathbb{C} := \gamma(t,q_y) = \mathrm{i}\frac{xt}{r^2} + \frac{z+h}{r}\sqrt{\frac{t^2}{r^2} - \frac{1}{\mathcal{V}^{+2}(q_y)}},$$

$$v \colon \left\{ t \in \mathrm{IR} \,|\, t_{h_1} < t < t_{h_2} \right\} \times \mathrm{IR} \mapsto \mathbb{C} := v(t,q_y) = -\mathrm{i}\left(\frac{z+h}{r} - \sqrt{\frac{1}{\mathcal{V}^{+2}(q_y)} - \frac{t^2}{r^2}} + \frac{x}{r^2}t \right),$$

$$q_0 \colon \mathrm{IR} \to \mathrm{IR} := q_0(t) = \sqrt{\left| \frac{t^2}{r^2} - \frac{1}{V^{+2}} \right|},$$

$$q_1 \colon \mathrm{IR} \to \mathrm{IR} := q_1(t) = \sqrt{\frac{1}{x^2}\left(t - (z+h)\sqrt{\frac{1}{V^{+2}} - \frac{1}{V_{\max}^2}} \right)^2 - \frac{1}{V_{\max}^2}}.$$

Remark 3.1. For the practical computation of the pressure, we won't have to explicitly compute the derivative of the function ξ_{ref}^+ (which would be rather tedious), since

$$p_{\mathrm{ref}}^+ * f = \partial_t \xi_{\mathrm{ref}}^+ * f = \xi_{\mathrm{ref}}^+ * f'.$$

Therefore, we will only have to compute the derivative of the source function f.

• $u_{Pf}^-(x,z,t)$ is the displacement of the transmitted Pf wave and satisfies:

$$\begin{cases} u_{Pf,x}^-(x,z,t) = -\dfrac{\mathcal{P}_{11}}{\pi^2} \displaystyle\int_0^{q_1(t)} \Re e\left[\mathrm{i}v(t,q)\mathcal{T}_{Pf}(v(t,q))\frac{\partial v}{\partial t}(t,q) \right] dq, \\[4mm] u_{Pf,z}^-(x,z,t) = \dfrac{\mathcal{P}_{11}}{\pi^2} \displaystyle\int_0^{q_1(t)} \Re e\left[\kappa_{Pf}^-(v(t,q))\mathcal{T}_{Pf}(v(t,q))\frac{\partial v}{\partial t}(t,q) \right] dq, \end{cases}$$

if $t_{h_1} < t \le t_0$ and $|\Im m[\gamma(t_0,0)]| > \frac{1}{V_{\max}}$,

$$
\begin{cases}
u_{Pf,x}^-(x,z,t) = -\dfrac{\mathcal{P}_{11}}{\pi^2} \displaystyle\int_0^{q_0(t)} \Re e\left[i\gamma(t,q)\mathcal{T}_{Pf}(\gamma(t,q))\dfrac{\partial\gamma}{\partial t}(t,q)\right] dq \\
\qquad\qquad\quad -\dfrac{\mathcal{P}_{11}}{\pi^2} \displaystyle\int_{q_0(t)}^{q_1(t)} \Re e\left[iv(t,q)\mathcal{T}_{Pf}(v(t,q))\dfrac{\partial v}{\partial t}(t,q)\right] dq, \\[2mm]
u_{Pf,z}^-(x,z,t) = \dfrac{\mathcal{P}_{11}}{\pi^2} \displaystyle\int_0^{q_0(t)} \Re e\left[\kappa_{Pf}^-(\gamma(t,q))\mathcal{T}_{Pf}(\gamma(t,q))\dfrac{\partial\gamma}{\partial t}(t,q)\right] dq \\
\qquad\qquad\quad +\dfrac{\mathcal{P}_{11}}{\pi^2} \displaystyle\int_{q_0(t)}^{q_1(t)} \Re e\left[\kappa_{Pf}^-(v(t,q))\mathcal{T}_{Pf}(v(t,q))\dfrac{\partial v}{\partial t}(t,q)\right] dq,
\end{cases}
$$

if $t_0 < t \le t_{h_2}$ and $|\Im m[\gamma(t_0,0)]| > \frac{1}{V_{\max}}$,

$$
\begin{cases}
u_{Pf,x}^-(x,z,t) = -\dfrac{\mathcal{P}_{11}}{\pi^2} \displaystyle\int_0^{q_0(t)} \Re e\left[i\gamma(t,q)\mathcal{T}_{Pf}(\gamma(t,q))\dfrac{\partial\gamma}{\partial t}(t,q)\right] dq, \\[2mm]
u_{Pf,z}^-(x,z,t) = \dfrac{\mathcal{P}_{11}}{\pi^2} \displaystyle\int_0^{q_0(t)} \Re e\left[\kappa_{Pf}^-(\gamma(t,q))\mathcal{T}_{Pf}(\gamma(t,q))\dfrac{\partial\gamma}{\partial t}(t,q)\right] dq,
\end{cases}
$$

if $t_{h_2} < t$ and $|\Im m[\gamma(t_0,0)]| > \frac{1}{V_{\max}}$ or if $t_0 < t$ and $|\Im m[\gamma(t_0,0)]| \le \frac{1}{V_{\max}}$ and $\boldsymbol{u}_{Pf}^-(x,z,t) = 0$ else.

t_0 denotes here the arrival time of the Pf body wave at point $(x,0,z)$ (we recall in appendix the computation of t_0),

$$
t_{h_1} = h\sqrt{\frac{1}{V^{+2}} - \frac{1}{V_{\max}^2}} - z\sqrt{\frac{1}{V_{Pf}^{-2}} - \frac{1}{V_{\max}^2}} + \frac{|x|}{V_{\max}} \tag{3.10}
$$

denotes the arrival time of the Pf head wave at point $(x,0,z)$,

$$
t_{h_2} = \frac{h^2 + z^2 - hz\left(\frac{c_2}{c_1} + \frac{c_1}{c_2}\right) + x^2}{\frac{h}{c_1} - \frac{z}{c_2}} \tag{3.11}
$$

denotes the time after which there is no longer head wave at point $(x,0,z)$, where

$$
c_1 = \sqrt{\frac{1}{V^{+2}} - \frac{1}{V_{\max}^2}} \quad \text{and} \quad c_2 = \sqrt{\frac{1}{V_{Pf}^2} - \frac{1}{V_{\max}^2}}.
$$

The function $q_0 : [t_0; +\infty] \mapsto \mathrm{IR}^+$ is the reciprocal function of $\tilde{t}_0 : \mathrm{IR}^+ \mapsto : [t_0, +\infty]$, where $\tilde{t}_0(q)$ is the arrival time at point $(x,0,z)$ of the fictitious Pf body wave, propagating at a

velocity $\mathcal{V}^+(q)$ in the top layer and at velocity $\mathcal{V}_{Pf}^-(q)$ in the bottom layer (we recall in appendix the computation of $\tilde{t}_0(q)$).

The function $q_1 : [t_1 ; t_0] \mapsto \mathrm{IR}^+$ is defined by

$$q_1(t) = \sqrt{\frac{1}{x^2}\left(t + z\sqrt{\frac{1}{V_{Pf}^{-2}} - \frac{1}{V_{\max}^2}} - h\sqrt{\frac{1}{V^{+2}} - \frac{1}{V_{\max}^2}}\right)^2 - \frac{1}{V_{\max}^2}}.$$

The function $\gamma : \{(t,q) \in \mathrm{IR}^+ \times \mathrm{IR}^+ \,|\, t > \tilde{t}_0(q)\} \mapsto \mathbb{C}$ is implicitly defined as the only root of the function

$$\mathcal{F}(\gamma,q,t) = -z\left(\frac{1}{\mathcal{V}_{Pf}^{-2}(q)} + \gamma^2\right)^{1/2} + h\left(\frac{1}{\mathcal{V}^{+2}(q)} + \gamma^2\right)^{1/2} + i\gamma x - t,$$

whose real part is positive.

The function $v : E_1 \cup E_2 \mapsto \mathbb{C}$ is implicitly defined as the only root of the function

$$\mathcal{F}(v,q,t) = -z\left(\frac{1}{\mathcal{V}_{Pf}^{-2}(q)} + v^2\right)^{1/2} + h\left(\frac{1}{\mathcal{V}^{+2}(q)} + v^2\right)^{1/2} + ivx - t,$$

such that $\Im m\,[\partial_t v(t,q)] < 0$, with

$$E_1 = \left\{(t,q) \in \mathrm{IR}^+ \times \mathrm{IR}^+ \,\middle|\, t_{h_1} < t < t_0 \text{ and } 0 < q < q_0(t)\right\},$$

$$E_2 = \left\{(t,q) \in \mathrm{IR}^+ \times \mathrm{IR}^+ \,\middle|\, t_0 < t < t_{h_1} \text{ and } q_0(t) < q < q_1(t)\right\}.$$

• $u_{Ps}^-(x,z,t)$ is the displacement of the transmitted Ps wave and satisfies:

$$\begin{cases} u_{Ps,x}^-(x,z,t) = -\dfrac{\mathcal{P}_{12}}{\pi^2}\displaystyle\int_0^{q_1(t)}\Re e\left[iv(t,q)\mathcal{T}_{Ps}(v(t,q))\frac{dv}{dt}(t,q)\right]dq, \\[2em] u_{Ps,z}^-(x,z,t) = \dfrac{\mathcal{P}_{12}}{\pi^2}\displaystyle\int_0^{q_1(t)}\Re e\left[\kappa_{Ps}^-(v(t),q)\mathcal{T}_{Ps}(v(t,q))\frac{dv}{dt}(t,q)\right]dq, \end{cases}$$

if $t_{h_1} < t \le t_0$ and $|\Im m[\gamma(t_0,0)]| > \frac{1}{V_{\max}}$,

$$
\begin{cases}
u^-_{Ps,x}(x,z,t) = -\dfrac{\mathcal{P}_{12}}{\pi^2} \displaystyle\int_0^{q_0(t)} \Re e\left[i\gamma(t,q)\mathcal{T}_{Pf}(\gamma(t,q))\dfrac{\partial\gamma}{\partial t}(t,q)\right]dq, \\[4mm]
\qquad\qquad -\dfrac{\mathcal{P}_{12}}{\pi^2}\displaystyle\int_{q_0(t)}^{q_1(t)}\Re e\left[iv(t,q)\mathcal{T}_{Ps}(v(t,q))\dfrac{dv}{dt}(t,q)\right]dq \\[4mm]
u^-_{Ps,z}(x,z,t) = \dfrac{\mathcal{P}_{12}}{\pi^2}\displaystyle\int_0^{q_0(t)}\Re e\left[\kappa^-_{Ps}(\gamma(t,q))\mathcal{T}_{Ps}(\gamma(t,q))\dfrac{d\gamma}{dt}(t,q)\right]dq \\[4mm]
\qquad\qquad +\dfrac{\mathcal{P}_{12}}{\pi^2}\displaystyle\int_{q_0(t)}^{q_1(t)}\Re e\left[\kappa^-_{Ps}(v(t),q)\mathcal{T}_{Ps}(v(t,q))\dfrac{dv}{dt}(t,q)\right]dq,
\end{cases}
$$

if $t_0 < t \le t_{h_2}$ and $|\Im m[\gamma(t_0,0)]| > \frac{1}{V_{\max}}$,

$$
\begin{cases}
u^-_{Ps,x}(x,z,t) = -\dfrac{\mathcal{P}_{12}}{\pi^2}\displaystyle\int_0^{q_0(t)}\Re e\left[i\gamma(t,q)\mathcal{T}_{Ps}(\gamma(t,q))\dfrac{d\gamma}{dt}(t,q)\right]dq, \\[4mm]
u^-_{Ps,z}(x,z,t) = \dfrac{\mathcal{P}_{12}}{\pi^2}\displaystyle\int_0^{q_0(t)}\Re e\left[\kappa^-_{Ps}(\gamma(t,q))\mathcal{T}_{Ps}(\gamma(t,q))\dfrac{d\gamma}{dt}(t,q)\right]dq,
\end{cases}
$$

if $t_{h_2} < t$ and $|\Im m[\gamma(t_0,0)]| > \frac{1}{V_{\max}}$ or if $t_0 < t$ and $|\Im m[\gamma(t_0,0)]| \le \frac{1}{V_{\max}}$ and $\boldsymbol{u}^-_{Ps}(x,z,t) = 0$ else. t_0 denotes here the arrival time of the Ps body wave at point $(x,0,z)$,

$$
t_{h_1} = h\sqrt{\frac{1}{V^{+2}} - \frac{1}{V^2_{\max}}} - z\sqrt{\frac{1}{V_{Ps}^{-2}} - \frac{1}{V^2_{\max}}} + \frac{|x|}{V_{\max}} \tag{3.12}
$$

denotes the arrival time of the Ps head wave at point $(x,0,z)$ and

$$
t_{h_2} = \frac{h^2 + z^2 - hz\left(\frac{c_2}{c_1} + \frac{c_1}{c_2}\right) + x^2}{\frac{h}{c_1} - \frac{z}{c_2}} \tag{3.13}
$$

denotes the time after which there is no longer head wave at point $(x,0,z)$, where

$$
c_1 = \sqrt{\frac{1}{V^{+2}} - \frac{1}{V^2_{\max}}} \quad \text{and} \quad c_2 = \sqrt{\frac{1}{V_{Ps}^2} - \frac{1}{V^2_{\max}}}.
$$

The function $q_0:[t_0;+\infty]\mapsto \mathrm{I\!R}^+$ is the reciprocal function of $\tilde{t}_0:\mathrm{I\!R}^+\mapsto:[t_0,+\infty]$, where $\tilde{t}_0(q)$ is the arrival time at point $(x,0,z)$ of the fictitious Ps body wave, propagating at a velocity $\mathcal{V}^+(q)$ in the top layer and at velocity $\mathcal{V}^-_{Ps}(q)$ in the bottom layer.

The function $q_1 : [t_1; t_0] \mapsto \mathbb{R}^+$ is defined by

$$q_1(t) = \sqrt{\frac{1}{x^2}\left(t + z\sqrt{\frac{1}{V_{Ps}^{-2}} - \frac{1}{V_{\max}^2}} - h\sqrt{\frac{1}{V^{+2}} - \frac{1}{V_{\max}^2}}\right)^2 - \frac{1}{V_{\max}^2}}.$$

The function $\gamma : \{(t,q) \in \mathbb{R}^+ \times \mathbb{R}^+ \,|\, t > \tilde{t}_0(q)\} \mapsto \mathbb{C}$ is implicitly defined as the only root of the function

$$\mathcal{F}(\gamma,q,t) = -z\left(\frac{1}{\mathcal{V}_{Ps}^{-2}(q)} + \gamma^2\right)^{1/2} + h\left(\frac{1}{\mathcal{V}^{+2}(q)} + \gamma^2\right)^{1/2} + i\gamma x - t,$$

whose real part is positive.

The function $v : E_1 \cup E_2 \mapsto \mathbb{C}$ is implicitly defined as the only root of the function

$$\mathcal{F}(v,q,t) = -z\left(\frac{1}{\mathcal{V}_{Ps}^{-2}(q)} + v^2\right)^{1/2} + h\left(\frac{1}{\mathcal{V}^{+2}(q)} + v^2\right)^{1/2} + ivx - t,$$

such that $\Im m[\partial_t v(t,q)] < 0$, with

$$E_1 = \left\{(t,q) \in \mathbb{R}^+ \times \mathbb{R}^+ \,|\, t_{h_1} < t < t_0 \text{ and } 0 < q < q_0(t)\right\},$$
$$E_2 = \left\{(t,q) \in \mathbb{R}^+ \times \mathbb{R}^+ \,|\, t_0 < t < t_{h_1} \text{ and } q_0(t) < q < q_1(t)\right\}.$$

• $u_S^-(x,z,t)$ is the displacement of the transmitted S wave and satisfies:

$$\begin{cases} u_{S,x}^-(x,z,t) = -\dfrac{1}{\pi^2}\displaystyle\int_0^{q_1(t)} \Re e\left[iv(t,q)\kappa_S^-(v(t,q))\mathcal{T}_S(v(t,q))\dfrac{dv}{dt}(t,q)\right]dq, \\[4mm] u_{S,z}^-(x,z,t) = \dfrac{1}{\pi^2}\displaystyle\int_0^{q_1(t)} \Re e\left[(v^2(t,q)+q^2)\mathcal{T}_S(v(t,q))\dfrac{dv}{dt}(t,q)\right]dq, \end{cases}$$

if $t_{h_1} < t \le t_0$ and $|\Im m[\gamma(t_0,0)]| > \frac{1}{V_{\max}}$,

$$\begin{cases} u_{S,x}^-(x,z,t) = -\dfrac{1}{\pi^2}\displaystyle\int_0^{q_0(t)} \Re e\left[i\gamma(t,q)\kappa_S^-(\gamma(t,q))\mathcal{T}_S(\gamma(t,q))\dfrac{d\gamma}{dt}(t,q)\right]dq \\[4mm] \qquad\qquad -\dfrac{1}{\pi^2}\displaystyle\int_{q_0(t)}^{q_1(t)} \Re e\left[iv(t,q)\kappa_S^-(v(t,q))\mathcal{T}_S(v(t,q))\dfrac{dv}{dt}(t,q)\right]dq, \\[4mm] u_{S,z}^-(x,z,t) = \dfrac{1}{\pi^2}\displaystyle\int_0^{q_0(t)} \Re e\left[(\gamma^2(t,q)+q^2)\mathcal{T}_S(\gamma(t,q))\dfrac{d\gamma}{dt}(t,q)\right]dq \\[4mm] \qquad\qquad +\dfrac{1}{\pi^2}\displaystyle\int_{q_0(t)}^{q_1(t)} \Re e\left[(v^2(t,q)+q^2)\mathcal{T}_S(v(t,q))\dfrac{dv}{dt}(t,q)\right]dq, \end{cases}$$

if $t_0 < t \leq t_{h_2}$ and $|\Im m[\gamma(t_0,0)]| > \frac{1}{V_{\max}}$,

$$
\begin{cases}
u_{S,x}^-(x,z,t) = -\frac{1}{\pi^2} \int_0^{q_0(t)} \Re e\left[i\gamma(t,q)\kappa_S^-(\gamma(t,q))\mathcal{T}_S(\gamma(t,q))\frac{d\gamma}{dt}(t,q) \right] dq, \\[3mm]
u_{S,z}^-(x,z,t) = \frac{1}{\pi^2} \int_0^{q_0(t)} \Re e\left[(\gamma^2(t,q)+q^2)\mathcal{T}_S(\gamma(t,q))\frac{d\gamma}{dt}(t,q) \right] dq,
\end{cases}
$$

if $t_{h_2} < t$ and $|\Im m[\gamma(t_0,0)]| > \frac{1}{V_{\max}}$ or if $t_0 < t$ and $|\Im m[\gamma(t_0,0)]| \leq \frac{1}{V_{\max}}$ and $\boldsymbol{u}_{P_S}^-(x,z,t) = 0$ else. t_0 denotes here the arrival time of the S body wave at point $(x,0,z)$ (we recall in appendix the computation of t_0),

$$
t_{h_1} = h\sqrt{\frac{1}{V^{+2}} - \frac{1}{V_{\max}^2}} - z\sqrt{\frac{1}{V_S^{-2}} - \frac{1}{V_{\max}^2}} + \frac{|x|}{V_{\max}} \tag{3.14}
$$

denotes the arrival time of the S head-wave at point $(x,0,z)$ and

$$
t_{h_2} = \frac{h^2 + z^2 - hz\left(\frac{c_2}{c_1} + \frac{c_1}{c_2}\right) + x^2}{\frac{h}{c_1} - \frac{z}{c_2}} \tag{3.15}
$$

denotes the time after which there is no longer head wave at point $(x,0,z)$, where

$$
c_1 = \sqrt{\frac{1}{V^{+2}} - \frac{1}{V_{\max}^2}} \quad \text{and} \quad c_2 = \sqrt{\frac{1}{V_S^2} - \frac{1}{V_{\max}^2}}.
$$

The function $q_0:[t_0;+\infty] \mapsto \mathbb{R}^+$ is the reciprocal function of $\tilde{t}_0:\mathbb{R}^+ \mapsto:[t_0,+\infty]$, where $\widetilde{t}_0(q)$ is the arrival time at point $(x,0,z)$ of the fictitious S body wave, propagating at a velocity $\mathcal{V}^+(q)$ in the top layer and at velocity $\mathcal{V}_S^-(q)$ in the bottom layer (we recall in appendix the computation of $\tilde{t}_0(q)$).

The function $q_1:[t_1;t_0] \mapsto \mathbb{R}^+$ is defined by

$$
q_1(t) = \sqrt{\frac{1}{x^2}\left(t + z\sqrt{\frac{1}{V_S^{-2}} - \frac{1}{V_{\max}^2}} - h\sqrt{\frac{1}{V^{+2}} - \frac{1}{V_{\max}^2}} \right)^2 - \frac{1}{V_{\max}^2}}.
$$

The function $\gamma:\{(t,q) \in \mathbb{R}^+ \times \mathbb{R}^+ \,|\, t > \tilde{t}_0(q)\} \mapsto \mathbb{C}$ is implicitly defined as the only root of the function

$$
\mathcal{F}(\gamma,q,t) = -z\left(\frac{1}{\mathcal{V}_S^{-2}(q)} + \gamma^2 \right)^{1/2} + h\left(\frac{1}{\mathcal{V}^{+2}(q)} + \gamma^2 \right)^{1/2} + i\gamma x - t,
$$

whose real part is positive.

The function $v\colon E_1\cup E_2\mapsto\mathbb{C}$ is implicitly defined as the only root of the function

$$\mathcal{F}(v,q,t)=-z\left(\frac{1}{\mathcal{V}_S^{-2}(q)}+v^2\right)^{1/2}+h\left(\frac{1}{\mathcal{V}^{+2}(q)}+v^2\right)^{1/2}+ivx-t,$$

such that $\Im m[\partial_t v(t,q)]<0$, with

$$E_1=\left\{(t,q)\in\mathbb{R}^+\times\mathbb{R}^+\,|\,t_{h_1}<t<t_0\text{ and }0<q<q_0(t)\right\},$$
$$E_2=\left\{(t,q)\in\mathbb{R}^+\times\mathbb{R}^+\,|\,t_0<t<t_{h_1}\text{ and }q_0(t)<q<q_1(t)\right\}.$$

4　Proof of the theorem

To prove the theorem, we use the Cagniard-de Hoop method (see [1, 2, 14–16]), which consists of three steps:

1. We apply a Laplace transform in time,

$$\tilde{u}(x,y,z,s)=\int_0^{+\infty}u(x,y,z,t)e^{-st}dt,$$

and a Fourier transform in the x and y variables,

$$\hat{u}(k_x,k_y,z,s)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\tilde{u}(x,y,s)e^{i(k_xx+k_yy)}dxdy$$

to (2.17) in order to obtain an ordinary differential system whose solution $\hat{\mathcal{G}}(k_x,k_y,z,s)$ can be explicitly computed (Section 4.1);

2. we apply an inverse Fourier transform in the x and y variables to $\mathcal{G}$ (we recall that we only need the solution at $y=0$:

$$\tilde{\mathcal{G}}(x,0,z,s)=\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\hat{\mathcal{G}}(k_x,k_y,z,s)e^{-ik_xx}dk_xdk_y.$$

Moreover, using tools of complex analysis, we turn the inverse Fourier transform in the x variable into the Laplace transform of some function $\mathcal{H}(x,k_y,z,t)$ (Section 4.2):

$$\tilde{\mathcal{G}}(x,0,z,s)=\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_0^{+\infty}\mathcal{H}(x,k_y,z,t)e^{-st}dtdk_y;\tag{4.1}$$

3. the last step of the method consists in inverting the order of integration in (4.1) to obtain

$$\tilde{\mathcal{G}}(x,0,z,s)=\frac{1}{4\pi^2}\int_0^{+\infty}\left(\int_{-q(t)}^{+q(t)}\mathcal{H}(x,k_y,z,t)dk_y\right)e^{-st}dt.$$

Then, using the injectivity of the Laplace transform, we identify $\mathcal{G}(x,0,z,t)$ to

$$\frac{1}{4\pi^2}\int_{-q(t)}^{+q(t)}\mathcal{H}(x,k_y,z,t)dk_y$$

(see Section 4.3).

4.1   The solution in the Laplace-Fourier plane

Let us first apply a Laplace transform in time and a Fourier transform in the x and y variables to (2.17) to obtain

$$
\begin{cases}
\left(\dfrac{s^2}{V^{+2}}+k_x^2+k_y^2\right)\hat{p}^+ -\dfrac{\partial^2\hat{p}^+}{\partial z^2}=\dfrac{\delta(z-h)}{V^{+2}}, & y>0, \\[3mm]
\left(\dfrac{s^2}{V_i^{-2}}+k_x^2+k_y^2\right)\hat{\Phi}_i^- -\dfrac{\partial^2\hat{\Phi}_i^-}{\partial z^2}=0, \ i\in\{Pf,Ps,S\}, & y<0, \\[3mm]
\hat{\mathcal{B}}(\hat{p}^+,\hat{\Phi}_{Pf}^-,\hat{\Phi}_{Pf}^-,\hat{\Phi}_S^-)=0, & y=0,
\end{cases}
\tag{4.2}
$$

where $\hat{\mathcal{B}}$ is the Laplace-Fourier transform of the operator $\mathcal{B}$.

From the two first equations of (4.2), we deduce that the solution $(\hat{p}^+,(\hat{\Phi}_i^-)_{i\in\{Pf,Ps,S\}})$ is such that

$$
\begin{cases}
\hat{p}^+=\hat{p}_{\text{inc}}^+ +\hat{p}_{\text{ref}}^+, \quad \hat{p}_{\text{ref}}^+=R(k_x,k_y,s)e^{-sz\kappa^+\left(\frac{k_x}{s},\frac{k_y}{s}\right)}, \\[3mm]
\hat{p}_{\text{inc}}^+=\dfrac{1}{sV^{+2}\kappa^+\left(\frac{k_x}{s},\frac{k_y}{s}\right)}e^{-s|z-h|\kappa^+\left(\frac{k_x}{s},\frac{k_y}{s}\right)}, \\[3mm]
\hat{\Phi}_i^-=T_i(k_x,k_y,s)e^{-s\left(z\kappa_i^-\left(\frac{k_x}{s}\frac{k_y}{s}\right)\right)}, \quad i\in\{Pf,Ps,S\},
\end{cases}
\tag{4.3}
$$

where the coefficients R and T_i are computed by using the last equation of (4.2):

$$
\hat{\mathcal{B}}(\hat{p}_{\text{ref}}^+,\hat{\Phi}_{Pf}^-,\hat{\Phi}_{Ps}^-,\hat{\Phi}_S^-)=-\hat{\mathcal{B}}(\hat{p}_{\text{inc}}^+,0,0,0),
$$

or, from(4.3):

$$
\hat{\mathcal{B}}\left[R(k_x,k_y,s),T_{Pf}(k_x,k_y,s),T_{Ps}(k_x,k_y,s),T_S(k_x,k_y,s)\right]=-\hat{\mathcal{B}}\left(\dfrac{e^{-sh\kappa^+\left(\frac{k_x}{s},\frac{k_y}{s}\right)}}{s\kappa^+\left(\frac{k_x}{s},\frac{k_y}{s}\right)},0,0,0\right).
$$

After some calculations that we do not detail here, we obtain that $R(k_x,k_y,s)$, $T_{Pf}(k_x,k_y,s)$, $T_{Ps}(k_x,k_y,s)$, and $T_S(k_x,k_y,s))$ are solution to

$$
\mathcal{A}\left(\frac{k_x}{s},\frac{k_y}{s}\right)
\begin{bmatrix}
R(k_x,k_y,s) \\[2mm]
s^2 T_{Pf}(k_x,k_y,s) \\[2mm]
s^2 T_{Ps}(k_x,k_y,s) \\[2mm]
s^3 T_S(k_x,k_y,s)
\end{bmatrix}
=-\dfrac{e^{-sh\kappa^+\left(\frac{k_x}{s},\frac{k_y}{s}\right)}}{2s\kappa^+\left(\frac{k_x}{s},\frac{k_y}{s}\right)V^{+2}}
\begin{bmatrix}
\dfrac{\kappa^+\left(\frac{k_x}{s},\frac{k_y}{s}\right)}{\rho^+} \\[3mm]
1 \\[2mm]
0 \\[2mm]
1
\end{bmatrix}.
\tag{4.4}
$$

From the definition of the reflection and transmission coefficients we deduce that

$$
\begin{bmatrix}
R(k_x,k_y,s) \\
s^2 T_{Pf}(k_x,k_y,s) \\
s^2 T_{Ps}(k_x,k_y,s) \\
s^3 T_S(k_x,k_y,s)
\end{bmatrix}
= \frac{1}{s}
\begin{bmatrix}
\mathcal{R}\left(\frac{k_x}{s},\frac{k_y}{s}\right) \\
\mathcal{T}_{Pf}\left(\frac{k_x}{s},\frac{k_y}{s}\right) \\
\mathcal{T}_{Ps}\left(\frac{k_x}{s},\frac{k_y}{s}\right) \\
\mathcal{T}_S\left(\frac{k_x}{s},\frac{k_y}{s}\right)
\end{bmatrix}
e^{-sh\kappa^+\left(\frac{k_x}{s}\right)}.
\tag{4.5}
$$

Finally, we obtain:

$$
\begin{cases}
\hat{p}^+ = \hat{p}^+_{\text{inc}} + \hat{p}^+_{\text{ref}}, \\[2mm]
\hat{p}^+_{\text{inc}} = \dfrac{1}{sV^{+2}\kappa^+\left(\frac{k_x}{s},\frac{k_y}{s}\right)} e^{-s|y-h|\kappa^+\left(\frac{k_x}{s},\frac{k_y}{s}\right)}, \\[3mm]
\hat{p}^+_{\text{ref}} = \dfrac{1}{s}\mathcal{R}\left(\dfrac{k_x}{s},\dfrac{k_y}{s}\right) e^{-s(z+h)\kappa^+\left(\frac{k_x}{s},\frac{k_y}{s}\right)}, \\[3mm]
\hat{\Phi}^-_i = \dfrac{1}{s^3}\mathcal{T}_i\left(\dfrac{k_x}{s},\dfrac{k_y}{s}\right) e^{-s\left(z\kappa^-_i\left(\frac{k_x}{s},\frac{k_y}{s}\right)-h\kappa^+\left(\frac{k_x}{s},\frac{k_y}{s}\right)\right)}, \quad i\in\{Pf,Ps\}, \\[3mm]
\hat{\Phi}^-_S = \dfrac{1}{s^4}\mathcal{T}_S\left(\dfrac{k_x}{s},\dfrac{k_y}{s}\right) e^{-s\left(z\kappa^-_S\left(\frac{k_x}{s},\frac{k_y}{s}\right)-h\kappa^+\left(\frac{k_x}{s},\frac{k_y}{s}\right)\right)}.
\end{cases}
\tag{4.6}
$$

and

$$
\begin{cases}
\hat{u}^+ = \hat{u}^+_{\text{inc}} + \hat{u}^+_{\text{ref}}, \\[2mm]
\hat{u}^+_{\text{inc},x} = i\dfrac{k_x}{\rho^+ s^2}\hat{p}^+_{\text{inc}}, \quad \hat{u}^+_{\text{inc},z} = \text{sign}(h-z)\dfrac{\kappa^+\left(\frac{k_x}{s},\frac{k_y}{s}\right)}{\rho^+ s}\hat{p}^+_{\text{inc}}, \\[3mm]
\hat{u}^+_{\text{ref},x} = i\dfrac{k_x}{\rho^+ s^2}\hat{p}^+_{\text{ref}}, \quad \hat{u}^+_{\text{ref},z} = \dfrac{\kappa^+\left(\frac{k_x}{s},\frac{k_y}{s}\right)}{\rho^+ s}\hat{p}^+_{\text{ref}}, \\[3mm]
\hat{u}^-_{sx} = -ik_x\mathcal{P}_{11}\hat{\Phi}^-_{Pf} - ik_x\mathcal{P}_{12}\hat{\Phi}^-_{Ps} - isk_x\kappa^-_S\left(\dfrac{k_x}{s}\right)\Phi^-_S, \\[3mm]
\hat{u}^-_{sz} = s\kappa^-_{Pf}\left(\dfrac{k_x}{s},\dfrac{k_y}{s}\right)\mathcal{P}_{11}\hat{\Phi}^-_{Pf} + s\kappa^-_{Ps}\left(\dfrac{k_x}{s},\dfrac{k_y}{s}\right)\mathcal{P}_{12}\hat{\Phi}^-_{Ps} + (k_x^2 + k_y^2)\Phi^-_S.
\end{cases}
\tag{4.7}
$$

In the following we only detail the computation of $\hat{u}^-_{sx,Ps} = -ik_x\mathcal{P}_{12}\hat{\Phi}^-_{Ps}$, since the computation of the other terms is very similar.

4.2 The Laplace transform of the solution

We apply an inverse Fourier transform in the x and y variable to $\hat{u}^-_{sx,Ps}$ and we set $k_x = q_x s$ and $k_y = q_y s$ to obtain (we recall that we consider $y = 0$)

$$\tilde{u}^-_{sx,Ps}(x,0,z,s) = -\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{iq_x \mathcal{P}_{12}}{4\pi^2} \mathcal{T}_{Ps}(q_x,q_y) e^{-s\left(-z\kappa^-_{Ps}(q_x,q_y)+h\kappa^+(q_x,q_y))+iq_x x\right)} dq_x dq_y$$

$$= -\frac{\mathcal{P}_{12}}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Xi(q_x,q_y) dq_x dq_y,$$

with

$$\Xi(q_x,q_y) = iq_x \mathcal{T}_{Ps}(q_x,q_y) e^{-s\left(-z\kappa^-_{Ps}(q_x,q_y)+h\kappa^+(q_x,q_y))+iq_x x\right)}.$$

Let us now focus on the integral over q_x for a fixed q_y

$$\int_{-\infty}^{+\infty} \Xi(q_x,q_y) dq_x = \int_{-\infty}^{+\infty} iq_x \mathcal{T}_{Ps}(q_x,q_y) e^{-s\left(-z\kappa^-_{Ps}(q_x,q_y)+h\kappa^+(q_x,q_y))+iq_x x\right)} dq_x \qquad (4.8)$$

This integral is very similar to the one we have obtained in 2D [7], therefore, using the same method, we have:

- if $|\gamma(q_y,t_0)| \leq \frac{1}{\mathcal{V}_{max}(q_y)}$,

$$\int_{-\infty}^{+\infty} \Xi(q_x,q_y) dq_x = 2\int_{\tilde{t}_0(q_y)}^{+\infty} \Re\left(i\gamma(q_y,t)\mathcal{T}_{Ps}(q_y,\gamma(q_y,t))\frac{\partial\gamma(q_y,t)}{\partial t}\right) e^{-st} dt;$$

- if $|\gamma(q_y,\tilde{t}_0(q_y))| > \frac{1}{\mathcal{V}_{max}(q_y)}$,

$$\int_{-\infty}^{+\infty} \Xi(q_x,q_y) dq_x = 2\int_{\tilde{t}_h(q_y)}^{\tilde{t}_0(q_y)} \Re\left(iv(q_y,t)\mathcal{T}_{Ps}(q_y,v(q_y,t))\frac{\partial v(q_y,t)}{\partial t}\right) e^{-st} dt$$

$$+ 2\int_{\tilde{t}_0(q_y)}^{+\infty} \Re\left(i\gamma(q_y,t)\mathcal{T}_{Ps}(q_y,\gamma(q_y,t))\frac{\partial\gamma(q_y,t)}{\partial t}\right) e^{-st} dt,$$

where $\mathcal{V}_{max}$ is greatest fictitious velocity defined by:

$$\mathcal{V}_{max} := \mathcal{V}_{max}(q) = V_{max}\sqrt{\frac{1}{1+V_{max}^2 q^2}},$$

$\tilde{t}_0$ is the fictitious arrival time of the Ps body wave we have defined in the theorem, $\tilde{t}_h(q_y)$ the fictitious arrival time of the Ps head wave defined by

$$\tilde{t}_h := \tilde{t}_h(q) = h\sqrt{\frac{1}{\mathcal{V}^{+2}(q)} - \frac{1}{\mathcal{V}_{max}^2(q)}} - z\sqrt{\frac{1}{\mathcal{V}_{Ps}^{-2}(q)} - \frac{1}{\mathcal{V}_{max}^2(q)}} + \frac{|x|}{\mathcal{V}_{max}(q)}.$$
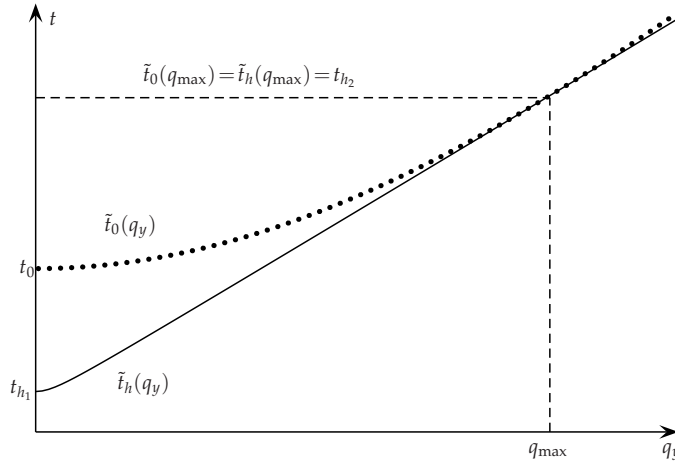
Figure 3: Functions $\tilde{t}_0$ (dotted line) and $\tilde{t}_h$ (solid line).

Let us recall [17] that the condition $|\gamma(q_y,\tilde{t}_0(q_y))| > \frac{1}{\mathcal{V}_{\max}(q_y)}$ is equivalent to

$$|\gamma(0,t_0)| > \frac{1}{V_{\max}} \quad \text{and} \quad |q_y| \leq q_{\max},$$

with

$$q_{\max} = \sqrt{\frac{r^2}{h\left(\frac{1}{V^{+2}} - \frac{1}{V_{\max}^2}\right)^{-1/2} - z\left(\frac{1}{V_{Ps}^{-2}} - \frac{1}{V_{\max}^2}\right)^{-1/2}} - \frac{1}{V_{\max}^2}}.$$

Moreover, $\tilde{t}_h$ is bijective from $[0;q_{\max}]$ to $[t_0;\tilde{t}_h(q_{\max})]$ and we denote its inverse by q_h:

$$q_h(t) = \sqrt{\frac{1}{x^2}\left(t + z\sqrt{\frac{1}{V_{Ps}^{-2}} - \frac{1}{V_{\max}^2}} - h\sqrt{\frac{1}{V^{+2}} - \frac{1}{V_{\max}^2}}\right)^2 - \frac{1}{V_{\max}^2}}.$$

Let us also recall that for $q_y = q_{\max}$, the arrival times of the fictitious head and body waves are the same: $\tilde{t}_h(q_{\max}) = \tilde{t}_0(q_{\max})$. As an illustration, we represent the functions $\tilde{t}_0$ (the dotted line) and $\tilde{t}_h$ (the solid line) in Fig. 3.

We then deduce that

- if $|\gamma(0,t_0)| \leq \frac{1}{V_{\max}}$,

$$\int_{\mathbb{R}^2} \Xi(q_x,q_y)\,dq_x dq_y$$
$$= 2\int_{\mathbb{R}} \int_{\tilde{t}_0(q_y)}^{+\infty} \Re e\left(i\gamma(q_y,t)\mathcal{T}_{Ps}(q_y,\gamma(q_y,t))\frac{\partial\gamma(q_y,t)}{\partial t}\right)e^{-st}\,dt\,dq_y;$$
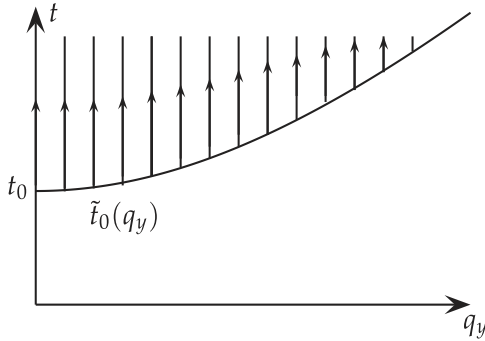
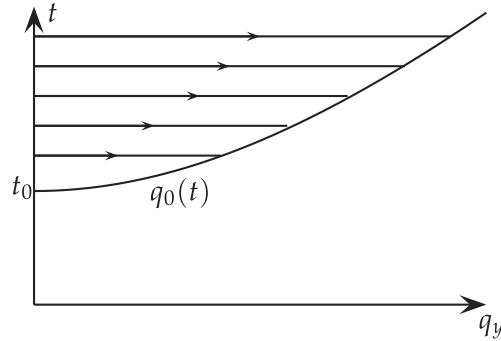Figure 4: Integration first over q_y then over t for the body wave.

Figure 5: Integration first over t then over q_y for the body wave.

- if $|\gamma(0,t_0)| > \frac{1}{V_{max}}$,

$$\int_{\mathrm{IR}^2} \Xi(q_x,q_y)\,dq_x dq_y$$

$$=2\int_{-q_{max}}^{+q_{max}} \int_{\tilde{t}_h(q_y)}^{\tilde{t}_0(q_y)} \Re\left(i v(q_y,t)\mathcal{T}_{Ps}(q_y,v(q_y,t))\frac{\partial v(q_y,t)}{\partial t} \right) e^{-st}\,dt dq_y$$

$$+2\int_{\mathrm{IR}} \int_{\tilde{t}_0(q_y)}^{+\infty} \Re\left(i\gamma(q_y,t)\mathcal{T}_{Ps}(q_y,\gamma(q_y,t))\frac{\partial \gamma(q_y,t)}{\partial t} \right) e^{-st}\,dt dq_y.$$

4.3 Inversion of the integrals

The key point of the method is the inversion of the integral with respect to q_y with the integral with respect to t. For the body wave we have (see Figs. 4 and 5), after having remark that the integrand is even with respect to q_y:

$$\int_{-\infty}^{+\infty} \int_{\tilde{t}_0(q_y)}^{+\infty} \Re\left(i\gamma(q_y,t)\mathcal{T}_{Ps}(q_y,\gamma(q_y,t))\frac{\partial \gamma(q_y,t)}{\partial t} \right) e^{-st}\,dt dq_y$$

$$=2\int_{t_0}^{+\infty} \int_{0}^{q_0(t)} \Re\left(i\gamma(q_y,t)\mathcal{T}_{Ps}(q_y,\gamma(q_y,t))\frac{\partial \gamma(q_y,t)}{\partial t} \right) e^{-st}\,dq_y dt;$$

and for the head wave (see Figs. 6 and 7):

$$\int_{-q_{max}}^{+q_{max}} \int_{\tilde{t}_h(q_y)}^{\tilde{t}_0(q_y)} \Re\left(i v(q_y,t)\mathcal{T}_{Ps}(q_y,v(q_y,t))\frac{\partial v(q_y,t)}{\partial t} \right) e^{-st}\,dt dq_y$$

$$=2\int_{t_{h_1}}^{t_0} \int_{0}^{q_h(t)} \Re\left(i v(q_y,t)\mathcal{T}_{Ps}(q_y,v(q_y,t))\frac{\partial v(q_y,t)}{\partial t} \right) e^{-st}\,dq_y dt$$

$$+2\int_{t_0}^{t_{h_2}} \int_{q_0(t)}^{q_h(t)} \Re\left(i v(q_y,t)\mathcal{T}_{Ps}(q_y,v(q_y,t))\frac{\partial v(q_y,t)}{\partial t} \right) e^{-st}\,dq_y dt.$$
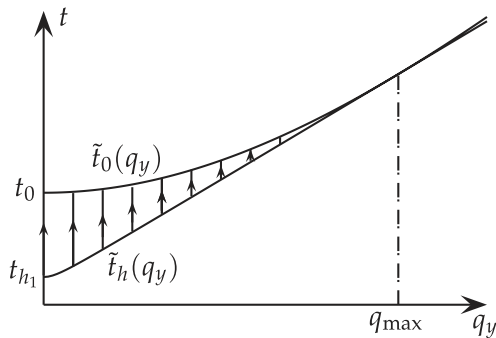
Figure 6: Integration first over q_y then over t for the head wave.
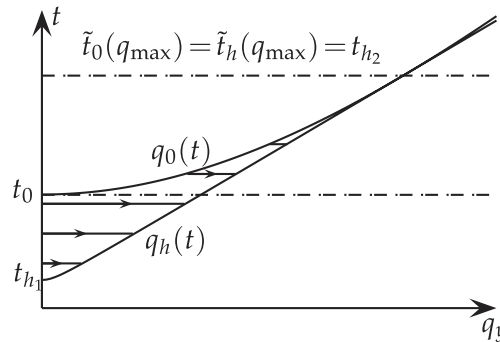

Figure 7: Integration first over t then over q_y for the head wave.

We thus have:

$$\tilde{u}_{sx,Ps}^-(x,0,z,s) = \int_0^{+\infty} u_{sx,Ps}^-(x,0,z,t)e^{-st}dt$$

and we conclude by using the injectivity of the Laplace transform.

5   Numerical illustration

To illustrate our results, we have computed the green function and the analytical solution to the following problem: we consider an acoustic layer with a density $\rho^+ = 1020\,\mathrm{kg/m^3}$ and a celerity $V^+ = 1500\,\mathrm{m/s}$ on top of a poroelastic layer whose characteristic coefficients are: the solid density $\rho_s^- = 2500\,\mathrm{kg/m^3}$; the fluid density $\rho_f^- = 1020\,\mathrm{kg/m^3}$; the porosity $\phi^- = 0.4$; the tortuosity $a^- = 2$; the solid bulk modulus $K_s^- = 16.0554\,\mathrm{GPa}$; the fluid bulk modulus $K_f^- = 2.295\,\mathrm{GPa}$; the frame bulk modulus $K_b^- = 10\,\mathrm{GPa}$; and the frame shear modulus $\mu^- = 9.63342\,\mathrm{GPa}$. As a result, the celerity of the waves in the poroelastic medium are: for the fast P wave, $V_{Pf}^- = 3677\,\mathrm{m/s}$; for the slow P wave, $V_{Ps}^- = 1060\,\mathrm{m/s}$; and for the ψ wave, $V_S^- = 2378\,\mathrm{m/s}$.

The source is located in the acoustic layer, at 500 m from the interface. It is a point source in space and a fifth derivative of a Gaussian of dominant frequency $f_0 = 15\,\mathrm{Hz}$:

$$f(t) = 2\frac{\pi^2}{f_0^2}\left[3 + 12\frac{\pi^2}{f_0^2}\left(t-\frac{1}{f_0}\right)^2 + 4\frac{\pi^4}{f_0^4}\left(t-\frac{1}{f_0}\right)^4\right]e^{-\frac{\pi^2}{f_0^2}\left(t-\frac{1}{f_0}\right)^2}.$$

We compute the solution at two receivers, the first one is in the acoustic layer, at 533 m from the interface; the first one is in the poroelastic layer, at 533 m from the interface; both are located on a vertical line at 400 m from the source (see Fig. 8). To compute the integrals over q and the convolution with the source function, we used a classical midpoint quadrature formula.
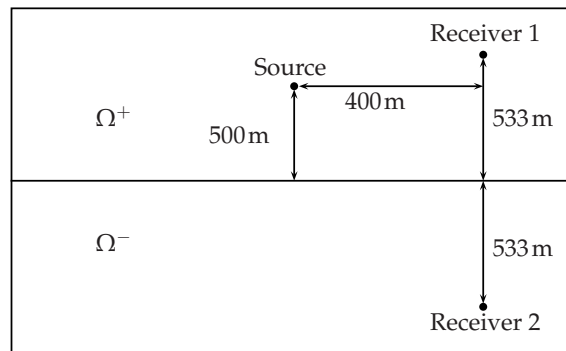
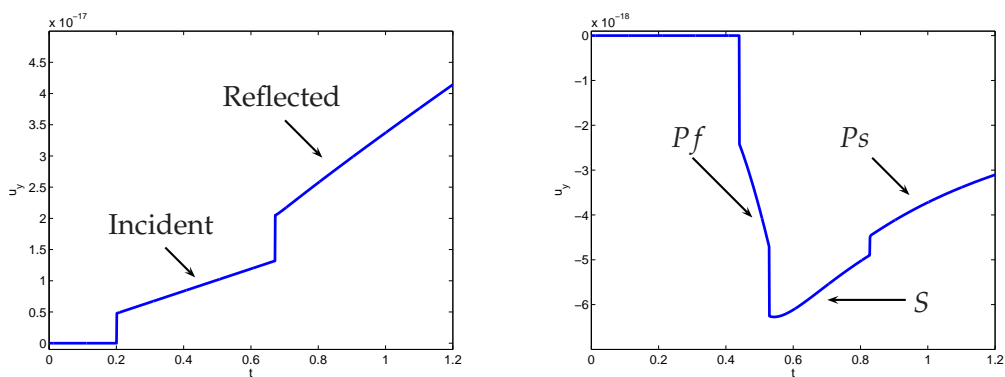Figure 8: Configuration of the experiment.

Figure 9: The z component of the green function associated to the displacement at receiver 1 (left picture) and 2 (right picture).
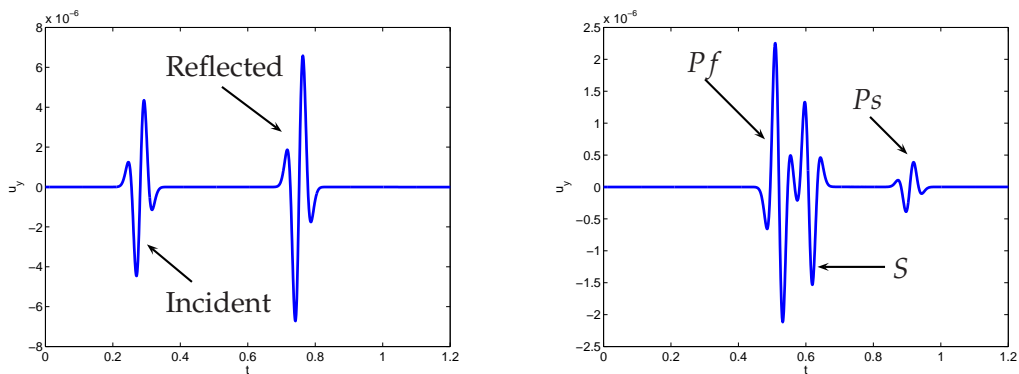

Figure 10: The z component of the displacement at receiver 1 (left picture) and 2 (right picture).

We represent the z component of the green function associated to the displacement from $t=0$ to $t=1.2$ s on Fig. 9 and the displacement in Fig. 10. The left picture represents the solution at receiver 1 while the right picture represents the solution at receiver 2. As

all the types of waves are computed independently, it is easy to distinguish all of them, as it is indicated in the figures. solution.

6   Conclusion

In this paper we have provided the complete solution (reflected and transmitted wave) of the propagation of wave in a stratified 3D medium composed of an acoustic and a poroelastic layer. In a forthcoming paper we will extend the method to the propagation of waves in bilayered poroelastic medium in three dimensions.

Acknowledgments

Appendix: Definition of the fictitious and real arrival times of the body waves

We detail in this section the computation of the fictitious and real arrival times of the transmitted Ps wave at point $(x,0,z)$. For a given $q_y \in \mathrm{I\!R}$, we first determine fastest path of the wave from the source to the point $(x,0,z)$, travelling at a velocity $\mathcal{V}^+(q_y)$ in the upper layer and at a velocity $\mathcal{V}^-_{Ps}(q_y)$ in the bottom layer: we search a point $\tilde{\xi}_0$ on the interface between the two media which minimizes the function

$$t(\xi) = \frac{\sqrt{\xi^2 + h^2}}{\mathcal{V}^+(q_y)} + \frac{\sqrt{(x-\xi)^2 + z^2}}{\mathcal{V}^-_{Ps}(q_y)}$$

(see Fig. 11). This leads us to find $\tilde{\xi}_0$ such that

$$t'(\tilde{\xi}_0) = \frac{\tilde{\xi}_0}{\mathcal{V}^+(q_y)\sqrt{\tilde{\xi}_0^2 + h^2}} + \frac{\tilde{\xi}_0 - x}{\mathcal{V}^-_{Ps}(q_y)\sqrt{(x-\tilde{\xi}_0)^2 + z^2}} = 0. \tag{A.1}$$

From a numerical point of view, the solution of this equation is done by computing the roots of the following fourth degree polynomial

$$\left(\frac{1}{\mathcal{V}^{+2}(q_y)} - \frac{1}{\mathcal{V}^{-2}_{Ps}(q_y)} \right) X^4 + 2x \left(\frac{1}{\mathcal{V}^{-2}_{Ps}(q_y)} - \frac{1}{\mathcal{V}^{+2}(q_y)} \right) X^3$$

$$+ \left(\frac{x^2 + z^2}{\mathcal{V}^{+2}(q_y)} - \frac{x^2 + h^2}{\mathcal{V}^{-2}_{Ps}(q_y)} \right) X^2 + \frac{xh^2}{\mathcal{V}^{-2}_{Ps}(q_y)} X + \frac{x^2 h^2}{\mathcal{V}^{-2}_{Ps}(q_y)},$$
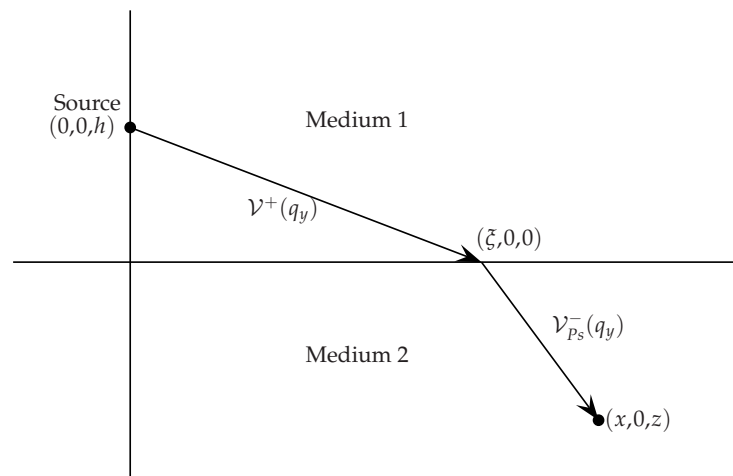
Figure 11: Path of the transmitted i wave.

ξ_0 is thus the only real root of this polynomial located between 0 and x which is also solution of (A.1). Once ξ_0 is computed, we can define

$$\tilde{t}_0(q_y) = \frac{\sqrt{\xi_0^2 + h^2}}{\mathcal{V}^+(q_y)} + \frac{\sqrt{(x-\xi_0)^2 + z^2}}{\mathcal{V}_{P_s}^-(q_y)} \text{ and } t_0 = \tilde{t}_0(0).$$

Let us remark that

Property A.1. Since the fictitious velocities are smaller than the real one, the fictitious arrival times are greater than the real one. Moreover, since the fictitious velocities are even functions decreasing on $\mathbb{R}^+$, $\tilde{t}_0$ is an even function, increasing on $\mathbb{R}^+$.

Corollaire A.1. The function $\tilde{t}_0$ is bijective from $\mathbb{R}^+$ to $\mathbb{R}^+$.

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