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Variational Formulation for Guided and Leaky Modes in Multilayer Dielectric Waveguides

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Abstract. The guided and leaky modes of a planar dielectric waveguide are eigensolutions of a singular Sturm-Liouville problem. The modes are the roots of a characteristic function which can be found with several methods that have been introduced in the past. However, the evaluation of the characteristic function suffers from numerical instabilities, and hence it is often difficult to find all modes in a given range. Here a new variational formulation is introduced, which, after discretization, leads either to a quadratic or a quartic eigenvalue problem. The modes can be computed with standard software for polynomial eigenproblems. Numerical examples show that the method is numerically stable and guarantees a complete set of solutions.

AMS subject classifications: 65N25, 65F15, 35J20

Key words: Dielectric waveguide, polynomial eigenvalue problem, variational formulation, finite elements, singular Sturm-Liouville problem, continuation method.

1 Introduction

Understanding the propagation properties of electromagnetic waves in layered dielectric media is essential for many applications in photonics. Multilayer dielectric structure can guide waves and are characterized by a refractive index that varies only in x-direction. Furthermore, in each layer the permittivity of each is a piecewise constant function, and a constant (with possibly different values) outside the finite interval $x \in [0,w]$. The two semi-infinite intervals are the cover and substrate and the finite interval is the stack of the waveguide. The magnetic permeability μ is constant.

The modes of such a structure have either transverse electric (TE) or transverse magnetic (TM) polarization. In the former case the electric field is of the form $\vec{E}(\vec{r},t) =$

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$\phi(x)\exp[i(\beta z - kt)]\cdot\vec{e}_y$ where ϕ satisfies the scalar equation

$$\phi''(x)+(k^2n^2(x)-\beta^2)\phi(x)=0, \quad x\in\mathbb{R}. \tag{1.1}$$

Here, $n(x)=\sqrt{\epsilon(x)\mu}$ is the refractive index.

In the TM case the magnetic field is of the form $\vec{H}(\vec{r},t)=\phi(x)\exp[i(\beta z-kt)]\cdot\vec{e}_y$ where ϕ satisfies the scalar equation

$$\left(\frac{1}{n^2(x)}\phi'(x)\right)' + \left(k^2 - \frac{\beta^2}{n^2(x)}\right)\phi(x)=0, \quad x\in\mathbb{R}. \tag{1.2}$$

Equations (1.1) and (1.2) are singular Sturm-Liouville problems, where the propagation constant β is the unknown eigenvalue.

The spectrum consists of a discrete part, corresponding to guided modes, and a continuous part, corresponding to radiation modes. These modes form a complete set, that is, any function in $L_2(\mathbb{R})$ is a superposition of a finite number of guided modes and a continuum of radiation modes [11]. Using the framework of thin-film translation matrices it is easy to obtain a characteristic function whose real roots are propagation constants of the guided modes. With the same framework one can also derive the form of the modes in the continuous spectrum [5].

There is a third type of eigensolutions, known as leaky modes. These are unbounded solutions corresponding to complex roots of the characteristic function whose modes radiate energy away from the stack. Although leaky waves have infinite energy they are physically significant and have been verified experimentally in finite regions of the waveguide [19]. The leaky modes form a discrete set of expansion functions in the stack and can therefore represent field solutions in this region [12]. Leaky-wave analysis has the advantage that in the representation of a field the superposition integral of radiation modes is replaced by a discrete sum of leaky modes. In practice, only a few modes are necessary to obtain good approximations. This type of analysis has been applied in several photonics applications, see, e.g., [10] and the references cited therein. For more information on leaky modes in planar waveguides we refer to the recently published survey article [8].

Since waves can travel in two directions in the semi-infinite layers of the waveguide, the characteristic function has two branch cuts in the complex plane. To avoid the difficulties of the iterative root finder caused by the branch cuts, Smith et. al. [15] suggest a change of variables in which the characteristic function is analytic in the complex plane except for the origin. To find all roots in a specified region of the complex plane one can use a method by Delves and Lyness [6], which is based on the argument principle of complex analysis. Similar techniques for finding the propagation constants of dielectric waveguides are discussed in [1, 2, 4] and [16]. A related method to find the roots is by continuation from a closed to an open waveguide [9].

The core of these methods is the evaluation of a characteristic function. Unfortunately, the function exhibits the exponential scaling which can lead to numerical instabilities when evaluating this function in floating point arithmetic. It is well known that the effect becomes noticeable if there are many or thick layers present, if the frequency is large or if there are lossy layers, see, e.g., [7,13,14].

An alternative to evaluating the characteristic function is a variational approach where the unknown eigenmode ϕ is approximated by either polynomial expansions [13] or finite elements [20]. This discretization scheme entails truncating (1.1) to a finite interval and imposing artificial boundary conditions at the endpoints. The result is a highly non-linear eigenvalue problem that must be solved by iteration. This approach avoids the numerical instabilities associated with the characteristic function, but does not guarantee that a complete set of solutions is found, since the solutions depend on the quality of the initial guess.

In this article we propose a new variational formulation for computing the guided and leaky modes. After discretization the variational form reduces to either a quadratic or a quartic eigenvalue problem, depending on whether the permittivity is equal or different in the two semi-infinite layers. The benefit of this method is that it does not rely on evaluating any characteristic function, thus the method is stable even for guiding structures consisting of multiple and thick layers. Furthermore, the method can be applied to waveguides with lossy materials and generalizes to the case where the stack has an arbitrary index profile. There are several numerical methods and software packages available for solving polynomial eigenvalue problems, they have been reviewed in [18].

The outline of this article is as follows. In Section 2 we briefly review the derivation of the characteristic function and explain the source of the numerical instabilities of its evaluation. Section 3 derives the new variational forms and their discretizations. Finally, in Section 5 we present numerical results with the variational formulation.

2 Characteristic function

A material consisting of $J+1$ layers is completely described by $J+1$ refractive indices $n_0,\cdots,n_J$ and the positions $x_1,\cdots,x_J$ of the J interfaces, c.f. Fig. 1. The layer with index 0 is the cover and layer J is the substrate. The other layers are referred to as interior layers. To simplify notations we shift the origin to the first interface, i.e., $x_1=0$ and write $x_J=w$.

Since the transverse wave numbers

$$\alpha_j=\sqrt{k^2n_j^2-\beta^2} \tag{2.1}$$

are constant in each layer, the solutions of (1.1) and (1.2) consist of a left- and a right going harmonic in the interior layers. The guided and leaky modes are characterized by *one* harmonic in the semi infinite layers. Hence such a mode has the form

$$\phi(x)=\begin{cases} \exp(i\alpha_0 x)\phi(0), & x\le 0,\\ \cos(\alpha_j(x-x_j))\phi(x_j)+\sin(\alpha_j(x-x_j))/\alpha_j\phi'(x_j), & x_j\le x\le x_{j+1},\\ \exp(-i\alpha_J(x-w))\phi(w), & w\le x, \end{cases} \tag{2.2}$$

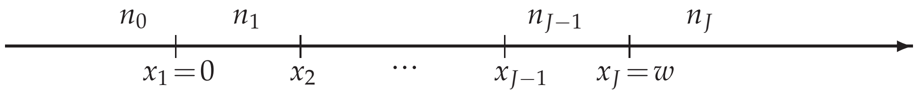

Figure 1: The geometry parameters.

In an interior layer $\phi(x)$ is an even function of the transverse wave number, therefore the branch choice of the square root in (2.1) is irrelevant. On the other hand, the branch choice in the semi-infinite layers determines the direction of propagation and whether a mode is exponentially decaying or increasing. The proper branch choice will be discussed later.

In the TE case, the values $\phi(x_j)$ and $\phi'(x_j)$ at the interfaces are determined from the condition that the function and the derivative are continuous. This condition gives rise to translation operators which describe how the state vectors are mapped from one layer to the next

$$\begin{bmatrix} \phi(x_{j+1}) \\ \phi'(x_{j+1}) \end{bmatrix} = \begin{bmatrix} \cos(\alpha_j w_j) & \sin(\alpha_j w_j)/\alpha_j \\ -\sin(\alpha_j w_j)\alpha_j & \cos(\alpha_j w_j) \end{bmatrix} \begin{bmatrix} \phi(x_j) \\ \phi'(x_j) \end{bmatrix}, \tag{2.3}$$

where $w_j = x_j - x_{j-1}$. In matrix-vector notation (2.3) is $\psi_j = T_j \psi_{j-1}$, where the subscript indicates a translation in positive x-direction through the j-th layer. Suppose the solution of (1.1) is normalized such that $\phi(x_1)=1$. Then it follows directly from (2.2) that $\phi'(x_1)=i\alpha_0$. In this case the state vectors at the other interfaces are products of translation operators

$$\begin{bmatrix} \phi(w) \\ \phi'(w) \end{bmatrix} = T_{J-1} \cdots T_1 \begin{bmatrix} 1 \\ i\alpha_0 \end{bmatrix}. \tag{2.4}$$

If $\phi(x)$ for a given propagation constant β is a solution of (1.1) then the derivatives of the solution given by (2.4) must match the derivative of the solution in the right semi-infinite layer. This leads to the condition $\phi'(w) = -i\alpha_J \phi(w)$. Thus the characteristic function is given by

$$F_{TE}(\beta) = i\alpha_J \phi(w) + \phi'(w). \tag{2.5}$$

For a given β the left hand side of (2.5) can be computed using (2.4), thus $F_{TE}(\beta)$ is a function of the propagation constant and we can solve $F_{TE}(\beta)=0$ numerically using a nonlinear solver such as Newton's method.

In the TM case, the functions $\phi(x)$ and $\phi'(x)/n^2(x)$ are continuous. A discussion similar to the TE case shows that the state vector must be transferred across the layer using the formula

$$\begin{bmatrix} \phi(w) \\ \phi'(w) \end{bmatrix} = T_{J-1} M_{J-1} \cdots T_1 M_1 \begin{bmatrix} 1 \\ i\alpha_0 \end{bmatrix},$$

where

$$M_j = \begin{bmatrix} 1 & 0 \\ 0 & n_j^2/n_{j-1}^2 \end{bmatrix}.$$

The characteristic function is given by

$$F_{TM}(\beta) = i\alpha_J \phi(w) + \frac{n_J^2}{n_{J-1}^2}\phi'(w),$$

where $\phi'(w)$ is the value of the derivative approached from layer $J-1$.

The translation matrix T_j can be diagonalized $T_j = S_j \Lambda_j S_j^{-1}$ where

$$S_j = \begin{bmatrix} -i & i \\ \alpha_j & \alpha_j \end{bmatrix} \quad \text{and} \quad \Lambda_j = \begin{bmatrix} e^{i\alpha_j w_j} & 0 \\ 0 & e^{-i\alpha_j w_j} \end{bmatrix}.$$

and thus there is an exponential scaling if α_j has a nonzero imaginary part.

To illustrate how this scaling can cause numerical instabilities consider the waveguide of Fig. 2.

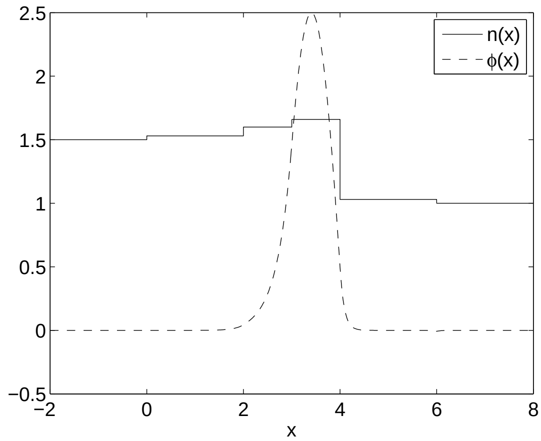

Figure 2: Refractive index and dominant guided mode of an unstable waveguide. The parameters are $w_1 = w_4 = 2$, $w_2 = w_3 = 1$, $n_0 = 1.5$, $n_1 = 1.53$, $n_2 = 1.66$, $n_3 = 1.6$, $n_4 = 1.03$, $n_5 = 1.0$, $k = 9$, $\beta/k \approx 1.63986$.

For $k = 9$, the dominant propagation constant is $\beta/k \approx 1.63986$. Since the refractive index in Layer 4 is smaller than this value, it follows from (2.1) that α_4 is purely imaginary and hence the translation operator T_4 has one exponentially large and one exponentially small eigenvalue.

On the other hand, it can be seen from Fig. 2 that the corresponding mode decreases in this layer. Therefore the transformed state vector $S_j^{-1}[\phi(x_4), \phi'(x_4)]^T$ must have one component that is close to zero as otherwise the solution $\phi(x)$ would grow large in the layer.

In the presence of floating point errors the exponentially growing solution will be excited, and thus errors will be significantly magnified when the state vector is translated across the layer. This effect becomes more noticeable when the width of the layer or the

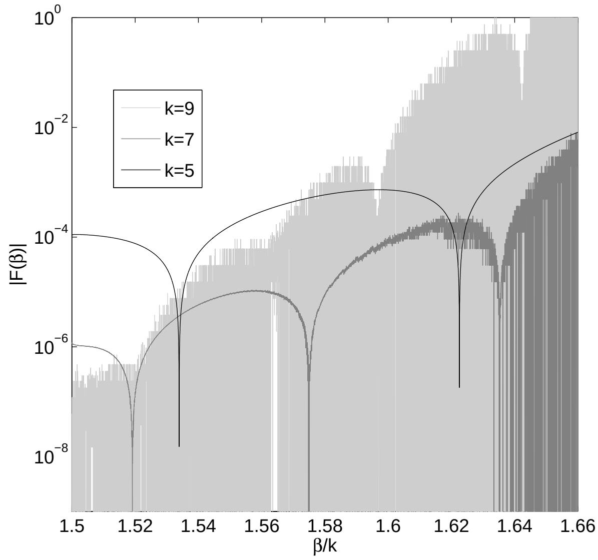

Figure 3: Numerically computed characteristic function of the unstable waveguide in Fig. 2. Three different frequencies. Double precision arithmetic.

frequency is increased. This is illustrated in Fig. 3 which shows the numerically computed characteristic function in the interval of β/k where the guided modes occur. When $k=7$ or $k=9$, the roots disappear in numerical noise.

The mentioned instability can also cause difficulties when computing eigenfunctions even if the computation of the root appears to be stable. Further examples in Section 5 will illustrate this behavior.

3 Variational formulation

We first derive the variational form for the TE polarization. Since the discussion of the TM polarization is completely analogous we will only present the final result at the end of this section.

From the representation (2.2) it follows that the guided and leaky modes are the solutions of the eigenvalue problem

$$\phi''(x)+(k^2n^2(x)-\beta^2)\phi(x)=0, \quad x\in(0,w), \tag{3.1}$$

$$\phi'(0)-i\alpha_0\phi(0)=0, \tag{3.2}$$

$$\phi'(w)+i\alpha_J\phi(w)=0. \tag{3.3}$$

Since the transverse wave numbers α_0 and α_J depend in a nonlinear way on the eigenvalue β^2 the system (3.1)-(3.3) does not constitute a regular Sturm-Liouville problem. The actual nature of this problem becomes more clear in variational form. Thus we multiply (3.1) by a test function and integrate. Using integration by parts gives

$$\begin{aligned} 0 &= \int_0^w \bar{\psi}\left(\phi'' + \left(k^2 n^2 - \beta^2\right)\phi\right) \\ &= \bar{\psi}(w)\phi'(w) - \bar{\psi}(0)\phi'(0) - \int_0^w \bar{\psi}'\phi' + \int_0^w \left(k^2 n^2 - \beta^2\right)\bar{\psi}\phi. \end{aligned}$$

Now we use boundary conditions (3.2) and (3.3) to obtain

$$-\int_0^w \bar{\psi}'\phi' + \int_0^w \left(k^2 n^2 - \beta^2\right)\bar{\psi}\phi - i\alpha_J \bar{\psi}(w)\phi(w) - i\alpha_0\bar{\psi}(0)\phi(0) = 0. \tag{3.4}$$

The ensuing discussion depends on whether the refractive indices in the semi infinite layers are the same or whether they are different. We begin with the former case as it is more straightforward.

3.1 Case 1: $n_0 = n_J$

From (2.1) it follows that $\beta^2 = k^2 n_0^2 - \alpha_0^2$. Furthermore we have $\alpha_0 = \alpha_J$. Letting $q = k^2\left(n^2 - n_0^2\right)$ in Eq. (3.4) gives

$$\int_0^w \bar{\psi}'\phi' - q\bar{\psi}\phi + i\alpha_0\left(\bar{\psi}(w)\phi(w) + \bar{\psi}(0)\phi(0)\right) - \alpha_0^2 \int_0^w \bar{\psi}\phi = 0. \tag{3.5}$$

We introduce the bilinear forms

$$\begin{aligned} a(\psi,\phi) &= \int_0^w \bar{\psi}'\phi' - q\bar{\psi}\phi, \\ a^+(\psi,\phi) &= \bar{\psi}(w)\phi(w) + \bar{\psi}(0)\phi(0), \end{aligned}$$

and write (ψ,ϕ) to denote the usual $L^2(0,w)$-inner product. By letting $z = i\alpha_0$ in equation (3.5) we obtain

$$a(\psi,\phi) + z a^+(\psi,\phi) + z^2(\psi,\phi) = 0. \tag{3.6}$$

This is a quadratic eigenvalue problem in variational form. The task is to find the values of z and the nontrivial ϕ such that (3.6) holds for all ψ. Once z is found the propagation constant can be recovered from (2.1)

$$\beta^2 = k^2 n_0^2 + z^2.$$

3.2 Case 2: $n_0 \neq n_J$

This case is more complicated since it involves a suitable transformation for both α_0 and α_J. To that end we again use (2.1), this time to obtain

$$\beta^2 = \frac{k^2(n_0^2 + n_J^2)}{2} - \frac{\alpha_0^2 + \alpha_J^2}{2} \tag{3.7}$$

and

$$\alpha_0^2-\alpha_J^2=k^2(n_0^2-n_J^2):=\delta^2. \tag{3.8}$$

Next we let $q(x)=k^2[n^2(x)-\frac{1}{2}(n_0^2+n_J^2)]$ so that

$$k^2n^2(x)=q(x)+\frac{k^2}{2}\left(n_0^2+n_J^2\right). \tag{3.9}$$

Using (3.7) and (3.9) we rewrite (3.4) as

$$\int_0^w \bar{\psi}'\phi'-q\bar{\psi}\phi+i\alpha_J\bar{\psi}(w)\phi(w)+i\alpha_0\bar{\psi}(0)\phi(0)-\frac{\alpha_0^2+\alpha_J^2}{2}\int_0^w \bar{\psi}\phi=0. \tag{3.10}$$

Now we introduce the following variables:

$$z_0=\frac{1}{2}(\alpha_0+\alpha_J), \quad z_1=\frac{1}{2}(\alpha_0-\alpha_J). \tag{3.11}$$

which is equivalent to

$$\alpha_0=z_0+z_1, \quad \alpha_J=z_0-z_1. \tag{3.12}$$

These transformations give, using (3.8)

$$z_0z_1=\frac{1}{4}(\alpha_0+\alpha_J)(\alpha_0-\alpha_J)=\frac{1}{4}\left(\alpha_0^2-\alpha_J^2\right)=\frac{1}{4}\delta^2 \tag{3.13}$$

and

$$\frac{\alpha_0^2+\alpha_J^2}{2}=\frac{1}{2}\left[(z_0+z_1)^2+(z_0-z_1)^2\right]=z_0^2+z_1^2. \tag{3.14}$$

As in Case 1, we introduce the bilinear forms

$$a(\psi,\phi)=\int_0^w \bar{\psi}'\phi'-q\bar{\psi}\phi, \tag{3.15}$$

$$a^{\pm}(\psi,\phi)=\bar{\psi}(w)\phi(w)\pm\bar{\psi}(0)\phi(0). \tag{3.16}$$

Then it follows from (3.14) that (3.10) is equivalent to

$$a(\psi,\phi)+iz_0a^+(\psi,\phi)-iz_1a^-(\psi,\phi)-(z_0^2+z_1^2)(\psi,\phi)=0.$$

Since $n_0\neq n_J$ it follows that $\delta^2\neq 0$ and therefore, by Eq. (3.13), z_0 and z_1 are nonzero. Substituting $z_1=\frac{\delta^2}{4z_0}$ into (3.10) leads to

$$a(\psi,\phi)+iz_0a^+(\psi,\phi)-i\frac{\delta^2}{4z_0}a^-(\psi,\phi)-\left(z_0^2+\frac{\delta^4}{16z_0^2}\right)(\psi,\phi)=0.$$

We let $z=iz_0$, multiply by z^2 and rearrange. The resulting equation is

$$\frac{\delta^4}{16}(\psi,\phi)+z\frac{\delta^2}{4}a^-(\psi,\phi)+z^2a(\psi,\phi)+z^3a^+(\psi,\phi)+z^4(\psi,\phi)=0. \tag{3.17}$$

This is a quartic eigenvalue problem. Once the eigenvalues z have been found, the transverse wave numbers and the propagation constant can be recovered from (3.7) and (3.12). This leads to

$$\begin{aligned}\alpha_0 &= i\left(\frac{\delta^2}{4z}-z\right),\\ \alpha_J &= \frac{1}{i}\left(\frac{\delta^2}{4z}+z\right),\\ \beta^2 &= \frac{k^2(n_0^2+n_J^2)}{2}+\frac{\delta^4}{16z^2}+z^2.\end{aligned}$$

The variational form for the TM polarization can be derived in a similar manner. We only state the result

$$\frac{\delta^4}{16}(\psi,\phi)_p+z\frac{\delta^2}{4}b^-(\psi,\phi)+z^2b(\psi,\phi)+z^3b^+(\psi,\phi)+z^4(\psi,\phi)_p=0, \tag{3.18}$$

where the bilinear forms are

$$\begin{aligned}b(\psi,\phi) &= \int_0^w \frac{1}{n^2(x)}\left[\bar{\psi}'\phi'-q\bar{\psi}\phi\right],\\ b^\pm(\psi,\phi) &= \frac{\bar{\psi}(w)\phi(w)}{n_J^2}\pm\frac{\bar{\psi}(0)\phi(0)}{n_0^2},\end{aligned}$$

and $(\cdot,\cdot)_p$ denotes the inner product with respect to weight function $p=\frac{1}{n^2(x)}$.

We conclude with some final comments that are obvious from the definitions.

1. If we let $\delta=0$ and $z\neq 0$ in Case 2 then variational form (3.17) reduces to the variational form for Case 1. The same is true for the TM polarization.
2. The bilinear forms in (3.17) and (3.18) are bounded in $H^1(0,w)$.
3. If the refractive index is real, then all bilinear forms are Hermitian. However, the derivation is also valid for the case of a complex index.

4 Discretization

Using a finite element method, we can approximate the eigenvalues of (3.6) and (3.17). We introduce a partition $\xi=\{x_1=\xi_0<\xi_1<\cdots<\xi_N=x_J\}$ with $N\gg n$. In our partitioning, we ensure that $\{x_j\}\in\xi$. For our basis functions, we use piecewise linear functions, φ_i, so that

$$\varphi_i(x)=\begin{cases}1, & x=\xi_i,\\ \dfrac{x-\xi_{i-1}}{\xi_i-\xi_{i-1}}, & \xi_{i-1}<x<\xi_i,\\ \dfrac{\xi_{i+1}-x}{\xi_{i+1}-\xi_i}, & \xi_i<x<\xi_{i+1},\\ 0, & \text{otherwise.}\end{cases}$$

The usual finite element formulation yields the quartic matrix eigenvalue problem: find z such that

$$A_0+zA_1+z^2A_2+z^3A_3+z^4A_4 \tag{4.1}$$

is singular. We note these matrices are sparse, since from (3.16)

$$\begin{aligned}A_1&=\frac{\delta^2}{4}diag(-1,0,0,\cdots,0,0,1),\\A_3&=diag(1,0,0,\cdots,0,0,1),\end{aligned}$$

and from (3.15)

$$\begin{aligned}A_0(i,j)&=\frac{\delta^4}{16}(\varphi_i,\varphi_j),\\A_2(i,j)&=(\varphi_i',\varphi_j')-(q\varphi_i,\varphi_j),\\A_4(i,j)&=(\varphi_i,\varphi_j).\end{aligned}$$

The discretizations of (3.6) and (3.17) are standard problems in numerical linear algebra, which can be solved with general purpose software packages.

5 Numerical examples

We have implemented the finite element discretization and use Matlab's polyeig routine to solve the eigenvalue problem (4.1). This routine converts the quartic eigenvalue problem into a generalized eigenvalue problem, which is solved using the QZ-factorization. Apparently, the routine does not exploit the sparsity of the matrices and it is conceivable that a Krylov subspace method is a more effective method for solving (4.1). However, for the computations presented here, the polyeig routine was able to find the eigenvalues within 75 seconds of CPU time.

The quartic eigenvalue problem has solutions corresponding to all branch choices of (2.1) in the semiinfinite layers $j=0,J$. Only some of the branches correspond to physical solutions, namely modes which radiate only in the cover, and those which radiate in both the cover and the substrate [11]. Specifically, if $n_J<\frac{Re(\beta)}{k}<n_0$, then we select solutions with $Re(\alpha_0)<0$, $Im(\alpha_0)>0$ and $Re(\alpha_J)>0$, $Im(\alpha_J)<0$. If $\frac{Re(\beta)}{k}<n_J$, we select modes leaking into both the substrate and the cover. Therefore $Re(\alpha_0)<0$, $Im(\alpha_J)>0$ and $Re(\alpha_J)<0$, $Im(\alpha_J)>0$.

We first consider a six-layer structure which has been studied in the literature [1,5]. For this particular guide the characteristic function can be computed in a stable manner: For $k=9.92918$ the dominant guided mode has the propagation $\beta/k=1.622729$, and hence α_2 is purely imaginary. However, $\alpha_2 w_2$ is relatively small and thus the translation operator T_2 is not very ill-conditioned. The same is true for T_3. The mode is shown in Fig. 4.

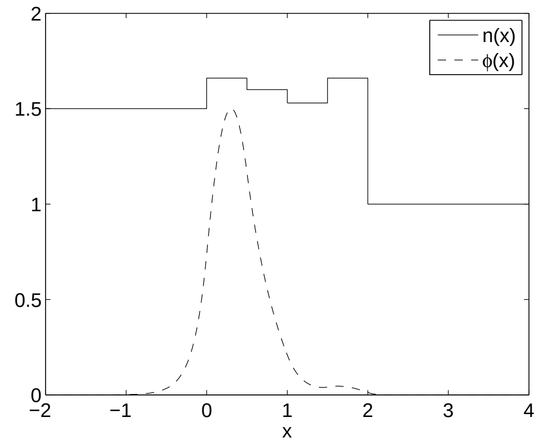

Figure 4: First guided mode, as computed using the characteristic function. The parameters are $w_1 = w_2 = w_3 = w_4 = 0.5$, $n_0 = 1.50$, $n_1 = 1.66$, $n_2 = 1.60$, $n_3 = 1.53$, $n_4 = 1.66$, $n_5 = 1.0$, $k = 9.92918$, $\beta/k \approx 1.63$.

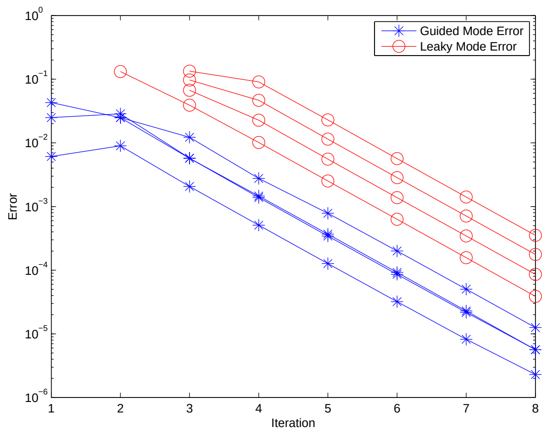

Figure 5: Convergence of eigenvalues as mesh is refined.

Unlike the methods that find the roots of the characteristic function, the finite element method only provides an approximation to the eigenvalue. The approximation is improved by refining the mesh width. In the case of piecewise linear elements the error decays like $\mathcal{O}(h^2)$ where h is the mesh width. The constant in the asymptotic estimates gets larger for the eigenvalues of higher order, see, e.g., [17]. We begin with a coarse uniform mesh of unit width and refine at each iteration by a factor of two, so that in the kth iteration $h = 2^{-k+1}$. In each step we compare the approximations with those computed in [1, 9]. The convergence results for the first eight modes are shown in Fig. 5.

Now we present more detailed results from application of our method to the six-layer waveguide studied above. Since in this example the evaluation of the characteristic function is numerically stable, we use approximate eigenvalues provided by our method as

Table 1: First modes of the six-layer waveguide shown in Fig. 4.

$n_0=1.5, n_1=1.66, n_2=1.60, n_3=1.53, n_4=1.66, n_5=1.0$ $w_1=w_2=w_3=w_4=.5, \lambda_0=.6328$					
	$\frac{Re(\beta)}{k}$	$\frac{Im(\beta)}{k}$	$\frac{Re(\beta^h)}{k}$	$\frac{Im(\beta^h)}{k}$	Newton its
TE_0	1.622729	0.000000	1.622722	0.000000	2
TE_1	1.605276	0.000000	1.607286	0.000000	2
TE_2	1.557136	0.000000	1.557197	0.000000	2
TE_3	1.503587	0.000000	1.504373	0.000000	3
TE_4	1.461857	0.007156	1.466823	0.006705	3
TE_5	1.382489	0.018166	1.388745	0.017075	2
TE_6	1.281364	0.035877	1.260756	0.031881	3
TE_7	1.142314	0.0528761	1.155354	0.026225	3
TE_8	1.003037	0.070771	1.052660	0.020142	4
TE_9	0.804025	0.155492	0.818605	0.070079	4
TE_{10}	0.492614	0.335904	0 .537435	0.132294	4
TE_{11}	0.298779	0.699429	0.161929	0.667641	3
TE_{12}	0.252121	1.005043	0.141075	0.972545	4
TE_{13}	0.222071	1.269684	0.122014	1.228778	4
TE_{14}	0.211786	1.516328	0.115630	1.467105	4

Table 2: First modes of a six-layer lossy waveguide.

$n_0=1.5, n_1=1.66, n_2=1.60, n_3=1.53-i1.53\times10^{-4}, n_4=1.66-i1.66\times10^{-4}, n_5=1.0$ $w_1=w_2=w_3=w_4=.5, \lambda_0=.6328$					
	$\frac{Re(\beta)}{k}$	$\frac{Im(\beta)}{k}\times10^{-4}$	$\frac{Re(\beta^h)}{k}$	$\frac{Im(\beta^h)}{k}\times10^{-4}$	Newton its
TE_0	1.622729	0.00673	1.622722	0.00745	2
TE_1	1.605276	1.66244	1.607286	1.65000	2
TE_2	1.557136	0.20880	1.557193	0.21147	2
TE_3	1.503587	0.55032	1.504373	0.61542	3

an initial guess for Newton's method to compute the propagation up to six significant digits. In Table 1, β^h is the eigenvalue approximation computed by our method. β results from using our approximation as an initial guess and applying Newton's method. We also include the number of Newton steps required to find β to within the specified tolerance. For this data we used $h=2^{-6}$. Note that, in general, as the mode order increases, more Newton steps are required. This is consistent with the theory illustrated by Fig. 5, i.e., for a fixed mesh size, as the mode order increases, the error in the finite element approximation will also increase. The results given in Table 1 agree within the displayed number of digits with those presented in [1,9].

Our method can also be applied to waveguides whose parameters are complex. The results presented in Table 2 also agree with [1].

The previous two examples are stable and hence one can reliably compute the modes

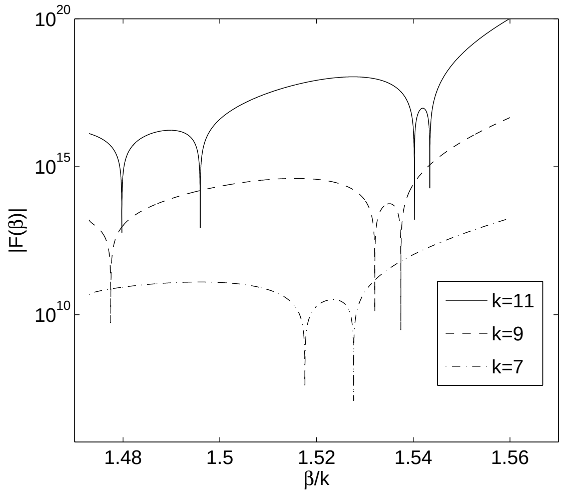

Figure 6: Characteristic function for the guide of Fig. 7. Three different frequencies.

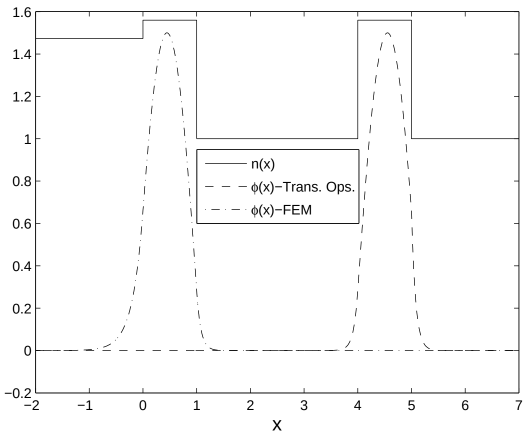

Figure 7: Comparison of first guided mode, computed by translation operators and FEM.The parameters are $w_1 = w_3 = 1$, $w_2 = 3$, $n_0 = 1.473$, $n_1 = 1.56$, $n_2 = 1.0$, $n_3 = 1.56$, $n_4 = 1.0$, $k = 11$, $\beta/k \approx 1.5434$.

with the characteristic function as well as with the variational approach. For an unstable guide, such as the one of Fig. 2 only the variational method leads to usable results. In our experimentations we also found examples where Newton's method applied to the characteristic function yields the proper propagation constants, but fails to provide the correct eigenfunction.

To illustrate the difficulties that may arise, we consider a two-channel waveguide similar to those studied in [3, 21]. The geometry of this structure is shown in Fig. 7. For $k = 11$ there are four guided modes. Newton's method applied to the characteristic function converges to the right roots, albeit it is impossible to reduce the residual to less than 10^3 in magnitude. This can be explained by the fact that the translation operator T_2

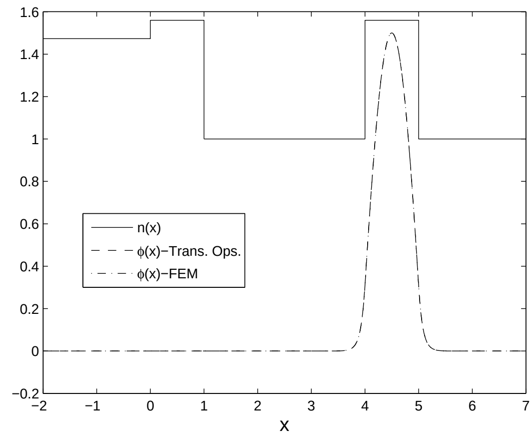

Figure 8: Comparison of second guided mode, computed by translation operators and FEM.

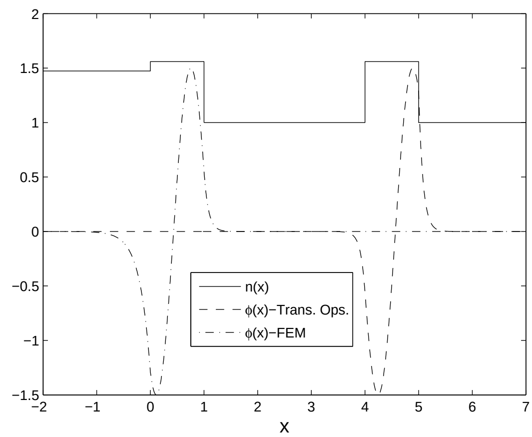

Figure 9: Comparison of third guided mode, computed by translation operators and FEM.

is unstable. However, since the other function values are in the range of $10^{15}-10^{20}$, the roots can be still identified; see Fig. 6.

Figs. 7-10 compare the four guided modes computed by the variational method with the modes computed using translation operators. Since the two channels are well separated, the modes are similar to the modes of each individual channel. In particular, the first mode of the two channel structure is mainly supported by one channel and the second mode by the other channel. Then, modes three and four are again supported by first and second channel respectively. We see that the variational method reproduces the expected behavior. On the other hand, all modes computed by translation matrices are supported by the right channel, which can be explained by the conditioning of the translation operator.

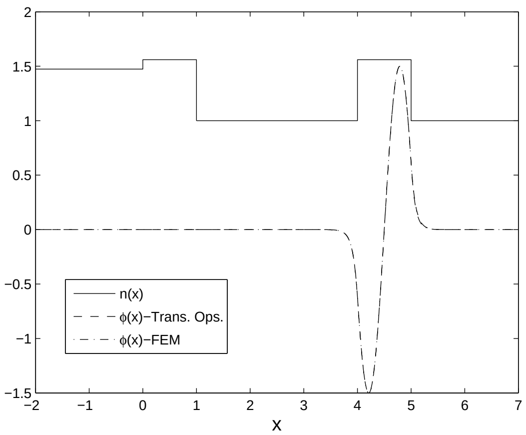

Figure 10: Comparison of fourth guided mode, computed by translation operators and FEM.

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