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Existence and Uniqueness of the Weak Solution of the Space-Time Fractional Diffusion Equation and a Spectral Method Approximation

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Abstract. In this paper, we investigate initial boundary value problems of the space-time fractional diffusion equation and its numerical solutions. Two definitions, i.e., Riemann-Liouville definition and Caputo one, of the fractional derivative are considered in parallel. In both cases, we establish the well-posedness of the weak solution. Moveover, based on the proposed weak formulation, we construct an efficient spectral method for numerical approximations of the weak solution. The main contribution of this work are threefold: First, a theoretical framework for the variational solutions of the space-time fractional diffusion equation is developed. We find suitable functional spaces and norms in which the space-time fractional diffusion problem can be formulated into an elliptic weak problem, and the existence and uniqueness of the weak solution are then proved by using existing theory for elliptic problems. Secondly, we show that in the case of Riemann-Liouville definition, the well-posedness of the space-time fractional diffusion equation does not require any initial conditions. This contrasts with the case of Caputo definition, in which the initial condition has to be integrated into the weak formulation in order to establish the well-posedness. Finally, thanks to the weak formulation, we are able to construct an efficient numerical method for solving the space-time fractional diffusion problem.

AMS subject classifications: 35S10, 35A05, 65M70, 65M12

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1 Introduction

Fractional partial differential equations (FPDEs) appear in the investigation of transport dynamics in complex systems which are characterized by the anomalous diffu-

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sion and nonexponential relaxation patterns [31]. Related equations of importance are the space/time fractional diffusion equations, the fractional advection-diffusion equation [17, 18] for anomalous diffusion with sources and sinks, and the fractional Fokker-Planck equation [4] for anomalous diffusion in an external field, etc. In fact, it has been shown (see, for example, [6,12,13,47,49]) that anomalous diffusion is ubiquitous in physical and biological systems where trapping and binding of particles can occur. Anomalous diffusion deviates from the standard Fichean description of Brownian motion, the main character of which is that its mean squared displacement is a nonlinear growth with respect to time, such as $\langle x^2(t)\rangle \sim t^\alpha$.

The space-time fractional diffusion equation (STFDE) considered in this paper is of interest not only in its own right, but also in that it constitutes the principal part in solving many other FPDEs. The physical background includes modeling turbulent flow, chaotic dynamics charge transport in amorphous semiconductors [43, 44], NMR diffusometry in disordered materials [32], and dynamics of a bead in polymer network [3]. In [33], Nigmatullin used the fractional diffusion equation to describe diffusion in media with fractal geometry. Mainardi [29] pointed out that the propagation of mechanical diffusive wave in viscoelastic media can be modeled by STFDE.

The universality of anomalous diffusion phenomenon in physical and biological experiments has led to an intensive investigation on the fractional differential equations in recent years. For example, the TFDE and related equations have been investigated in analytical and numerical frames by a number of authors [14, 26, 45, 54]. Schneider and Wyss [45] and Wyss [54] investigated the Green functions and their properties for the time fractional diffusion wave equations. Gorenflo et al. [14, 15] used the similarity method and Laplace transform to obtain the scale invariant solution of TFDE in terms of the Wright function. The work done on the numerical solution of the TFDE includes finite difference methods by Liu et al. [27], Sun and Wu [51], Langlands and Henry [20] and so on. More recently, Lin and Xu [23] proposed a finite difference scheme in time and Legendre spectral method in space for TFDE. A convergence rate of $(2-\alpha)$-order in time and spectral accuracy in space of the method was rigourously proved. In [10, 11, 41], Ervin and Roop presented a Galerkin finite element approximation for variational solution to the steady state fractional advection dispersion equations. Very recently, Li and Xu [22] proposed a time-space spectral method for TFDE based on a weak formulation, and detailed error analysis was carried out.

A suitable variational formulation is the starting point of many numerical methods, such as finite element methods and spectral methods. The existence and uniqueness of the variational solution is thus essential for these methods to be efficient. The construction of the variational formulation strongly relies on the choice of suitable spaces and norms. The main contribution of this paper includes: First, we establish the well-posedness of the weak formulation of STFDE, with the help of the introduction of suitable fractional Sobolev spaces and norms. We clearly distinguish two different definitions of the fractional derivative: Riemann-Liouville derivative and Caputo one. We find that in the case of Riemann-Liouville definition there is no need to impose any explicit initial conditions for the well-posedness of the problem, while in the Caputo case an usual initial condition is required to guarantee the uniqueness. Secondly, we construct an efficient spectral method for numerical approximations of the weak solution. Based on the weak formulation and the polynomial approximation results in the related Sobolev spaces, we are able to derive some error estimates. We present also an implementation technique of the algorithm, and some numerical results to confirm the theoretical statements. Finally, we generalize the space-time spectral method to the nonlinear Fokker-Planck equation. A combination of the BICGSTAB and Newton iteration method is proposed to solve the resulting nonsymmetric nonlinear system. Some numerical tests are carried out to demonstrate the efficiency of the method.

The outline of the paper is as follows: In next section we first describe the problems and then introduce necessary functional spaces and investigate their properties. In Section 3, we construct the weak formulations both in space and time directions for the space-time fractional diffusion equation. The existence and uniqueness of the problems are proved. In Section 4 we propose the space-time spectral methods based on the weak formulations and carry out the error analysis. We give some implementation details and present the numerical results in Section 5. In Section 6, we consider a generalization of the space-time spectral method to the nonlinear Fokker-Planck equation. Some concluding remarks are given in Section 7. Finally we give in the appendix an evaluation technique for computing the integrals.

2 Problems and functional spaces

2.1 Notation

We first introduce some notations that will be used throughout the paper. Let

$$\Omega=(-1,1)^d, \qquad I=(0,T), \qquad Q=\Omega\times I,$$

where $d\geq 1$ is the space dimension. We use the symbol $\mathcal{O}$ to denote a domain which may stand for Ω, I, Q or $\mathbb{R}$. Let $L^2(\mathcal{O})$ be the space of measurable functions whose square is Lebesgue integrable in $\mathcal{O}$. The inner product and norm of $L^2(\mathcal{O})$ are defined by

$$(u,v)_{\mathcal{O}}=\int_{\mathcal{O}} uv\,\mathrm{d}\mathcal{O}, \qquad \|u\|_{0,\mathcal{O}}=(u,u)_{\mathcal{O}}^{1/2}, \quad \forall\, u,v\in L^2(\mathcal{O}).$$

For a nonnegative real number s, we use $H^s(\mathcal{O})$ and $H_0^s(\mathcal{O})$ to denote the usual Sobolev spaces, whose norms are denoted by $\|\cdot\|_{s,\mathcal{O}}$ (see [2,24]). Let $C_0^\infty(\mathcal{O})$ stand for the space of all functions having continuous derivatives of all orders and compactly supported in $\mathcal{O}$. For the Sobolev space X with norm $\|\cdot\|_X$, let

$$H^s(I;X):=\big\{v;\ \|v(\cdot,t)\|_X\in H^s(I)\big\},$$

endowed with the norm

$$\|v\|_{H^s(I;X)}:=\big\|\,\|v(\cdot,t)\|_X\big\|_{s,I}.$$

Particularly, when X is $H^\sigma(\Omega)$ or $H_0^\sigma(\Omega)$, $\sigma\geqslant 0$, the norm of the space $H^s(I;X)$ will be denoted by $\|\cdot\|_{\sigma,s,Q}$.

Hereafter, in cases where no confusion would arise, the domain symbols Ω, I, or Q may be dropped from the notations.

We then recall some definitions of fractional derivatives and fractional integrals (see [36, 38]). Let $\Gamma(\cdot)$ denote the Gamma function. For any positive integer n and $n-1\le s<n$, the Caputo derivative, Riemann-Liouville derivative, and fractional integral of order s are respectively defined as

- left Caputo derivative:

$$ {}^C D_t^s v(t)=\frac{1}{\Gamma(n-s)}\int_0^t \frac{v^{(n)}(\tau)\mathrm{d}\tau}{(t-\tau)^{s-n+1}}, \qquad \forall\, t\in[0,T], \tag{D1}$$

- right Caputo derivative:

$$ {}_t^C D^s v(t)=\frac{(-1)^n}{\Gamma(n-s)}\int_t^T \frac{v^{(n)}(\tau)\mathrm{d}\tau}{(\tau-t)^{s-n+1}}, \qquad \forall\, t\in[0,T], \tag{D2}$$

- left Riemann-Liouville derivative:

$$ {}^R D_t^s v(t)=\frac{1}{\Gamma(n-s)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_0^t \frac{v(\tau)\mathrm{d}\tau}{(t-\tau)^{s-n+1}}, \qquad \forall\, t\in[0,T], \tag{D3}$$

- right Riemann-Liouville derivative:

$$ {}_t^R D^s v(t)=\frac{(-1)^n}{\Gamma(n-s)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_t^T \frac{v(\tau)\mathrm{d}\tau}{(\tau-t)^{s-n+1}}, \qquad \forall\, t\in[0,T], \tag{D4}$$

- fractional integral:

$$ I_t^s v(t)=\frac{1}{\Gamma(s)}\int_0^t \frac{v(\tau)\mathrm{d}\tau}{(t-\tau)^{1-s}}, \qquad \forall\, t\in[0,T]. \tag{I1}$$

Same definitions apply for the spatial variable x in place of t in (D1)-(I1).

Let c stand for a generic positive constant independent of any functions and of any discretization parameters. We use the expression $A\lesssim B$ to mean that $A\leqslant cB$, and use the expression $A\cong B$ to mean that $A\lesssim B\lesssim A$.

2.2 Problems and preparations

For $0<\alpha<1$, $1<\beta<2$, we consider the following two space-time fractional diffusion problems:

$$\begin{cases} {}^R D_t^\alpha u(x,t)-p_1 {}^R D_x^\beta u(x,t)-p_2\, {}_x^R D^\beta u(x,t)=f(x,t), & \forall\,(x,t)\in Q,\\ u(x,t)|_{\partial\Omega}=0, & \forall\, t\in I,\\ I_t^{1-\alpha}u(x,0)=0, & \forall\, x\in\Omega, \end{cases} \tag{2.1}$$

and

$$\begin{cases} {}^C D_t^\alpha u(x,t) - p_1 {}^R D_x^\beta u(x,t) - p_2 \, {}_x^R D^\beta u(x,t) = f(x,t), & \forall \, (x,t) \in Q, \\ u(x,t)|_{\partial\Omega} = 0, & \forall \, t \in I, \\ u(x,0) = u_0(x), & \forall \, x \in \Omega, \end{cases} \tag{2.2}$$

where p_1, p_2 are two constants satisfying

$$p_1 + p_2 = 1, \qquad 0 < p_1, p_2 < 1.$$

Note that the assumption on the upper-bound of p_1 and p_2 is technique, and general p_1 and p_2 can be treated by a simple scaling. Note also that different fractional time-derivatives, Riemann-Liouville and Caputo, correspond to different initial conditions, the reason of which will become clear after the well-posedness analysis in the following.

The first part of this paper concerns the investigation of the well-posedness of the problems (2.1) and (2.2). This was motivated by a previous investigation [22] in which we aimed to provide efficient methods for the numerical solution of the time fractional diffusion equation. In order to relax the storage restriction due to the "global time dependence" of the fractional derivative, a spectral method was proposed to solve the fractional diffusion equation. It is known that a suitable variational formulation is essential for the spectral method to be efficient. That was in the paper [22], we started to address the existence and uniqueness of a weak solution of the time fractional diffusion equation. In this paper we follow the same idea and aim to clarify some of the issues raised in the study of the FPDEs. In particular, we improve some basic results given in [22] by providing less restrictive assumptions on the regularity, and thanks to that we will be able to prove that in the case of Riemann-Liouville definition, the boundary value problem of STFDE is well-posed in a weak sense without any explicit initial conditions. By contrast, the well-posedness of STFDE with Caputo definition requests a suitable initial condition, treated weakly in a way similar to a Neumann boundary condition for integer order differential equations. The two definitions will be discussed separately in the next section.

In order to establish the weak formulation of the problems (2.1) and (2.2), we need some preparations. We start with defining some useful functional spaces and giving some properties related to these spaces. Let $\Lambda = (a,b)$, which may stand for I or Ω. For any real $s \geq 0$, we define the spaces

$${}^l H^s(\Lambda) := \{ v;\ \|v\|_{{}^l H^s(\Lambda)} < \infty \}, \tag{2.3}$$

with

$$\|v\|_{{}^l H^s(\Lambda)} := \left(\|v\|_{0,\Lambda}^2 + |v|_{{}^l H^s(\Lambda)}^2 \right)^{\frac{1}{2}}, \qquad |v|_{{}^l H^s(\Lambda)} := \left\| {}^R D_z^s v \right\|_{0,\Lambda}, \tag{2.4}$$

and

$${}^r H^s(\Lambda) := \{ v;\ \|v\|_{{}^r H^s(\Lambda)} < \infty \}, \tag{2.5}$$

with

$$\|v\|_{{}^rH^s(\Lambda)} := \left(\|v\|_{0,\Lambda}^2 + |v|_{{}^rH^s(\Lambda)}^2\right)^{\frac{1}{2}}, \qquad |v|_{{}^rH^s(\Lambda)} := \left\|{}_z^R D^s v\right\|_{0,\Lambda}. \tag{2.6}$$

Let ${}^lH_0^s(\Lambda)$ and ${}^rH_0^s(\Lambda)$ be the closures of $C_0^\infty(\Lambda)$ with respect to the norms $\|v\|_{{}^lH^s(\Lambda)}$ and $\|v\|_{{}^rH^s(\Lambda)}$ respectively. In the above notations the exponents "l" and "r" have been used to indicate respectively the Left and Right fractional derivatives in the norm definitions. The definitions of these spaces differ from the ones of the usual Sobolev spaces. However, as we are going to see, these spaces are indeed equivalent for $s \neq n-1/2$ in a sense to be specified later. To this end, in the usual Sobolev space $H_0^s(\Lambda)$, we also define

$$|v|_{H_0^s(\Lambda)}^* := \left(\frac{({}^RD_z^s v, {}_z^R D^s v)_\Lambda}{\cos(\pi s)}\right)^{\frac{1}{2}}, \qquad \forall\, v \in H_0^s(\Lambda).$$

We will prove that this functional is well defined and equivalent to the usual seminorm $|\cdot|_{H_0^s(\Lambda)}$ in the sense that

$$|v|_{H_0^s(\Lambda)}^* \lesssim |v|_{H_0^s(\Lambda)} \lesssim |v|_{H_0^s(\Lambda)}^*, \qquad \forall\, v \in H_0^s(\Lambda).$$

In fact, a more general result will be proved in Lemma 2.6. Nevertheless, we emphasize that $|\cdot|_{H_0^s(\Lambda)}^*$ is not a seminorm as it does not satisfy the triangular inequality. We first recall the following result.

Lemma 2.1. ([22]) *Let $s>0$, $s \neq n-1/2$. Then the seminorms $|\cdot|_{{}^lH^s(\Lambda)}$, $|\cdot|_{{}^rH^s(\Lambda)}$ and $|\cdot|_{H_0^s(\Lambda)}$ are all equivalent to $|\cdot|_{H_0^s(\Lambda)}^*$ in space $C_0^\infty(\Lambda)$.*

We derive below a number of useful properties related to the Caputo fractional derivatives, Riemann-Liouville derivatives and integrals.

Lemma 2.2. *If $0<p<1/2$, $v \in L^2(\Lambda)$, or if $1/2 \leq p < 1$, $v \in H^s(\Lambda)$, $p-1/2<s<1/2$, then it holds that*

$$\left|I_z^{1-p} v(z)\right|_{z=a^+} = 0. \tag{2.7}$$

Proof. If $0<p<1/2$, $v \in L^2(\Lambda)$, then

$$\begin{aligned}\left|I_z^{1-p} v(z)\right|_{z=a^+} &= \lim_{z\to a^+}\left|\frac{1}{\Gamma(1-p)}\int_a^z \frac{v(\tau)}{(z-\tau)^p}\mathrm{d}\tau\right| \\ &\lesssim \|v\|_{L^2(a,z)} \lim_{z\to a^+}\left|\int_a^z \frac{1}{(z-\tau)^{2p}}\mathrm{d}\tau\right|^{1/2} = 0.\end{aligned}$$

If $1/2\le p<1$, $v\in H^s(\Lambda)$, $s>p-1/2$, we take $r=2/(1+2s)$ and $r'=2/(1-2s)$, such that $1/r+1/r'=1$. Then

$$
\begin{aligned}
\lim_{z\to a^+}\left|I_z^{1-p}v(z)\right| &= \lim_{z\to a^+}\left|\frac{1}{\Gamma(1-p)}\int_a^z \frac{v(\tau)}{(z-\tau)^p}\mathrm{d}\tau\right|\\
&\lesssim \lim_{z\to a^+}\|v(\tau)\|_{L^{r'}(a,z)}\left\|\frac{1}{(z-\tau)^p}\right\|_{L^r(a,z)}\\
&\lesssim \lim_{z\to a^+}\|v(\tau)\|_{H^s(a,z)}\left\|\frac{1}{(z-\tau)^p}\right\|_{L^r(a,z)}=0.
\end{aligned}
$$

In the above deduction, we have used the following embedding result [2,40]

$$
H^s(a,z)\hookrightarrow L^{r'}(a,z),\qquad r'=\frac{2}{1-2s},\qquad 0<s<\frac{1}{2}.
$$

Thus we get (2.7). $\square$

One of the remarkable properties of the Riemann-Liouville fractional derivative is given in the following lemma.

Lemma 2.3. *For real* s, $0<s<1$, *if*

$$
w\in {}^lH^s(\Lambda)\cap H^s(\Lambda),\quad v\in C^\infty(\Lambda),
$$

then

$$
\left({}^RD_z^s w(z),\, v(z)\right)_\Lambda=\left(w(z),\, {}_z^RD^s v(z)\right)_\Lambda. \tag{2.8}
$$

Proof. By using integration by parts, we get

$$
{}_z^RD^s v(z)=\frac{v(b)}{\Gamma(1-s)(b-z)^s}+{}_z^CD^s v(z).
$$

In fact, we have

$$
\begin{aligned}
\mathrm{LHS}&=\frac{-1}{\Gamma(1-s)}\frac{\mathrm{d}}{\mathrm{d}z}\int_z^b\frac{v(\xi)}{(\xi-z)^s}\,\mathrm{d}\xi\\
&=\frac{-1}{\Gamma(1-s)}\left\{\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{v(\xi)(\xi-z)^{1-s}}{1-s}\bigg|_z^b-\frac{1}{1-s}\int_z^b v'(\xi)(\xi-z)^{1-s}\,\mathrm{d}\xi\right]\right\}\\
&=\frac{-1}{\Gamma(1-s)}\left\{\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{v(b)(b-z)^{1-s}}{1-s}\right]-\frac{1}{1-s}\frac{\mathrm{d}}{\mathrm{d}z}\int_z^b v'(\xi)(\xi-z)^{1-s}\,\mathrm{d}\xi\right\}\\
&=\frac{-1}{\Gamma(1-s)}\left\{\frac{-v(b)}{(b-z)^s}-\frac{1}{1-s}\int_z^b\frac{\mathrm{d}}{\mathrm{d}z}\left[v'(\xi)(\xi-z)^{1-s}\right]\,\mathrm{d}\xi+\frac{1}{1-s}v'(\xi)(\xi-z)^{1-s}\big|_{\xi=z}\right\}\\
&=\frac{v(b)}{\Gamma(1-s)(b-z)^s}+\frac{-1}{\Gamma(1-s)}\int_z^b\frac{v'(\xi)}{(\xi-z)^s}\,\mathrm{d}\xi\\
&=\mathrm{RHS}.
\end{aligned}
$$

On the other hand, for $w\in H^s(\Lambda)$, we have, by Lemma 2.2,

$$\lim_{z\to a^+}\int_a^z \frac{w(\tau)}{(z-\tau)^s}\,\mathrm{d}\tau=0.$$

Then, by employing again integration by parts and using the above two equalities, we obtain

$$\begin{aligned}\Big({}^RD_z^s w(z),\,v(z)\Big)_\Lambda &= \frac{1}{\Gamma(1-s)}\int_a^b \frac{\mathrm{d}}{\mathrm{d}z}\int_a^z \frac{w(\tau)}{(z-\tau)^s}\,\mathrm{d}\tau\; v(z)\,\mathrm{d}z\\ &=\frac{v(z)}{\Gamma(1-s)}\int_a^z \frac{w(\tau)}{(z-\tau)^s}\,\mathrm{d}\tau\Big|_a^b-\frac{1}{\Gamma(1-s)}\int_a^b\int_a^z \frac{w(\tau)}{(z-\tau)^s}\,\mathrm{d}\tau\; v'(z)\,\mathrm{d}z\\ &=\frac{v(b)}{\Gamma(1-s)}\int_a^b \frac{w(\tau)}{(b-\tau)^s}\,\mathrm{d}\tau-\frac{1}{\Gamma(1-s)}\int_a^b\int_\tau^b \frac{v'(z)}{(z-\tau)^s}\,\mathrm{d}z\; w(\tau)\,\mathrm{d}\tau\\ &=\int_a^b w(\tau)\Big[\frac{v(b)}{\Gamma(1-s)(b-\tau)^s}+\frac{-1}{\Gamma(1-s)}\int_\tau^b \frac{v'(z)}{(z-\tau)^s}\,\mathrm{d}z\Big]\,\mathrm{d}\tau\\ &=\Big(w(\tau),\,{}_\tau^RD^s v(\tau)\Big)_\Lambda.\end{aligned}$$

So, the lemma is proved. □

For general positive real s, we have the following result.

Lemma 2.4. *For all positive real s, if $w\in {}^lH^s(\Lambda)$, $v\in C_0^\infty(\Lambda)$, then*

$$\Big({}^RD_z^s w(z),\,v(z)\Big)_\Lambda=\Big(w(z),\,{}_z^RD^s v(z)\Big)_\Lambda. \tag{2.9}$$

Proof. Let n be the integer such that $n-1\le s<n$. By repeating integration by parts n times, we get

$${}_z^RD^s v(z)={}_z^CD^s v(z)+\sum_{j=0}^{n-1}\frac{v^{(j)}(b)(b-z)^{j-s}}{\Gamma(1+j-s)}={}_z^CD^s v(z). \tag{2.10}$$

In virtue of the definition of ${}^RD_z^s w$, we have, for the left hand side of (2.9)

$$\begin{aligned}\Big({}^RD_z^s w(z),\,v(z)\Big)_\Lambda &=\frac{1}{\Gamma(n-s)}\int_a^b \frac{\mathrm{d}^n}{\mathrm{d}z^n}\int_a^z \frac{w(\tau)}{(z-\tau)^{s-n+1}}\,\mathrm{d}\tau\; v(z)\,\mathrm{d}z\\ &=\frac{(-1)^n}{\Gamma(n-s)}\int_a^b\int_a^z \frac{w(\tau)}{(z-\tau)^{s-n+1}}\,\mathrm{d}\tau\; v^{(n)}(z)\,\mathrm{d}z\\ &=\frac{(-1)^n}{\Gamma(n-s)}\int_a^b\int_\tau^b \frac{v^{(n)}(z)}{(z-\tau)^{s-n+1}}\,\mathrm{d}z\; w(\tau)\,\mathrm{d}\tau\\ &=\Big(w(\tau),\,{}_\tau^CD^s v(\tau)\Big)_\Lambda.\end{aligned}$$

Finally, using (2.10) gives (2.9). □

For a given positive real s, the fractional derivative ${}^R D_z^s v$ can be generalized for all $v \in L^2(\Lambda)$ in the following way: for $v \in L^2(\Lambda)$, we define the linear functional, denoted still by ${}^R D_z^s v$: $C_0^\infty(\Lambda) \to \mathbb{R}$, through

$$ {}^R D_z^s v(\phi) := \int_\Lambda v \, {}_z^R D^s \phi \mathrm{d}z, \quad \forall \, \phi \in C_0^\infty(\Lambda). \tag{2.11} $$

Then it can be verified that ${}^R D_z^s v(\phi)$ is continuous in $C_0^\infty(\Lambda)$. In fact, for all $\phi_j \in C_0^\infty(\Lambda)$, such that

$$ \|\phi_j^{(m)}\|_\infty \to 0, \quad \forall \, m \in Z, \quad \text{as } j \to \infty, $$

we have, with n being the integer such that $n-1 \le s < n$,

$$ \begin{aligned} \left| {}^R D_z^s v(\phi_j) \right| &= \left| \int_\Lambda v \, {}_z^R D^s \phi_j \, \mathrm{d}z \right| \le \|v\|_{L^2} \left\| {}_z^R D^s \phi_j \right\|_{L^2} = \|v\|_{L^2} \left\| {}_z^C D^s \phi_j \right\|_{L^2} \\ &= \|v\|_{L^2} \left\| \frac{1}{\Gamma(n-s)} \int_z^b \frac{\phi_j^{(n)}(\tau)}{(\tau - z)^{s-n+1}} \mathrm{d}\tau \right\|_{L^2} \\ &\lesssim \|v\|_{L^2} \left\| \phi_j^{(n)}(z) \right\|_\infty \left\| \int_z^b \frac{1}{(\tau - z)^{s-n+1}} \mathrm{d}\tau \right\|_{L^2} \\ &\lesssim \|v\|_{L^2} \left\| \phi_j^{(n)}(z) \right\|_\infty \left\| (b-z)^{n-s} \right\|_{L^2} \to 0, \quad \text{as } j \to \infty. \end{aligned} $$

Thus ${}^R D_z^s v$ is a distribution. This means that for $v \in L^2(\Lambda)$, ${}^R D_z^s v$ can be defined as a distribution, which, by virtue of Lemma 2.4, coincides with the standard definition (D3) if $v \in {}^l H^s(\Lambda)$.

Thanks to these properties, we are able to prove the following fundamental results.

Lemma 2.5. *For all positive real s, spaces ${}^l H^s(\Lambda)$ and ${}^r H^s(\Lambda)$ are complete.*

Proof. We only give a proof for ${}^l H^s(\Lambda)$. The completeness of ${}^r H^s(\Lambda)$ can be proved in a similar way. Let v_n be a Cauchy sequence under norm $\|\cdot\|_{{}^l H^s}$, then there exist v, w such that

$$ v_n \to v, \qquad \text{in } L^2(\Lambda), \tag{2.12} $$

$$ {}^R D_z^s v_n \to w, \qquad \text{in } L^2(\Lambda). \tag{2.13} $$

In the following, we want to prove that ${}^R D_z^s v = w$.

On one hand, by (2.13), we have

$$ \int_\Lambda {}^R D_z^s v_n \phi \mathrm{d}z \to \int_\Lambda w \phi \mathrm{d}z, \quad \forall \, \phi \in C_0^\infty(\Lambda). \tag{2.14} $$

On the other hand, by Lemma 2.4 and (2.12), we have

$$ \int_\Lambda {}^R D_z^s v_n \phi \mathrm{d}z = \int_\Lambda v_n {}_z^R D^s \phi \mathrm{d}z \to \int_\Lambda v {}_z^R D^s \phi \mathrm{d}z. $$

Then by (2.11), we obtain

$$\int_\Lambda {}^R D_z^s v_n \phi \mathrm{d}z \to {}^R D_z^s v(\phi). \tag{2.15}$$

By combining (2.14) with (2.15), we conclude

$${}^R D_z^s v = w.$$

This completes the proof. $\square$

Lemma 2.6. *For $s>0$, $s\neq n-1/2$, the spaces ${}^l H_0^s(\Lambda)$, ${}^r H_0^s(\Lambda)$ and $H_0^s(\Lambda)$ are equal and their seminorms are all equivalent to $|\cdot|^*_{H_0^s(\Lambda)}$.*

Proof. It is well known that $H^s(\Lambda)$ is complete. Thus from Lemma 2.5 all the three spaces ${}^l H^s(\Lambda)$, ${}^r H^s(\Lambda)$ and $H^s(\Lambda)$ are complete. In virtue of Lemma 2.1, all the three seminorms $|\cdot|_{{}^l H^s(\Lambda)}$, $|\cdot|_{{}^r H^s(\Lambda)}$ and $|\cdot|_{H_0^s(\Lambda)}$ are equivalent in $C_0^\infty(\Lambda)$. As a consequence we know that the spaces ${}^l H_0^s(\Lambda)$, ${}^r H_0^s(\Lambda)$ and $H_0^s(\Lambda)$ are equal in the sense that their seminorms are equivalent.

It remains to prove $|v|_{{}^l H^s(\Lambda)} \cong |v|^*_{H_0^s(\Lambda)}$. For all $v\in H_0^s(\Lambda)$, there exists a sequence $v_n \in C_0^\infty(\Lambda)$, such that

$$|v-v_n|_{{}^l H^s(\Lambda)} \cong |v-v_n|_{{}^r H^s(\Lambda)} \cong |v-v_n|_{H_0^s(\Lambda)} \to 0, \quad \text{as } n\to+\infty. \tag{2.16}$$

By Lemma 2.1, we have

$$\Big(\frac{({}^R D_z^s v_n, {}_z^R D^s v_n)_{L^2(\Lambda)}}{\cos\pi s}\Big)^{\frac{1}{2}} \cong |v_n|_{{}^l H^s(\Lambda)}. \tag{2.17}$$

Then by applying triangle inequality and (2.16), we get

$$\begin{aligned}
&\Big|\frac{({}^R D_z^s v_n, {}_z^R D^s v_n)_{L^2(\Lambda)}}{\cos\pi s} - \frac{({}^R D_z^s v, {}_z^R D^s v)_{L^2(\Lambda)}}{\cos\pi s}\Big| \\
&\lesssim |v_n|_{{}^l H^s(\Lambda)} |v-v_n|_{{}^r H^s(\Lambda)} + |v|_{{}^r H^s(\Lambda)} |v-v_n|_{{}^l H^s(\Lambda)} \\
&\to 0, \quad \text{as } n\to+\infty,
\end{aligned}$$

and

$$\Big||v|_{{}^l H^s(\Lambda)} - |v_n|_{{}^l H^s(\Lambda)}\Big| \le |v-v_n|_{{}^l H^s(\Lambda)} \to 0, \quad \text{as } n\to+\infty.$$

Thus taking limit in both sides of (2.17) yields

$$\Big(\frac{({}^R D_z^s v, {}_z^R D^s v)_{L^2(\Lambda)}}{\cos\pi s}\Big)^{\frac{1}{2}} \cong |v|_{{}^l H^s(\Lambda)}.$$

Consequently,

$$|v|_{{}^l H^s(\Lambda)} \cong |v|_{{}^r H^s(\Lambda)} \cong |v|_{H_0^s(\Lambda)} \cong |v|^*_{H_0^s(\Lambda)}.$$

This completes the proof. $\square$

Lemma 2.7. *For $0<s<2$, $s\neq 1$, $w\in H_0^{\frac{s}{2}}(\Lambda)$, it holds that*

$$ {}^RD_z^s w = {}^RD_z^{\frac{s}{2}}\,{}^RD_z^{\frac{s}{2}} w, \tag{2.18}$$

in the distribution sense, i.e.,

$$\Big\langle {}^RD_z^s w(z),\phi(z)\Big\rangle_\Lambda = \Big\langle {}^RD_z^{\frac{s}{2}}\,{}^RD_z^{\frac{s}{2}} w(z),\phi(z)\Big\rangle_\Lambda, \quad \forall\, \phi\in C_0^\infty(\Lambda).$$

Furthermore,

$$ {}^RD_z^s w\in H^{-\frac{s}{2}}(\Lambda). \tag{2.19}$$

Proof. By definition (2.11), we have $\forall\phi\in C_0^\infty(\Lambda)$,

$$\begin{aligned}
&\Big\langle {}^RD_z^s w(z),\, \phi(z)\Big\rangle_\Lambda = \big(w(z),\, {}_z^RD^s\phi(z)\big)_\Lambda,\\
&\Big\langle {}^RD_z^{\frac{s}{2}}\,{}^RD_z^{\frac{s}{2}} w(z),\, \phi(z)\Big\rangle_\Lambda = \big({}^RD_z^{\frac{s}{2}} w(z),\, {}_z^RD^{\frac{s}{2}}\phi(z)\big)_\Lambda.
\end{aligned}$$

Applying the decomposition property [36] and Lemma 2.3, we obtain

$$\Big(w(z),\, {}_z^RD^s\phi(z)\Big)_\Lambda = \Big(w(z),\, {}_z^RD^{\frac{s}{2}}\,{}_z^RD^{\frac{s}{2}}\phi(z)\Big)_\Lambda = \Big({}^RD_z^{\frac{s}{2}} w(z),\, {}_z^RD^{\frac{s}{2}}\phi(z)\Big)_\Lambda. \tag{2.20}$$

Thus the equality (2.18) holds in the distribution sense. Furthermore, we have from (2.20) that

$$\begin{aligned}
\big|\big\langle {}^RD_z^s w(z),\, \phi(z)\big\rangle_\Lambda\big| &= \big|\big({}^RD_z^{\frac{s}{2}} w(z),\, {}_z^RD^{\frac{s}{2}}\phi(z)\big)_\Lambda\big|\\
&\leqslant \big\|{}^RD_z^{\frac{s}{2}} w(z)\big\|_{L^2(\Lambda)}\big\|{}_z^RD^{\frac{s}{2}}\phi(z)\big\|_{L^2(\Lambda)}, \quad \forall\, \phi\in C_0^\infty(\Lambda).
\end{aligned}$$

Since $C_0^\infty(\Lambda)$ is dense in $H_0^{\frac{s}{2}}(\Lambda)$, the above inequality remains true for all $v\in H_0^{\frac{s}{2}}(\Lambda)$. Thus (2.19) holds. Hence, the lemma is proved. □

Lemma 2.8. *If $0<s<2$, $s\neq 1$, $w,v\in H_0^{\frac{s}{2}}(\Lambda)$, then*

$$\Big\langle {}^RD_z^s w(z),\, v(z)\Big\rangle_\Lambda = \Big({}^RD_z^{\frac{s}{2}} w(z),\, {}_z^RD^{\frac{s}{2}} v(z)\Big)_\Lambda, \tag{2.21a}$$

$$\Big\langle {}_z^RD^s w(z),\, v(z)\Big\rangle_\Lambda = \Big({}_z^RD^{\frac{s}{2}} w(z),\, {}^RD_z^{\frac{s}{2}} v(z)\Big)_\Lambda. \tag{2.21b}$$

Proof. By the definition of $H_0^{\frac{s}{2}}(\Lambda)$, there exists a sequence $v_n\in C_0^\infty(\Lambda)$, such that

$$\|v_n - v\|_{H^{\frac{s}{2}}(\Lambda)} \to 0, \quad \text{as } n\to+\infty.$$

For all $w\in H_0^{\frac{s}{2}}(\Lambda)$, by virtue of Lemma 2.7, we have

$$\Big\langle {}^RD_z^s w(z),\, v_n(z)\Big\rangle_\Lambda = \Big\langle {}^RD_z^{\frac{s}{2}}\,{}^RD_z^{\frac{s}{2}} w(z),\, v_n(z)\Big\rangle_\Lambda = \Big({}^RD_z^{\frac{s}{2}} w(z),\, {}_z^RD^{\frac{s}{2}} v_n(z)\Big)_\Lambda. \tag{2.22}$$

On one side, it holds

$$\left|\left\langle {}^{R}D_{z}^{s}w,v_{n}\right\rangle_{\Lambda}-\left\langle {}^{R}D_{z}^{s}w,v\right\rangle_{\Lambda}\right|\lesssim\|v-v_{n}\|_{H^{\frac{s}{2}}(\Lambda)}\to 0,\quad \text{as } n\to+\infty.$$

On the other side, applying Lemma 2.6 yields

$$\begin{aligned}&\left|\left({}^{R}D_{z}^{\frac{s}{2}}w,\,{}_{z}^{R}D^{\frac{s}{2}}v_{n}\right)_{\Lambda}-\left({}^{R}D_{z}^{\frac{s}{2}}w,\,{}_{z}^{R}D^{\frac{s}{2}}v\right)_{\Lambda}\right|\\ \leqslant&\left\|{}^{R}D_{z}^{\frac{s}{2}}w\right\|_{0}\left|v-v_{n}\right|_{{}^{r}H^{\frac{s}{2}}(\Lambda)}\\ \lesssim&\left\|{}^{R}D_{z}^{\frac{s}{2}}w\right\|_{0}\left\|v-v_{n}\right\|_{H^{\frac{s}{2}}(\Lambda)}\to 0,\quad \text{as } n\to+\infty.\end{aligned}$$

Then by taking limit on both sides of (2.22), we obtain (2.21a). The result (2.21b) can be derived similarly. □

Remark 2.1. Properties (2.21a) has been proved in Lemma 2.6 of [22] under a stronger assumption, i.e., assuming w a function in $H^1(\Lambda)$ with $w(a)=0$. The present paper improves the corresponding results by removing this restriction. This is an important point which allows establishing suitable weak formulations for the problems under consideration.

3 Existence and uniqueness of the weak solution

We first define the space

$$B^{s,\sigma}(Q):=H^{s}\left(I;L^{2}(\Omega)\right)\cap L^{2}\left(I;H_{0}^{\sigma}(\Omega)\right),$$

equipped with the norm

$$\|v\|_{B^{s,\sigma}}:=\left(\|v\|^{2}_{H^{s}\left(I;L^{2}(\Omega)\right)}+\|v\|^{2}_{L^{2}\left(I;H^{\sigma}(\Omega)\right)}\right)^{1/2}.$$

3.1 Riemann Liouville weak formulation

We consider the weak formulation of problem (2.1) as follows: for $f\in B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)'$, find $u\in B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)$, such that

$$\mathcal{A}(u,v)=\mathcal{F}_{1}(v),\qquad \forall\, v\in B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q), \tag{3.1}$$

where the bilinear form $\mathcal{A}(\cdot,\cdot)$ is defined by

$$\mathcal{A}(u,v):=\left({}^{R}D_{t}^{\frac{\alpha}{2}}u,\,{}_{t}^{R}D^{\frac{\alpha}{2}}v\right)_{Q}-p_{1}\left({}^{R}D_{x}^{\frac{\beta}{2}}u,\,{}_{x}^{R}D^{\frac{\beta}{2}}v\right)_{Q}-p_{2}\left({}_{x}^{R}D^{\frac{\beta}{2}}u,\,{}^{R}D_{x}^{\frac{\beta}{2}}v\right)_{Q}, \tag{3.2}$$

and the functional $\mathcal{F}_1(\cdot)$ is given by

$$\mathcal{F}_{1}(v):=\langle f,v\rangle_{Q}.$$

As usual, the notation $B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)'$ has been used to mean the dual space of $B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)$. $\langle\cdot,\cdot\rangle_Q$ stands for the duality between $B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)'$ and $B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)$.

Thanks to the preparation in the previous section, we are able to prove the well-posedness of the weak problem (3.1), which is presented in the next theorem.

Theorem 3.1. *For all $0<\alpha<1$, $1<\beta<2$ and $f\in B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)'$, problem* (3.1) *admits a unique solution. Furthermore, if u is its solution, then it holds*

$$\|u\|_{B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)}\lesssim\|f\|_{B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)'}. \tag{3.3}$$

Proof. The existence and uniqueness of the solution is guaranteed by the well-known Lax-Milgram lemma. It consists in verifying the continuity and coercivity of the bilinear form $\mathcal{A}$ in $B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)\times B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)$. First, it follows from Lemma 2.6, $\forall\, u,v\in B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)$, that

$$\begin{aligned}
&|\mathcal{A}(u,v)|\\
\le&\|{}^RD_t^{\frac{\alpha}{2}}u\|_{L^2(Q)}\|{}_t^RD^{\frac{\alpha}{2}}v\|_{L^2(Q)}+\|{}^RD_x^{\frac{\beta}{2}}u\|_{L^2(Q)}\|{}_x^RD^{\frac{\beta}{2}}v\|_{L^2(Q)}+\|{}_x^RD^{\frac{\beta}{2}}u\|_{L^2(Q)}\|{}^RD_x^{\frac{\beta}{2}}v\|_{L^2(Q)}\\
\lesssim&\|u\|_{H^{\frac{\alpha}{2}}(I;L^2(\Omega))}\|v\|_{H^{\frac{\alpha}{2}}(I;L^2(\Omega))}+\|u\|_{L^2(I;H^{\frac{\beta}{2}}(\Omega))}\|v\|_{L^2(I;H^{\frac{\beta}{2}}(\Omega))}\\
\lesssim&\|u\|_{B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)}\|v\|_{B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)}.
\end{aligned}$$

This means the continuity of $\mathcal{A}$. Then for the coercivity, we have from the same lemma, for all $v\in B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)$,

$$\begin{aligned}
&\mathcal{A}(v,v)\\
=&\big({}^RD_t^{\frac{\alpha}{2}}v,\,{}_t^RD^{\frac{\alpha}{2}}v\big)_Q-p_1\big({}^RD_x^{\frac{\beta}{2}}v,\,{}_x^RD^{\frac{\beta}{2}}v\big)_Q-p_2\big({}_x^RD^{\frac{\beta}{2}}v,\,{}^RD_x^{\frac{\beta}{2}}v\big)_Q\\
\cong&\cos\Big(\frac{\pi\alpha}{2}\Big)\big({}^RD_t^{\frac{\alpha}{2}}v,\,{}^RD_t^{\frac{\alpha}{2}}v\big)_Q-p_1\cos\Big(\frac{\pi\beta}{2}\Big)\big({}^RD_x^{\frac{\beta}{2}}v,\,{}^RD_x^{\frac{\beta}{2}}v\big)_Q-p_2\cos\Big(\frac{\pi\beta}{2}\Big)\big({}^RD_x^{\frac{\beta}{2}}v,\,{}^RD_x^{\frac{\beta}{2}}v\big)_Q\\
\gtrsim&\|v\|^2_{B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)}.
\end{aligned}\tag{3.4}$$

To derive the stability, we take $v=u$ in (3.1), then use (3.4) to get

$$\|u\|^2_{B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)}\lesssim\langle f,u\rangle_Q\lesssim\|f\|_{B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)'}\|u\|_{B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)}. \tag{3.5}$$

The estimate (3.3) is then obtained. □

The link between variational formulation (3.1) and problem (2.1) is stated below.

Theorem 3.2. *For all $0<\alpha<1$, $1<\beta<2$, $f\in B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)'$, if u is a solution of problem* (2.1)*, then u is also a solution of weak form* (3.1)*. Reciprocally, if u is the solution of weak form* (3.1)*, then it is also a solution of problem* (2.1) *in the distribution sense.*

Proof. First, if u is a solution of problem (2.1), then obviously $u\in B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)$. By multiplying the first equation of (2.1) by any $v\in B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)$, integrating the resulting equation over Q, and then using Lemma 2.8 respectively with $z=t$ for the first term and with $z=x$ for the second and third terms, we obtain (3.1).

Inversely, if u is the solution of weak form (3.1), then by using Lemma 2.8, we get the first equation of (2.1) in the distribution sense. The boundary condition is guaranteed by the fact that $u(\cdot,t)\in H_0^{\beta/2}(\Omega)$ for almost every $t\in I$. The initial condition is derived from Lemma 2.2. ☐

3.2 Caputo weak formulation

The construction of weak formulations for the problem with Caputo derivative is more delicate. We may think about defining a similar space as ${}^lH^s(I)$ and ${}^rH^s(I)$ in (2.3) and (2.5) with Caputo derivatives. For example, we may try to define the weak solution space for Caputo problem (2.2) as follows

$${}^l\tilde{H}^s(I):=\{v;\ \|v\|_{{}^l\tilde{H}^s(I)}<\infty\},$$

with

$$\|v\|_{{}^l\tilde{H}^s(I)}:=\left(\|v\|_{0,I}^2+|v|_{{}^l\tilde{H}^s(I)}^2\right)^{\frac{1}{2}},\qquad |v|_{{}^l\tilde{H}^s(I)}:=\|{}^CD_t^s v\|_{0,I},$$

where we have used the Caputo derivative ${}^CD_t^s$ instead of ${}^RD_t^s$ (see (2.4)) to define the norm. This idea seems quick natural, but would not work for a reason we will see below. That is, we will prove that, for $0<s<1/2$, the space ${}^l\tilde{H}^s(I)$ is not complete. To this end, let's consider the sequence $\{v_n\}$

$$v_n(t)=\begin{cases} nt, & t\in[0,\,1/n],\\ 1, & t\in[1/n,\,1].\end{cases}$$

A direct calculation shows that

$${}^CD_t^s v_n(t)=\begin{cases} \frac{1}{\Gamma(1-s)}\frac{n}{1-s}t^{1-s}, & t\in[0,\,1/n],\\ \frac{n}{\Gamma(2-s)}\left[t^{1-s}-\left(t-\frac{1}{n}\right)^{1-s}\right], & t\in[1/n,\,1].\end{cases}$$

It is verified that $v_n\in L^2(I)$ and ${}^CD_t^s v_n\in L^2(I)$, thus $v_n\in {}^l\tilde{H}^s(I)$. Clearly, if v_n converges in ${}^l\tilde{H}^s(I)$ to a function v, then $v\equiv 1$, ${}^CD_t^s v\equiv 0$, but ${}^CD_t^s v_n \nrightarrow 0$. This means that v_n is not convergent in ${}^l\tilde{H}^s(I)$.

Next we prove that $\{v_n\}$ is in fact a Cauchy sequence. Without loss of generality, we assume that $n \leqslant m \leqslant 2n$, then

$$ {}^CD_t^s v_m - {}^CD_t^s v_n = \begin{cases} \frac{m}{\Gamma(2-s)} t^{1-s} - \frac{n}{\Gamma(2-s)} t^{1-s}, & t \in [0,\, 1/m], \\ \frac{m}{\Gamma(2-s)} \left[t^{1-s} - \left(t - \frac{1}{m}\right)^{1-s} \right] - \frac{n}{\Gamma(2-s)} t^{1-s}, & t \in [1/m,\, 1/n], \\ \frac{m}{\Gamma(2-s)} \left[t^{1-s} - \left(t - \frac{1}{m}\right)^{1-s} \right] - \frac{n}{\Gamma(2-s)} \left[t^{1-s} - \left(t - \frac{1}{n}\right)^{1-s} \right], & t \in [1/n,\, 1]. \end{cases} $$

In different subintervals $[0,\, 1/m]$,$[1/m,\, 1/n]$, and $[1/n,\, 1]$, we have

$$ \begin{aligned} \left\| {}^CD_t^s v_m - {}^CD_t^s v_n \right\|^2_{L^2(0,\frac{1}{m})} &\leqslant 2 \left\| {}^CD_t^s v_m \right\|^2_{L^2(0,\frac{1}{m})} + 2 \left\| {}^CD_t^s v_n \right\|^2_{L^2(0,\frac{1}{m})} \\ &= \frac{m^2}{\Gamma(2-s)^2} \frac{2}{(3-2s)m^{3-2s}} + \frac{n^2}{\Gamma(2-s)^2} \frac{2}{(3-2s)m^{3-2s}} \\ &< \frac{4}{\Gamma(2-s)^2 (3-2s) n^{1-2s}} \to 0, \quad \text{as } m,n \to \infty, \end{aligned} \tag{3.6a} $$

$$ \begin{aligned} \left\| {}^CD_t^s v_m - {}^CD_t^s v_n \right\|^2_{L^2(\frac{1}{m},\frac{1}{n})} &\leqslant 2 \left\| {}^CD_t^s v_m \right\|^2_{L^2(\frac{1}{m},\frac{1}{n})} + 2 \left\| {}^CD_t^s v_n \right\|^2_{L^2(\frac{1}{m},\frac{1}{n})} \\ &\leqslant 2 \left\| {}^CD_t^s v_m \right\|^2_{L^2(\frac{1}{m},\frac{1}{n})} + 2 \left\| {}^CD_t^s v_n \right\|^2_{L^2(0,\frac{1}{n})} \\ &\leqslant 2 \left\| {}^CD_t^s v_m \right\|^2_{L^2(\frac{1}{m},\frac{1}{n})} + \frac{2}{\Gamma(2-s)^2 (3-2s) n^{1-2s}}. \end{aligned} \tag{3.6b} $$

We observe that

$$ t^{1-s} - \left(t - \frac{1}{m}\right)^{1-s} = \frac{(1-s)\xi^{-s}}{m} < \frac{(1-s)m^s}{m}, \qquad \xi \in \left[1/m,\, 1/n\right]. $$

This leads to

$$ \left\| {}^CD_t^s v_m \right\|^2_{L^2(\frac{1}{m},\frac{1}{n})} < \frac{1}{\Gamma(1-s)^2} m^{2s} \left(\frac{1}{n} - \frac{1}{m} \right) < \frac{1}{\Gamma(1-s)^2} m^{2s-1}. $$

Thus

$$ \left\| {}^CD_t^s v_m - {}^CD_t^s v_n \right\|^2_{L^2(\frac{1}{m},\frac{1}{n})} \leqslant \frac{2m^{2s-1}}{\Gamma(1-s)^2} + \frac{2n^{2s-1}}{\Gamma(1-s)^2(3-2s)} \to 0, \quad \text{as } m,n \to \infty. \tag{3.7} $$

For the interval $\left[1/n,\, 1\right]$, we first prove that for fixed $t \in \left[1/n,\, 1\right]$, ${}^CD_t^s v_m$ is decreasing with respect to m when $m \geqslant n$. In fact, if we treat m as a real number, then for fixed t in $\left[1/n,\, 1\right]$, we have

$$ \begin{aligned} \frac{d}{dm} {}^CD_t^s v_m &= \frac{d}{dm} \left\{ \frac{m}{\Gamma(1-s)} \left[t^{1-s} - \left(t - \frac{1}{m}\right)^{1-s} \right] \right\} \\ &= \frac{1}{\Gamma(1-s)} \left[t^{1-s} - \left(t - \frac{1}{m}\right)^{1-s} \right] - \frac{1}{m\Gamma(1-s)} \left(t - \frac{1}{m}\right)^{-s} < 0. \end{aligned} $$

So we can only consider the case $m=2n$. Taking $m=2n$ gives

$$ {}^C D_t^s v_{2n} - {}^C D_t^s v_n = \frac{n}{\Gamma(1-s)} \Big[t^{1-s} - 2\big(t-\frac{1}{2n}\big)^{1-s} + \big(t-\frac{1}{n}\big)^{1-s} \Big]. $$

We now divide $[1/n,\ 1]$ into two subintervals $[1/n,\ 2/n]$ and $[2/n,\ 1]$. In $[1/n,\ 2/n]$, similar to (3.7), we have

$$ \left\| {}^C D_t^s v_{2n} - {}^C D_t^s v_n \right\|_{L^2(\frac{1}{n},\frac{2}{n})} \to 0, \quad \text{as } \ n \to \infty. $$

For $t \in [2/n,\ 1]$, it is observed that

$$ \begin{aligned} \Big| t^{1-s} - 2\big(t-\frac{1}{2n}\big)^{1-s} + \big(t-\frac{1}{n}\big)^{1-s} \Big| &\leqslant \frac{1}{4n^2} s(1-s)\xi(t)^{-s-1} + \frac{1}{2n^2} s(1-s)\eta(t)^{-s-1} \\ &< \frac{1}{n^2}\frac{3}{4} s(1-s)\Big(t-\frac{1}{n}\Big)^{-s-1}, \quad \xi(t),\eta(t) \in \Big[t-\frac{1}{n},t\Big]. \end{aligned} $$

This results in

$$ \begin{aligned} \left\| {}^C D_t^s v_{2n} - {}^C D_t^s v_n \right\|^2_{L^2(\frac{2}{n},1)} &< \frac{1}{\Gamma(1-s)^2} \frac{9}{16} \int_{\frac{2}{n}}^1 \frac{s^2}{n^2} \Big(t-\frac{1}{n}\Big)^{-2s-2} \mathrm{d}t \\ &< \frac{1}{\Gamma(1-s)^2} \int_{\frac{1}{n}}^1 \frac{s^2 t^{-2s-2}}{n^2} \mathrm{d}t \\ &= \frac{1}{\Gamma(1-s)^2} \frac{s^2}{n^2(1+2s)} (n^{2s+1}-1) \to 0, \quad \text{as } \ n \to \infty. \end{aligned} $$

Thus we proved that $\{v_n\}$ is a Cauchy sequence, which does not converge in ${}^l\tilde{H}^s(I)$.

The above investigation indicates that the space ${}^l\tilde{H}^s(I)$ is not a suitable solution space for Caputo problem, and therefore we are led to consider the following weak formulation for problem (2.2): for $f \in B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)'$, find $u \in B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)$, such that

$$ \mathcal{A}(u,v) = \mathcal{F}_2(v), \qquad \forall\, v \in B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q), \tag{3.8} $$

where $\mathcal{A}(\cdot,\cdot)$ is defined in (3.2), and the functional $\mathcal{F}_2(\cdot)$ is given by

$$ \mathcal{F}_2(v) := \langle f,v \rangle_Q + \Big(\frac{u_0(x)t^{-\alpha}}{\Gamma(1-\alpha)},\, v \Big)_Q. $$

In order to establish the link between variational formulation (3.8) and problem (2.2), we first derive the following lemma.

Lemma 3.1. *If* $0<\alpha<1$, $w \in H^1(I)$, $v \in H_0^{\frac{\alpha}{2}}(I)$, *then*

$$ \Big({}^C D_t^\alpha w(t),\, v(t) \Big)_I = \Big({}^R D_t^{\frac{\alpha}{2}} w(t),\, {}_t^R D^{\frac{\alpha}{2}} v(t) \Big)_I - \Big(\frac{w(0)t^{-\alpha}}{\Gamma(1-\alpha)},\, v(t) \Big)_I. \tag{3.9} $$

Proof. For $0<\alpha<1$, $w\in H^1(I)$, it can be checked

$$ {}^{C}D_t^{\alpha}w(t)={}^{R}D_t^{\alpha}w(t)-\frac{w(0)t^{-\alpha}}{\Gamma(1-\alpha)}. $$

Then we get (3.9) by applying Lemma 2.8 to the second term below

$$ \Big({}^{C}D_t^{\alpha}w(t),\,v(t)\Big)_I=\Big({}^{R}D_t^{\alpha}w(t),\,v(t)\Big)_I-\Big(\frac{w(0)t^{-\alpha}}{\Gamma(1-\alpha)},\,v(t)\Big)_I,\quad \forall\, v\in H_0^{\frac{\alpha}{2}}(I). $$

So, the proof is completed. $\square$

The following known result will also be useful.

Lemma 3.2. (Hardy-Littlewood Lemma, [36]) *For $0<s<1$, $1<p<1/s$, the factional integration operator $I_t^s v(t)$ is bounded from L^p into L^q with $q=p/(1-sp)$.*

The link between weak problem (3.8) and strong problem (2.2) is given in the theorem below.

Theorem 3.3. *Suppose $0<\alpha<1$, $1<\beta<2$. If u is a classical solution of problem* (2.2)*, then u is a weak solution of* (3.8)*. Reciprocally, if u is a weak solution of* (3.8)*, and for almost every x, $u(x,\cdot)\in H^1(I)$, ${}^{R}D_x^{\beta}u(x,\cdot)$, ${}_x^RD^{\beta}u(x,\cdot)$ and $f(x,\cdot)\in L^{\frac{1}{\alpha}}(I)$, then u is also a solution of* (2.2)*.*

Proof. If u is a classical solution of problem (2.2), then we have

$$ \big({}^{C}D_t^{\alpha}u,v\big)_Q-p_1\big({}^{R}D_x^{\beta}u,v\big)_Q-p_2\big({}_x^RD^{\beta}u,v\big)_Q=\langle f,v\rangle_Q,\qquad \forall\, v\in B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q). $$

Then (3.8) can be derived by employing Lemma 3.1 with respect to t to the first term, and Lemma 2.8 with respect to x to the second and third terms.

Inversely, if u is a solution of weak form (3.8), i.e., u satisfies

$$ \begin{aligned} &\Big({}^{R}D_t^{\frac{\alpha}{2}}u,\,{}_t^RD^{\frac{\alpha}{2}}v\Big)_Q-p_1\Big({}^{R}D_x^{\frac{\beta}{2}}u,\,{}_x^RD^{\frac{\beta}{2}}v\Big)_Q-p_2\Big({}_x^RD^{\frac{\beta}{2}}u,\,{}^{R}D_x^{\frac{\beta}{2}}v\Big)_Q\\ =&\langle f,v\rangle_Q+\Big(\frac{u_0(x)t^{-\alpha}}{\Gamma(1-\alpha)},v\Big)_Q,\qquad \forall\, v\in B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q), \end{aligned} $$

then employing Lemmas 3.1 and 2.8 yields

$$ \begin{aligned} &\Big({}^{C}D_t^{\alpha}u,\,v\Big)_Q-p_1\Big\langle{}^{R}D_x^{\beta}u,\,v\Big\rangle_Q-p_2\Big\langle{}_x^RD^{\beta}u,\,v\Big\rangle_Q\\ =&\langle f,v\rangle_Q-\Big(\frac{(u(x,0)-u_0(x))t^{-\alpha}}{\Gamma(1-\alpha)},\,v\Big)_Q,\qquad \forall\, v\in C_0^{\infty}(Q). \end{aligned} $$

Thus it holds in the distribution sense

$$ {}^{C}D_t^{\alpha}u-p_1{}^{R}D_x^{\beta}u-p_2\,{}_x^RD^{\beta}u=f-\frac{\big(u(x,0)-u_0(x)\big)t^{-\alpha}}{\Gamma(1-\alpha)}. $$

Furthermore, if $u(x,\cdot)\in H^1(I)$, then $\partial_t u(x,\cdot)\in L^2(I)$, and by Hardy-Littlewood Lemma 3.2 with

$$s=1-\alpha,\quad \text{and}\quad 1<p<\min\{1/1-\alpha,\,2\},$$

we have

$${}^C D_t^\alpha u = I_t^{1-\alpha}\partial_t u(x,t)\in L^{\frac{p}{1-(1-\alpha)p}}(I)\subset L^{\frac{1}{\alpha}}(I),\quad \text{for a.e. } x\in\Omega.$$

This result, together with the assumption on ${}^R D_x^\beta u(x,\cdot)$, ${}_x^R D^\beta u(x,\cdot)$, and $f(x,\cdot)$, leads to

$$\frac{\big(u(x,0)-u_0(x)\big)t^{-\alpha}}{\Gamma(1-\alpha)} = f - {}^C D_t^\alpha u + p_1 {}^R D_x^\beta u + p_2\, {}_x^R D^\beta u \in L^{\frac{1}{\alpha}}(I),\quad \text{for a.e. } x\in\Omega.$$

On the other side, it is readily seen that $t^{-\alpha}$ does not belong to $L^{\frac{1}{\alpha}}(I)$. Thus, it holds necessarily

$$u(x,0)=u_0(x),\quad \text{for a.e. } x\in\Omega.$$

The other direct consequence of the above is

$${}^C D_t^\alpha u - p_1 {}^R D_x^\beta u - p_2\, {}_x^R D^\beta u = f.$$

We finally conclude that u satisfies (2.2), since the boundary condition

$$u(x,t)|_{\partial\Omega}=0,\quad \forall\, t\in I,$$

is evident. □

Remark 3.1. In the weak solution space for both problems (3.1) and (3.8) the solutions are not required to satisfy any initial conditions. In fact there is no sense to define the trace at time $t=0$ for functions in $B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)$ with $\alpha<1$. The initial conditions imposed in (2.1) and (2.2) are obtained only if the weak solutions are regular enough. In the Riemann case, this is guaranteed by Lemma 2.2 as long as $u(x,\cdot)\in H^{\alpha/2}(I)$ for fixed $x\in\Omega$. In the Caputo case, a regularity requirement sufficient to guarantee the initial condition is given in Theorem 3.3.

Remark 3.2. As for partial differential equations of integer order, the regularity of the weak solutions of the STFDE is an another important subject worthy of profound understanding. For the time being, we have very limited knowledge about that.

Theorem 3.4. *For all $0<\alpha<1$ and $f\in B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)'$, problem* (3.8) *admits a unique solution. Furthermore, if u is its solution, then it holds*

$$\|u\|_{B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)} \lesssim \|f\|_{B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)'} + \|u_0\|_{L^2(\Omega)}\|t^{-\alpha}\|_{L^q(I)}, \tag{3.10}$$

where $q=2/(1+\alpha)$.

Proof. The proof of the existence and uniqueness of the solution is similar to that in Theorem 3.1. We only need to derive the stability (3.10). By taking $v=u$ in (3.8), then using (3.4) and Hölder inequality, we get

$$\begin{aligned}
\|u\|^2_{B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)} &\lesssim \langle f,u\rangle_Q + \frac{1}{\Gamma(1-\alpha)}\left(u_0(x)t^{-\alpha},\, u\right)_Q \\
&\lesssim \|f\|_{B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)'}\|u\|_{B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)} + \int_I t^{-\alpha}\int_\Omega u_0(x)u(x,t)\mathrm{d}x\mathrm{d}t \\
&\lesssim \|f\|_{B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)'}\|u\|_{B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)} + \int_I t^{-\alpha}\|u(\cdot,t)\|_{L^2(\Omega)}\|u_0\|_{L^2(\Omega)}\mathrm{d}t \\
&\lesssim \|f\|_{B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)'}\|u\|_{B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)} + \|u_0\|_{L^2(\Omega)}\|t^{-\alpha}\|_{L^q(I)}\|u\|_{L^{q'}(I;L^2(\Omega))},
\end{aligned} \tag{3.11}$$

where

$$q=2/(1+\alpha), \qquad q'=2/(1-\alpha).$$

Furthermore, by the Embedding Theorem [2], we know that

$$H^{\frac{\alpha}{2}}(I)\hookrightarrow L^{q'}(I).$$

Thus

$$\|u\|_{L^{q'}(I;L^2(\Omega))}\lesssim \|u\|_{H^{\frac{\alpha}{2}}(I;L^2(\Omega))}\leq \|u\|_{B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)}. \tag{3.12}$$

Finally, combining (3.11) and (3.12) yields (3.10). □

4 Spectral Galerkin method

In this section we propose and analyze a spectral Galerkin method to solve the initial boundary value problems of STFDE expressed in the weak forms. For the sake of simplification, we only consider the problem with Riemann derivative (3.1). We mention that a similar approach in the time variable has already been analyzed in [22] with slightly different approximation spaces for approximating solutions vanishing at $t=0$. The method to be presented below will work for non-homogeneous initial conditions.

We define $P_M(\Omega)$ (resp. $P_N(I)$) as the polynomials spaces of degree less than or equal to M (resp. N) with respect to x (resp. t). Let

$$P^0_M(\Omega):=P_M(\Omega)\cap H_0^{\frac{\beta}{2}}(\Omega).$$

Then we define the spectral approximation space

$$S_{M,N}:=P^0_M(\Omega)\otimes P_N(I).$$

We now consider the following space-time spectral method to problem (3.1): find $u_{M,N} \in S_{M,N}$, such that

$$\mathcal{A}(u_{M,N}, v_{M,N}) = \mathcal{F}(v_{M,N}), \qquad \forall\, v_{M,N} \in S_{M,N}. \tag{4.1}$$

Since $S_{M,N}$ is a subspace of $B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)$, the well-posedness of problem (4.1) is immediate.

Theorem 4.1. *For all $0<\alpha<1$, $1<\beta<2$ and $f \in L^2(Q)$, discrete problem* (4.1) *admits a unique solution. Furthermore, if $u_{M,N}$ is the solution of* (4.1)*, then $u_{M,N}$ satisfies*

$$\left\| u_{M,N} \right\|_{B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)} \lesssim \|f\|_{0,Q}. \tag{4.2}$$

In order to derive the error estimate for the numerical solution, we need to introduce some approximation operators and investigate their approximation properties.

Theorem 4.2. ([28]) *Let p and s be two real numbers, such that*

$$p \neq n+1/2, \qquad 0 \leq s \leq p.$$

Then there exists an operator $\Pi^{s,0}_{p,N}$, from $H^p \cap H^s_0$ onto P^s_N, such that for any $\varphi \in H^\sigma \cap H^s_0$ with $\sigma \geq p$, we have

$$\|\varphi - \Pi^{s,0}_{p,N}\varphi\|_\nu \lesssim N^{\nu-\sigma}\|\varphi\|_\sigma, \quad \forall\, 0 \leq \nu \leq p. \tag{4.3}$$

In the next lemma, we study the property of the composite approximation operator $\Pi^{0,0}_{\frac{\alpha}{2},N}\Pi^{\frac{\beta}{2},0}_{\frac{\beta}{2},M}$, where the operation $\Pi^{\frac{\beta}{2},0}_{\frac{\beta}{2},M} v(x,t)$ acts on the space variable, while $\Pi^{0,0}_{\frac{\alpha}{2},N}$ acts on the time variable.

Lemma 4.1. *For $0<\alpha<1$, $1<\beta<2$, $\gamma>1$, $\sigma \geqslant 1$. If*

$$v \in H^{\frac{\alpha}{2}}\big(I; H^\sigma(\Omega)\big) \cap H^\gamma\big(I; H_0^{\frac{\beta}{2}}(\Omega)\big),$$

then we have

$$\begin{aligned}
&\left\| {}^R D_x^{\frac{\beta}{2}} \big(v - \Pi^{0,0}_{\frac{\alpha}{2},N}\Pi^{\frac{\beta}{2},0}_{\frac{\beta}{2},M} v\big) \right\|_{0,0} \lesssim M^{\frac{\beta}{2}-\sigma}\|v\|_{\sigma,0} + N^{-\gamma}\|v\|_{\frac{\beta}{2},\gamma}, \\
&\left\| {}^R D_t^{\frac{\alpha}{2}} \big(v - \Pi^{0,0}_{\frac{\alpha}{2},N}\Pi^{\frac{\beta}{2},0}_{\frac{\beta}{2},M} v\big) \right\|_{0,0} \lesssim N^{\frac{\alpha}{2}-\gamma}\|v\|_{0,\gamma} + M^{-\sigma}\|v\|_{\sigma,\frac{\alpha}{2}}.
\end{aligned}$$

Proof. By using estimate (4.3), we get

$$\begin{aligned}
\left\| {}^R D_x^{\frac{\beta}{2}} \big(v - \Pi^{0,0}_{\frac{\alpha}{2},N}\Pi^{\frac{\beta}{2},0}_{\frac{\beta}{2},M} v\big) \right\|_{0,0} &\lesssim \left\| {}^R D_x^{\frac{\beta}{2}} \big(v - \Pi^{\frac{\beta}{2},0}_{\frac{\beta}{2},M} v\big) \right\|_{0,0} + \left\| {}^R D_x^{\frac{\beta}{2}} \Pi^{\frac{\beta}{2},0}_{\frac{\beta}{2},M} \big(v - \Pi^{0,0}_{\frac{\alpha}{2},N} v\big) \right\|_{0,0} \\
&\lesssim M^{\frac{\beta}{2}-\sigma}\|v\|_{\sigma,0} + \left\| \Pi^{\frac{\beta}{2},0}_{\frac{\beta}{2},M} \big(v - \Pi^{0,0}_{\frac{\alpha}{2},N} v\big) \right\|_{\frac{\beta}{2},0} \\
&\lesssim M^{\frac{\beta}{2}-\sigma}\|v\|_{\sigma,0} + \left\| v - \Pi^{0,0}_{\frac{\alpha}{2},N} v \right\|_{\frac{\beta}{2},0} \\
&\lesssim M^{\frac{\beta}{2}-\sigma}\|v\|_{\sigma,0} + N^{-\gamma}\|v\|_{\frac{\beta}{2},\gamma},
\end{aligned}$$

$$\begin{aligned}\left\|{}^{R}D_t^{\frac{\alpha}{2}}\left(v-\Pi^{0,0}_{\frac{\alpha}{2},N}\Pi^{\frac{\beta}{2},0}_{\frac{\beta}{2},M}v\right)\right\|_{0,0}&\lesssim\left\|v-\Pi^{0,0}_{\frac{\alpha}{2},N}v\right\|_{0,\frac{\alpha}{2}}+\left\|\Pi^{0,0}_{\frac{\alpha}{2},N}\left(v-\Pi^{\frac{\beta}{2},0}_{\frac{\beta}{2},M}v\right)\right\|_{0,\frac{\alpha}{2}}\\&\lesssim N^{\frac{\alpha}{2}-\gamma}\|v\|_{0,\gamma}+\left\|v-\Pi^{\frac{\beta}{2},0}_{\frac{\beta}{2},M}v\right\|_{0,\frac{\alpha}{2}}\\&\lesssim N^{\frac{\alpha}{2}-\gamma}\|v\|_{0,\gamma}+M^{-\sigma}\|v\|_{\sigma,\frac{\alpha}{2}}.\end{aligned}$$

So, the lemma is proved. □

We are now in a position to derive the error estimate for the solution of the space-time spectral approximation.

Theorem 4.3. *Let* $0<\alpha<1$, $1<\beta<2$, $\gamma>1$, $\sigma\geqslant1$, *and let* u, $u_{M,N}$ *be respectively the solutions of* (2.1) *and* (4.1). *If*

$$u\in H^{\frac{\alpha}{2}}\left(I;H^{\sigma}(\Omega)\right)\cap H^{\gamma}\left(I;H_0^{\frac{\beta}{2}}(\Omega)\right),$$

then we have

$$\left\|u-u_{M,N}\right\|_{B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)}\lesssim N^{\frac{\alpha}{2}-\gamma}\|u\|_{0,\gamma}+M^{-\sigma}\|u\|_{\sigma,\frac{\alpha}{2}}+M^{\frac{\beta}{2}-\sigma}\|u\|_{\sigma,0}+N^{-\gamma}\|u\|_{\frac{\beta}{2},\gamma}. \tag{4.4}$$

Proof. Following the standard procedure of error estimation for Galerkin methods, we arrive at

$$\left\|u-u_{M,N}\right\|_{B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)}\leqslant\inf_{v_{M,N}\in S_{M,N}}\left\|u-v_{M,N}\right\|_{B^{\frac{\alpha}{2},\frac{\beta}{2}}(Q)}.$$

By taking

$$v_{M,N}=\Pi^{0,0}_{\frac{\alpha}{2},N}\Pi^{\frac{\beta}{2},0}_{\frac{\beta}{2},M}u,$$

in the right hand side and by using Lemma 4.1, we can obtain the estimate (4.4). □

5 Numerical results

In this section, we present some numerical results to demonstrate the efficiency of the proposed space-time spectral method. We start with an implementation technique.

5.1 Implementation

All the integrals involved in (3.1) will be evaluated by using suitable numerical quadratures.For the reason that the time and space fractional derivatives make the integrants non polynomial, we will introduce some Gauss-Lobatto-Jacobi (GLJ) quadratures to exactly evaluate the integrals in partial directions depending on terms. We denote by L_M the Legendre polynomial of degree M. The points of the Gauss-Lobatto formula, denoted by ξ_i^M, are defined by

$$\xi_0^M=-1,\quad \xi_M^M=1,\quad L_M^{'}(\xi_i^M)=0,\quad i=1,\cdots,M-1,$$

where $\xi_0^M < \xi_1^M < \cdots < \xi_M^M$. The associated weights of the Gauss-Lobatto formula are denoted by ρ_i^M, $0 \le i \le M$. The $(M+1)\times(N+1)$ GLL points in Q are then defined by

$$(x_i,t_j) := \big(\xi_i^M,\, (\xi_j^N+1)T/2\big), \quad i=0,1,\cdots,M; \quad j=0,1,\cdots,N.$$

The corresponding weights are

$$\rho_i^M \hat{\rho}_j^N \quad \text{with } \hat{\rho}_j^N := \frac{1}{2}T\rho_j^N, \quad i=0,1,\cdots,M; \quad j=0,1,\cdots,N.$$

We now approximate problem (3.1) by: find $u_{M,N} \in S_{M,N}$, such that

$$\begin{aligned}
&\left({}^R D_t^{\frac{\alpha}{2}} u_{M,N},\, {}_t^R D^{\frac{\alpha}{2}} v_{M,N}\right)_M - p_1\left({}^R D_x^{\frac{\beta}{2}} u_{M,N},\, {}_x^R D^{\frac{\beta}{2}} v_{M,N}\right)_N \\
&\quad - p_2\left({}_x^R D^{\frac{\beta}{2}} u_{M,N},\, {}^R D_x^{\frac{\beta}{2}} v_{M,N}\right)_N = (f, v_{M,N})_{M,N}, \quad \forall\, v_{M,N} \in S_{M,N},
\end{aligned} \tag{5.1}$$

where $(\cdot,\cdot)_M$, $(\cdot,\cdot)_N$ and $(\cdot,\cdot)_{M,N}$ are defined as, for $u,v \in C^0(\bar{Q})$

$$(u,v)_M := \sum_{i=0}^{M} \int_I u(x_i,t) v(x_i,t) \rho_i^M \mathrm{d}t, \tag{5.2}$$

$$(u,v)_N := \sum_{j=0}^{N} \int_\Omega u(x,t_j) v(x,t_j) \hat{\rho}_j^N \mathrm{d}x, \tag{5.3}$$

$$(u,v)_{M,N} := \sum_{i=0}^{M}\sum_{j=0}^{N} u(x_i,t_j) v(x_i,t_j) \rho_i^M \hat{\rho}_j^N. \tag{5.4}$$

It remains to compute the integrals in (5.1) term by term. Let $\{h_i^x : i=0,\cdots,M\}$ be the Lagrangian polynomials associated with GLL points $\{x_i : i=0,\cdots,M\}$ and $\{h_j^t : j=0,1,\cdots,N\}$ be for $\{t_j : j=0,1,\cdots,N\}$. That is, $h_i^x \in P_M(\Omega)$, $h_j^t \in P_N(I)$, such that

$$h_i^x(x_k) = \delta_{ik}, \qquad h_j^t(t_k) = \delta_{jk},$$

where δ denotes the Kronecker function. It is seen that the set $\{h_i^x h_j^t,\, i=1,\cdots,M-1;\, j=0,\cdots,N\}$ forms a basis of $P_M^0(\Omega)\otimes P_N(I)$:

$$P_M^0(\Omega)\otimes P_N(I) = \mathrm{span}\Big\{h_i^x(x)h_j^t(t),\, i=1,\cdots,M-1;\, j=0,\cdots,N\Big\}.$$

By expressing $u_{M,N}$ in this basis

$$u_{M,N}(x,t) = \sum_{i=1}^{M-1}\sum_{j=0}^{N} u_{ij} h_i^x(x) h_j^t(t),$$

and letting the test function $v_{M,N}$ go through all $h_m^x(x)h_n^t(t)$, $m=1,\cdots,M-1$; $n=0,\cdots,N$, we arrive at the matrix form of (5.1)

$$A\boldsymbol{u}B^T - p_1 E\boldsymbol{u}H^T - p_2 E^T\boldsymbol{u}H^T = \boldsymbol{f}, \tag{5.5}$$

where $\boldsymbol{u}=(u_{ij})_{(M-1)\times(N+1)}$ is the unknown vector, $\cdot^T$ means transpose of matrix. The dimension of the matrices A and E are $(M-1)\times(M-1)$, while the matrices B and H are of dimension $(N+1)\times(N+1)$. These matrices are computed as follows:

$$A_{m,i} = \left(h_i^x, h_m^x\right)_M = \sum_{k=0}^{M} h_i^x(x_k)h_m^x(x_k)\rho_k^M = \delta_{im}\rho_m^M, \tag{5.6a}$$

$$H_{n,j} = \left(h_j^t, h_n^t\right)_N = \sum_{k=0}^{N} h_n^t(t_k)h_j^t(t_k)\hat{\rho}_k^N = \delta_{jn}\hat{\rho}_n^N, \tag{5.6b}$$

$$B_{n,j} = \left({}^R D_t^{\frac{\alpha}{2}} h_j^t(t),\ {}_t^R D^{\frac{\alpha}{2}} h_n^t(t)\right)_I = \int_0^T {}^R D_t^{\frac{\alpha}{2}} h_j^t(t)\ {}_t^R D^{\frac{\alpha}{2}} h_n^t(t)\mathrm{d}t, \tag{5.6c}$$

$$E_{m,i} = \left({}^R D_x^{\frac{\beta}{2}} h_i^x(x),\ {}_x^R D^{\frac{\beta}{2}} h_m^x(x)\right)_\Omega = \int_{-1}^1 {}^R D_x^{\frac{\beta}{2}} h_i^x(x)\ {}_x^R D^{\frac{\beta}{2}} h_m^x(x)\mathrm{d}x. \tag{5.6d}$$

The right hand side vector $\boldsymbol{f}$ takes form $(f_{mn})_{(M-1)\times(N+1)}$ with

$$f_{mn} = (f, h_m^x h_n^t)_{M,N} = \sum_{i=0}^{M}\sum_{j=0}^{N} f(x_i,t_j)h_m^x(x_i)h_n^t(t_j)\rho_i^M\hat{\rho}_j^N = f(x_m,t_n)\rho_m^M\hat{\rho}_n^N.$$

The most expensive computation in the above system is the evaluation of the matrices B and E, in which the integrals (5.6c) and (5.6d) must be efficiently computed. The presence of the time and space fractional derivatives make the computations nontrivial. We will present in the appendix a method to efficiently calculate these integrals.

Finally, we found that the nonsymmetric system (5.5) can be solved by using the well known BICGSTAB [40] iteration methods.

5.2 Some tests

To confirm the theoretical result predicted by the error estimate (4.4), we carry out a numerical experiment by considering a (2+1)-D problem

$$\begin{aligned}&{}^R D_t^\alpha u(x,y,t) - \frac{1}{2}{}^R D_x^\beta u(x,y,t) - \frac{1}{2}{}_x^R D^\beta u(x,y,t) - \frac{1}{2}{}^R D_y^\beta u(x,y,t) - \frac{1}{2}{}_y^R D^\beta u(x,y,t)\\ &= f(x,y,t), \qquad \forall\ (x,y,t)\in(-1,1)^2\times(0,1),\end{aligned}$$

with the exact analytical solution

$$u(x,y,t) = \exp(0.3t)\exp(-0.3x^2)\sin\pi y.$$

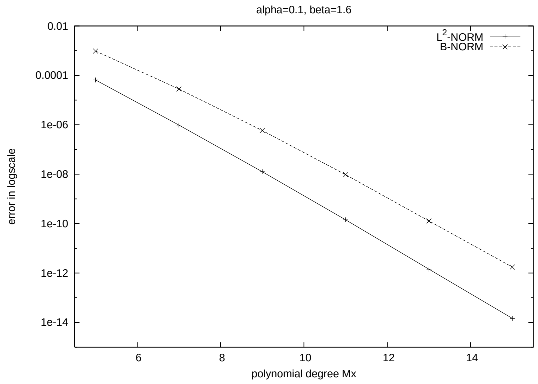

Figure 1: B- and L^2-errors versus M_x with $N=16$, $M_y=20$; $\alpha=0.1$, $\beta=1.6$.

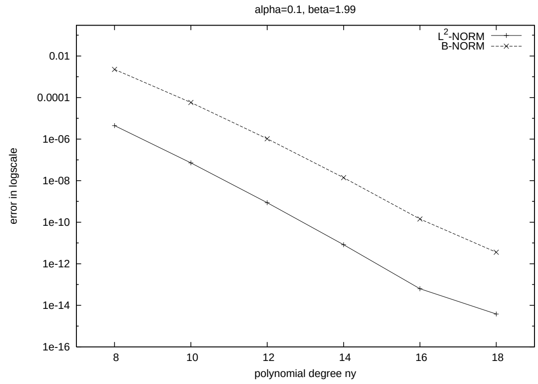

Figure 2: B- and L^2-errors versus M_y with $N=16$, $M_x=18$; $\alpha=0.1$, $\beta=1.99$.

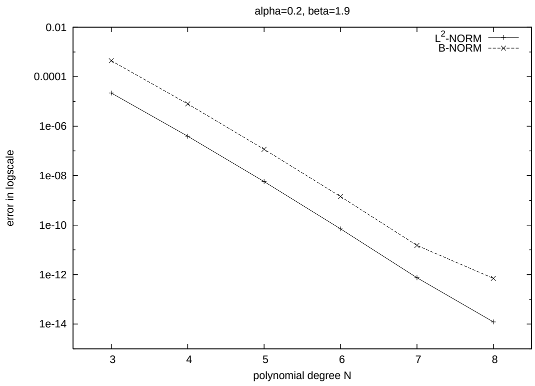

Figure 3: B- and L^2-errors versus N with $M_x=18$, $M_y=20$; $\alpha=0.2$, $\beta=1.9$.

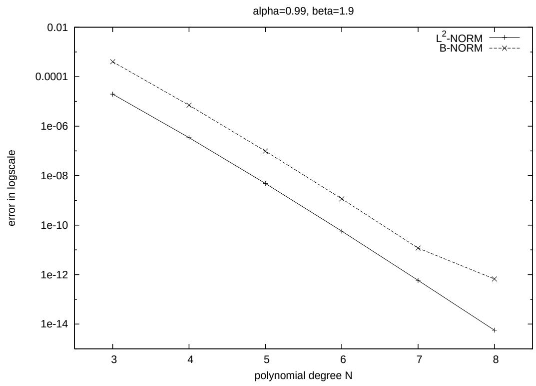

Figure 4: B- and L^2-errors versus N with $M_x=18$, $M_y=20$; $\alpha=0.99$, $\beta=1.9$.

We investigate the convergence behavior of numerical solutions with respect to the polynomial degrees M_x,M_y and N for different α and β. In Figs. 1-4, we plot the L^2-errors and $B^{\frac{\alpha}{2},\frac{\beta}{2}}$-errors in semi-log scale. First, we fix $N=16$, a value large enough such that the time errors are negligible as compared with the space errors. In Fig. 1, we plot the errors as functions of the polynomial degrees M_x for $\alpha=0.1$, $\beta=1.6$. Fig. 2 shows the errors versus M_y for $\alpha=0.1$, $\beta=1.99$. Secondly, we fix $M_x=18$, $M_y=20$ for investigation of the time errors. In Figs. 3-4, we plot the errors as functions of N for values $\alpha=0.2$, 0.99. As expected, all errors show an exponential decay, since in these semi-log representations one observes that the error variations are linear versus the degrees of polynomial. We emphasize that the proposed method seems work too for α close to 1 or β close to 2, even though our theoretical analysis assumes $\alpha<1$ and $\beta<2$.

Next test is related to an examination of the sharpness of the estimate given in (4.4).

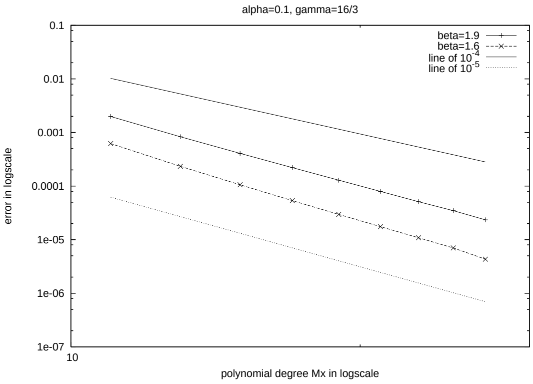

Figure 5: B-errors versus M_x with $N=5$, $M_y=20$ for varying β.

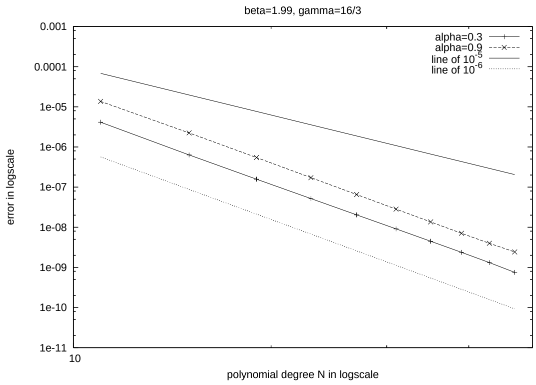

Figure 6: B-errors versus N with $M_x=8$, $M_y=8$ for varying α.

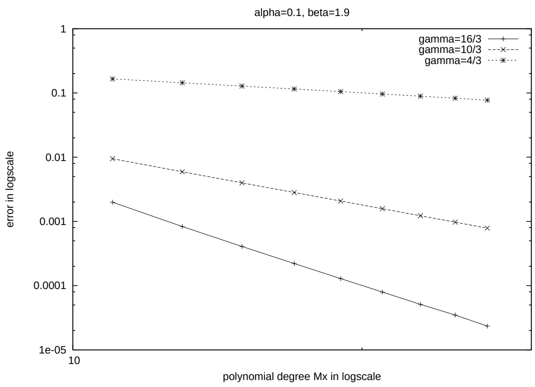

Figure 7: B-errors versus M_x with $N=5$, $M_y=20$ for varying γ.

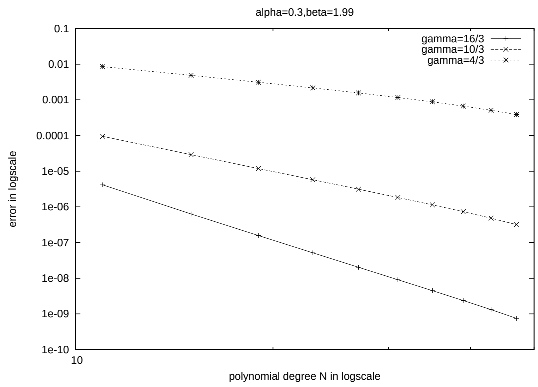

Figure 8: B-errors versus N with $M_x=8$, $M_y=8$ for varying γ.

To this end, we consider the following two exact solutions having limited regularity

$$u(x,t)=t^3(1-x^2)x^\gamma\sin(\pi y),$$
$$u(x,t)=(t+1)(t-1/2)^\gamma(1-x^2)^3(1-y^2)^3,$$

where γ is a constant. It can be verified that the first solution belongs to $H^{\gamma+1/2}$ with respect to the space variable, while the second is a $H^{\gamma+1/2}$ function on the time if γ is not an integer. We plot in Fig. 5 the error decay rates in the $B^{\frac{\alpha}{2},\frac{\beta}{2}}$-norm versus the polynomial degrees M_x with $\alpha=0.1,\gamma=16/3$ for two different values of $\beta=1.6,1.9$. The M_x^{-4} and M_x^{-5} decay rates are also shown for comparison. The error behavior as a function of the polynomial degrees N is plotted in Fig. 6 with β=1.99,γ=16/3 for α=0.3,0.9. Also shown are the N^{-5} and N^{-6} decay rates. It is observed that all the error curves are straight lines in this log-log representation, which indicates the algebraic convergence for these two solutions of limited regularity. Moreover it is seen that the errors decrease with rates conform to the estimate (4.4), which predicts, in Fig. 5, $M_x^{-5.0}$ decay rate for $\beta=1.6$ and $M_x^{-4.8}$ decay rate for $\beta=1.9$, and in Fig. 6, $N^{-5.7}$ decay rate for $\alpha=0.3$ and $N^{-5.3}$ decay rate for $\alpha=0.9$.

The investigation of the convergence behavior for less regular solutions is done by decreasing γ. We plot in Fig. 7 the errors versus M_x and in Fig. 8 the errors versus N for three different values of $\gamma=16/3$, $10/3$, $4/3$. As expected the convergence rate slows down as γ, i.e., the regularity of the solution, decreases.

6 Application to the nonlinear Fokker-Planck equations

One of representative nonlinear Fokker-Planck equations (NLFPE) for describing anomalous diffusion reads

$$\partial_t u(x,t) = -\partial_x [F(x)u(x,t)] + \mathcal{N}\partial_x^2 u^q(x,t), \quad x\in\mathbb{R}, \ t>0. \tag{6.1}$$

This equation was suggested by Plastino-Plastino [37] to describe anomalous diffusion. In (6.1), $u(x,t)$ is the probability density function. $\mathcal{N}$ is the noise amplitude of the system. $F(x)$ denotes the external force. We intend to consider here a specific (but very common) drift, namely characterized by

$$F(x) = k_1 - k_2 x,$$

with k_1 and k_2 two constants. The anomalous diffusion is described by parameter q: $q>1$ corresponds to sub-diffusion, $q<1$ means sup-diffusion. The particular case $F(x)=0$ (no drift) has been considered by Spohn [50] for arbitrary q. Eq. (6.1) has found wide applications in gas and fluid flows in porous media, see, for example, thin saturated regions in porous media $(q=2)$ [39], a solid-on-solid model for surface growth $(q=3)$, thin liquid films spreading under gravity $(q=4)$ [7], and plasma flows $(q<1)$ [9,42].

On the other hand, fractional Fokker-Planck equations (FFPEs) have been introduced as a complementary tool in the description of anomalous transport [31]. They are sometimes considered as alternative approaches to continuous time random walk models. The FFPEs describing the Lévy flights and subdiffusion are derived in [30,31] for anomalous diffusion in the presence of external field. They read

$$\partial_t u(x,t) = {}^R D_t^{1-\alpha}\left[-\partial_x\big(F(x)u(x,t)\big) + \mathcal{K}_\alpha^\beta {}^R D_x^\beta u(x,t)\right], \tag{6.2}$$

where $0<\alpha\leqslant 1$, $1<\beta\leqslant 2$, $\mathcal{K}_\alpha^\beta$ is the modified diffusion coefficient. The anomalous diffusion is related to parameters α and β. $0<\alpha<1$, $\beta=2$ is subdiffusion. $\alpha=1$, $1<\beta<2$ is Lévy flights, and $0<\alpha<1$, $1<\beta<2$ corresponds to the competition between Lévy flights and subdiffusion.

The present paper addresses the unification of both equations as follows:

$$\partial_t u(x,t) = {}^R D_t^{1-\alpha}\left[-\partial_x\big(F(x)u(x,t)\big) + \mathcal{K}_\alpha^\beta {}^R D_x^\beta u^q(x,t)\right]. \tag{6.3}$$

The exact solution of this equation for $\alpha=1$ has been studied in [5,21,48,52,53]. A number of authors have investigated analytic solutions for $\alpha\le 1$ [1,16,34,35,46]. The work on the numerical solution of the nonlinear fractional Fokker-Planck equation is relatively sparse [8,25].

Here, as we are going to see, that the space-time spectral method proposed in the previous section can be directly applied to solve Eq. (6.3) or in an alternative form

$$\begin{cases} {}^{C}D_t^{\alpha}u(x,t)+\partial_x\big(F(x)u(x,t)\big)-{}^{R}D_x^{\beta}u^q(x,t)=f(x,t), & \forall\ (x,t)\in Q,\\ u(-1,t)=u(1,t)=0, & \forall\ t\in I,\\ u(x,0)=0, & \forall\ x\in\Omega, \end{cases} \tag{6.4}$$

where the diffusion coefficient $\mathcal{K}_{\alpha}^{\beta}$ has been dropped down for simplification.

6.1 Numerical method and implementation

We propose the space-time spectral method to (6.4) as follows: find $u_{M,N}\in S_{M,N}$, such that

$$\begin{aligned} &\big({}^{R}D_t^{\frac{\alpha}{2}}u_{M,N},\,{}_t^{R}D^{\frac{\alpha}{2}}v_{M,N}\big)_M+\big(\partial_x[F(x)u_{M,N}],\,v_{M,N}\big)_{M,N}-\big({}^{R}D_x^{\frac{\beta}{2}}u_{M,N}^q,\,{}_x^{R}D^{\frac{\beta}{2}}v_{M,N}\big)_N\\ &=(f,v_{M,N})_{M,N},\qquad \forall\ v_{M,N}\in S_{M,N}. \end{aligned} \tag{6.5}$$

By expressing $F(x)u_{M,N}$ and $u_{M,N}^q$ in form

$$\begin{aligned} F(x)u_{M,N}(x,t)&=\sum_{i=1}^{M-1}\sum_{j=0}^{N}F(x_i)u_{ij}h_i^x(x)h_j^t(t),\\ u_{M,N}^q(x,t)&=\sum_{i=1}^{M-1}\sum_{j=0}^{N}u_{ij}^q h_i^x(x)h_j^t(t), \end{aligned}$$

and taking $h_m^x(x)h_n^t(t)$, $m=1,\cdots,M-1$; $n=0,\cdots,N$, as the test functions in (6.5), we obtain

$$A\boldsymbol{u}B^T+C(F\boldsymbol{u})H^T-E\boldsymbol{u}^qH^T=\boldsymbol{f}, \tag{6.6}$$

where $\boldsymbol{u}=(u_{ij})_{(M-1)\times(N+1)}$ is the unknown vector, $(F\boldsymbol{u})_{ij}=F(x_i)u_{ij}$, $\boldsymbol{u}^q=(u_{ij}^q)_{(M-1)\times(N+1)}$. The matrices A,B,H, and E are same as in (5.5). The matrix C is of dimension $(M-1)\times(M-1)$, which is computed by

$$C_{m,i}=\big(\partial_x h_i^x,h_m^x\big)_M=\sum_{k=0}^{M}\partial_x h_i^x(x_k)h_m^x(x_k)\rho_k^M=\delta_{im}D_{m,i}\rho_m^M, \tag{6.7}$$

where $D_{ij}=h_j'(x_i)$.

Note that (6.6) is a nonsymmetric nonlinear system, we suggest to use Jacobian-free Newton-Krylov methods [19] to solve it. The process is as follows: we first rewrite (6.6) as

$$\mathcal{F}(\boldsymbol{u})=0,$$

where $\mathcal{F}(\boldsymbol{u})=A\boldsymbol{u}B^T+C(F\boldsymbol{u})H^T-E\boldsymbol{u}^qH^T-\boldsymbol{f}$. Then the iteration looks like:

- initial guess $\boldsymbol{u}^{(0)}$ given;
- solve the linear system

$$\mathcal{J}_{\mathcal{F}}(\boldsymbol{u}^{(k)})\delta\boldsymbol{u}^{(k)}=-\mathcal{F}(\boldsymbol{u}^{(k)}),$$

by using the Bi-Conjugate-Gradient-stab (BICGSTAB) [40]. Here, k is the iteration index, $\mathcal{F}(\boldsymbol{u}^k)$ is the residual, $\mathcal{J}_{\mathcal{F}}(\boldsymbol{u}^{(k)})$ is its associated Jacobian matrix at $\boldsymbol{u}^{(k)}$. Solving this linear system requires the matrix-vector product:

$$\begin{aligned}
\mathcal{J}_{\mathcal{F}}(\boldsymbol{u}^{(k)})\delta\boldsymbol{u}^{(k)}&=\lim_{\epsilon\to 0}\frac{\mathcal{F}(\boldsymbol{u}^{(k)}+\epsilon\delta\boldsymbol{u}^{(k)})-\mathcal{F}(\boldsymbol{u}^{(k)})}{\epsilon}\\
&=\lim_{\epsilon\to 0}\frac{A(\boldsymbol{u}^{(k)}+\epsilon\delta\boldsymbol{u}^{(k)})B^T-A(\boldsymbol{u}^{(k)})B^T}{\epsilon}\\
&\quad+\lim_{\epsilon\to 0}\frac{C(F\boldsymbol{u}^{(k)}+\epsilon F\delta\boldsymbol{u}^{(k)})H^T-C(F\boldsymbol{u}^{(k)})H^T}{\epsilon}\\
&\quad-\lim_{\epsilon\to 0}\frac{E(\boldsymbol{u}^{(k)}+\epsilon\delta\boldsymbol{u}^{(k)})^qH^T-E((\boldsymbol{u}^{(k)})^q)H^T}{\epsilon}\\
&=A(\delta\boldsymbol{u}^{(k)})B^T+C(F\delta\boldsymbol{u}^{(k)})H^T-E(q(\boldsymbol{u}^{(k)})^{q-1}*\delta\boldsymbol{u}^{(k)})H^T,
\end{aligned}$$

where $*$ acting on two vectors means $(\boldsymbol{u}*\boldsymbol{v})_i=u_i*v_i$;

- update $\boldsymbol{u}^{(k)}$ by

$$\boldsymbol{u}^{(k+1)}=\boldsymbol{u}^{(k)}+\omega\delta\boldsymbol{u}^{(k)},$$

where $0<\omega\le 1$ is a relaxation parameter, which is selected such

$$\|\mathcal{F}(\boldsymbol{u}^{(k+1)})\|<(1-\lambda\omega)\|\mathcal{F}(\boldsymbol{u}^{(k)})\|,$$

where λ is a scale parameter, fixed to 10^{-1}. In our calculation, we try $\omega=1,\frac{1}{4},\frac{1}{16},\cdots$, until the above criterion is met;

- stop if

$$\frac{\|\delta\boldsymbol{u}^{(k)}\|}{\|\boldsymbol{u}^{(k)}\|}<10^{-15}.$$

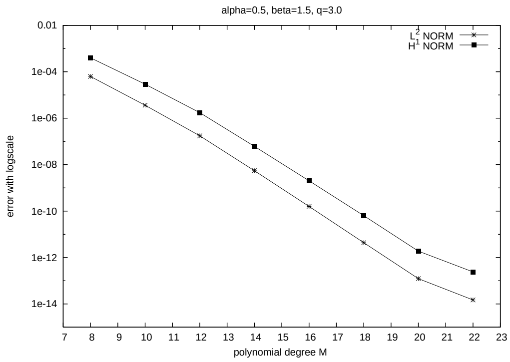

Figure 9: H^1- and L^2-errors versus M with $N=12$.

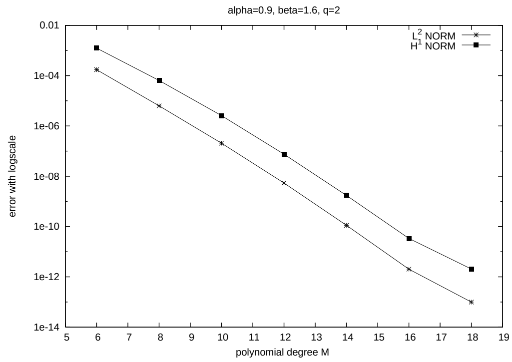

Figure 10: H^1- and L^2-errors versus M with $N=12$.

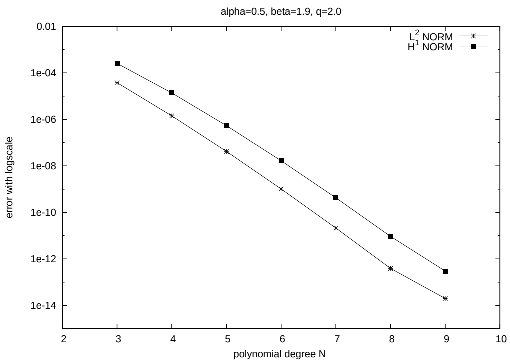

Figure 11: H^1- and L^2-errors versus N with $M=25$.

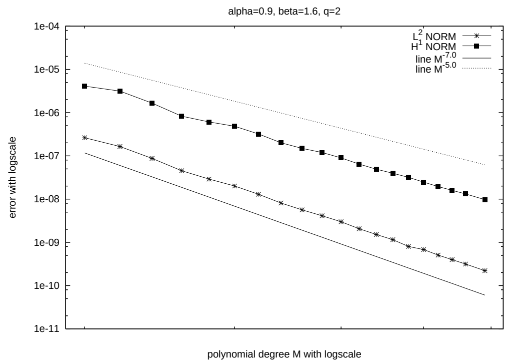

Figure 12: H^1- and L^2-errors versus M with $N=15$.

6.2 Numerical results

We present some numerical results for the proposed method to the nonlinear Fokker-Planck equation corresponding to the so called Uhlenbeck-Ornstein process, i.e.,

$$F(x)=-k_2x,$$

with $k_2=1$. We test for the exact analytical solution

$$u(x,t)=\exp(0.3t)\Big(\exp(-0.3x^2)-\exp(-0.3)\Big), \quad \text{in } (-1,1)\times(0,1).$$

In Figs. 9 and 10, we plot the L^2- errors and H^1-errors in semi-log scale as functions of the polynomial degrees M for two groups of parameters: $\alpha=0.5$, $\beta=1.5$, $q=3.0$ and $\alpha=0.9$, $\beta=1.6$, $q=2.0$. The errors with respect to N is presented in Fig. 11 for $\alpha=0.5$, $\beta=1.9$, $q=2.0$. Once again, the errors show exponential decay as expected.

For a reason mentioned in Section 5.2, we now study the convergence behavior for the exact solutions with limited regularity in $(0,2)\times(0,1)$

$$u(x,t)=(t^2+1)(2x^{10/3}-x^{13/3}), \qquad u(x,t)=(t^{16/3}+1)(2x^3-x^4).$$

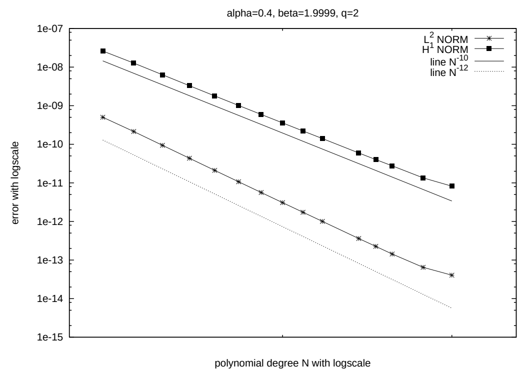

Figure 13: H^1- and L^2-errors versus N with $M_x=15$.

In Fig. 12, we plot the errors versus the polynomial degrees M for $\alpha=0.9$, $\beta=1.6$, $q=2.0$. The M^{-5} and M^{-7} decay rates are also shown for comparison reason. The convergence history with respect to the polynomial degrees N is presented in Fig. 13, together with lines of decay rates N^{-10} and N^{-12}. We observe here the algebraic convergence rates conform to the regularities of the solutions.

Finally we examine the efficiency of the Newton method by presenting the iteration numbers needed to reach the convergence. Particularly, we are interested in the impact of different parameters on the convergence. In Tables 1-4, we list the numbers of Newton iteration as functions of equation parameters α,β, and q, as well as the discretization parameters M and N. It is found that the convergence of the Newton iteration is more sensitive to the non-linearity q than to the discretization parameters M and N. A good point here is, as it can be observed throughout these tables, that the convergence history seems to be independent or has very slight dependence on M and N.

Table 1: Iteration history of Newton method for problem in Fig. 9.

Iteration	$M=8$	$M=15$	$M=22$
1	0.37	0.35	0.35
8	0.38	0.36	0.36
16	0.39	0.35	0.34
20	0.61	0.48	0.47
23	5.76D-2	2.85	0.45
24	1.15D-3	0.33	8.80D-2
25	1.42D-6	8.98D-2	1.25D-2
26	2.96D-12	4.20D-2	6.54D-4
27	2.01D-16	3.85D-3	9.81D-7
28		4.80D-5	1.43D-12
29		9.88D-9	1.23D-16
30		2.49D-16	

Parameters: $\alpha=0.5$, $\beta=1.5$, $q=3.0$, $\lambda=0.1$, $N=12$.

Table 2: Iteration history of Newton method for problem in Fig. 11.

Iteration	$N=4$	$N=7$	$N=10$
1	0.91	0.91	0.91
2	0.70	0.70	0.71
3	0.36	0.36	0.36
4	0.14	0.14	0.14
5	5.87D-2	5.91D-2	5.92D-2
6	2.55D-2	2.56D-2	2.57D-2
7	1.11D-2	1.12D-2	1.12D-2
8	4.20D-3	4.24D-3	4.24D-3
9	9.39D-4	9.53D-4	9.56D-4
10	6.27D-5	6.46D-5	6.51D-5
11	3.47D-7	3.69D-7	3.73D-7
12	1.30D-11	1.41D-11	1.43D-11
13	3.43D-16	5.34D-16	4.26D-16

Parameters: $\alpha=0.5$, $\beta=1.9$, $q=2.0$, $\lambda=0.1$, $M=25$.

Table 3: Iteration history of Newton method for problem in Fig. 12.

Iteration	$M=14$	$M=34$	$M=51$
1	0.28	0.28	0.28
2	5.44D-2	5.42D-2	5.42D-2
3	1.81D-2	1.40D-2	1.37D-2
4	6.69D-3	9.28D-3	8.15D-3
5	1.33D-3	6.32D-3	5.67D-3
6	3.15D-4	1.81D-3	1.27D-4
7	5.61D-6	2.85D-4	8.98D-5
8	1.89D-9	4.62D-5	2.65D-6
9	7.13D-16	2.88D-7	9.79D-9
10		2.12D-11	8.02D-14
11		1.79D-15	

Parameters: $\alpha=0.9$, $\beta=1.6$, $q=2.0$, $\lambda=0.1$, $N=10$.

7 Conclusions

We have set up a general variational framework for the space-time fractional diffusion equation. Some suitable fractional Sobolev spaces and norms are introduced and investigated. We proved the well-posedness of the weak formulation of the space-time fractional diffusion equation. Based on this weak formulation, we were able to construct and analyze a space-time spectral method for the numerical solution. A rigorous error estimate is provided, together with a detailed implementation and numerical confirmation. Furthermore the space-time spectral method has been generalized to solve the nonlinear Fokker-Planck equation. The numerical results have shown the applicability of the proposed method.

Table 4: Iteration history of Newton method for problem in Fig. 13.

iteration	$N=10$	$N=16$	$N=22$
1	0.27	0.27	0.27
2	5.73D-2	5.71D-2	5.70D-2
3	8.24D-3	8.07D-3	8.08D-3
4	4.30D-3	4.35D-3	4.40D-3
5	7.38D-4	7.27D-4	8.04D-4
6	7.99D-5	2.81D-5	1.92D-5
7	1.24D-5	1.77D-6	9.26D-7
8	3.58D-7	1.02D-8	3.29D-9
9	3.01D-10	3.40D-13	3.97D-14
10	1.17D-15	9.40D-16	

Parameters: $\alpha=0.4$, $\beta=1.9999$, $q=2.0$, $\lambda=0.1$, $M=12$.

Appendix

Here we present the numerical quadratures for fast evaluations of the integrals

$$\int_0^T {}^R D_t^{\frac{\alpha}{2}} h_j^t(t)\, {}_t^R D^{\frac{\alpha}{2}} h_n^t(t)\mathrm{d}t, \qquad \int_a^b {}^R D_x^{\frac{\beta}{2}} h_i^x(x)\, {}_x^R D^{\frac{\beta}{2}} h_m^x(x)\mathrm{d}x,$$

involved in the matrices B and E, which are the most expensive calculation in our methods. First, in order to compute the integral

$$\int_a^b {}^R D_x^{\frac{\beta}{2}} h_i^x(x)\, {}_x^R D^{\frac{\beta}{2}} h_m^x(x)\mathrm{d}x,$$

we need to compute the left Riemann-Liouville derivative of $h_i^x(x)$ and right Riemann-Liouville derivative of $h_m^x(x)$. This is done by

$$\begin{aligned} {}^R D_x^{\frac{\beta}{2}} h_i^x(x) &= \frac{1}{\Gamma(1-\frac{\beta}{2})}\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x \frac{h_i^x(\xi)\mathrm{d}\xi}{(x-\xi)^{\frac{\beta}{2}}} = \frac{1}{\Gamma(1-\frac{\beta}{2})}\int_a^x \frac{\frac{\mathrm{d}}{\mathrm{d}\xi}h_i^x(\xi)}{(x-\xi)^{\frac{\beta}{2}}}\mathrm{d}\xi \\ &= \frac{1}{\Gamma(1-\frac{\beta}{2})}\int_{-1}^1 \frac{\frac{\mathrm{d}}{\mathrm{d}\eta}h_i^x\big(\xi(\eta)\big)(x-a)\mathrm{d}\eta}{2\big(x-\xi(\eta)\big)^{\frac{\beta}{2}}}, \end{aligned}$$

where

$$\xi(\eta) = \frac{(x-a)\eta+a+x}{2} = \frac{1}{\Gamma(1-\frac{\beta}{2})}\Big(\frac{x-a}{2}\Big)^{1-\frac{\beta}{2}}\int_{-1}^1 \frac{\mathrm{d}}{\mathrm{d}\eta}h_i^x\Big(\frac{(x-a)\eta+a+x}{2}\Big)\frac{\mathrm{d}\eta}{(1-\eta)^{\frac{\beta}{2}}},$$

$$
\begin{aligned}
{}_x^R D^{\frac{\beta}{2}} h_m^x(x) &= \frac{1}{\Gamma(1-\frac{\beta}{2})}\frac{\mathrm{d}}{\mathrm{d}x}\int_x^b \frac{h_m^x(\xi)\mathrm{d}\xi}{(\xi-x)^{\frac{\beta}{2}}} = \frac{1}{\Gamma(1-\frac{\beta}{2})}\int_x^b \frac{\frac{\mathrm{d}}{\mathrm{d}\xi}h_m^x(\xi)\mathrm{d}\xi}{(\xi-x)^{\frac{\beta}{2}}} \\
&= \frac{1}{\Gamma(1-\frac{\beta}{2})}\int_{-1}^1 \frac{\frac{\mathrm{d}}{\mathrm{d}\eta}h_m^x(\xi(\eta))(b-x)\mathrm{d}\eta}{2(\xi(\eta)-x)^{\frac{\beta}{2}}},
\end{aligned}
$$

with

$$
\xi(\eta) = \frac{(b-x)\eta+b+x}{2} = \frac{1}{\Gamma(1-\frac{\beta}{2})}\Big(\frac{b-x}{2}\Big)^{1-\frac{\beta}{2}}\int_{-1}^1 \frac{\mathrm{d}}{\mathrm{d}\eta}h_m^x\Big(\frac{(b-x)\eta+b+x}{2}\Big)\frac{\mathrm{d}\eta}{(1+\eta)^{\frac{\beta}{2}}}.
$$

Let

$$
\phi_i(x) := \frac{x-a}{\Gamma(1-\frac{\beta}{2})2^{1-\frac{\beta}{2}}}\int_{-1}^1 \frac{\mathrm{d}}{\mathrm{d}\eta}h_i^x\Big(\frac{(x-a)\eta+a+x}{2}\Big)\frac{\mathrm{d}\eta}{(1-\eta)^{\frac{\beta}{2}}}, \tag{A.1}
$$

$$
\psi_m(x) := \frac{b-x}{\Gamma(1-\frac{\beta}{2})2^{1-\frac{\beta}{2}}}\int_{-1}^1 \frac{\mathrm{d}}{\mathrm{d}\eta}h_m^x\Big(\frac{(b-x)\eta+b+x}{2}\Big)\frac{\mathrm{d}\eta}{(1+\eta)^{\frac{\beta}{2}}}. \tag{A.2}
$$

Then it is readily seen that both $\phi_i(x)$ and $\psi_m(x)$ are polynomials satisfying

$$
\int_a^b {}^R D_x^{\frac{\beta}{2}} h_i^x(x)\, {}_x^R D^{\frac{\beta}{2}} h_m^x(x)\mathrm{d}x = \int_a^b \phi_i(x)\psi_m(x)(x-a)^{-\frac{\beta}{2}}(b-x)^{-\frac{\beta}{2}}\mathrm{d}x. \tag{A.3}
$$

We are now led to compute the integrals in the right hand sides of (A.1), (A.2), and (A.3). We denote by $J_{M+1}^{\alpha,\beta}(\eta)$ the Jacobi polynomial of degree $M+1$ with respect to weight $w^{\alpha,\beta}(\eta)=(1-\eta)^\alpha(1+\eta)^\beta$. Let $\{\eta_k^{\alpha,\beta}\}_{k=0}^{M+1}$ be the points of the GLJ quadrature formula, defined by

$$
\eta_0^{\alpha,\beta}=-1, \quad \eta_{M+1}^{\alpha,\beta}=1, \quad \frac{\mathrm{d}}{\mathrm{d}\eta}J_{M+1}^{\alpha,\beta}(\eta_k^{\alpha,\beta})=0, \quad k=1,\cdots,M.
$$

The associated weights of the GLJ quadrature formula are denoted by $\rho_k^{\alpha,\beta}$, $0\le k\le M+1$.

We then define the GLJ quadrature points ξ_k and weights w_k in the interval $\Lambda=[a,b]$ by

$$
\begin{aligned}
\xi_k &= \frac{(b-a)\eta_k^{-\frac{\beta}{2},-\frac{\beta}{2}}+b+a}{2}, \qquad k=0,1,\cdots,M+1, \\
w_k &= (\frac{b-a}{2})^{1-\beta}\rho_k^{-\frac{\beta}{2},-\frac{\beta}{2}}, \qquad k=0,1,\cdots,M+1.
\end{aligned}
$$

Then the numerical quadrature

$$
\int_a^b u(x)v(x)(x-a)^{-\frac{\beta}{2}}(b-x)^{-\frac{\beta}{2}}\mathrm{d}x \simeq \sum_{k=0}^{M+1} u(\xi_k)v(\xi_k)w_k,
$$

is exact for all functions u,v such that $uv\in P_{2M+1}(\Lambda)$. As a result, it holds

$$\int_a^b \phi_i(x)\psi_m(x)(x-a)^{-\frac{\beta}{2}}(b-x)^{-\frac{\beta}{2}}\mathrm{d}x=\sum_{k=0}^{M+1}\phi_i(\xi_k)\psi_m(\xi_k)w_k,\quad i,m=1,2,\cdots,M-1, \tag{A.4}$$

since $\phi_i\psi_m\in P_{2M}(\Lambda)$, for all $i,m=1,2,\cdots,M-1$.

In (A.4), $\phi_i(\xi_k)$ and $\psi_m(\xi_k)$ are evaluated by

$$\begin{aligned}\phi_i(\xi_k)&=\frac{\xi_k-a}{\Gamma(1-\frac{\beta}{2})2^{1-\frac{\beta}{2}}}\int_{-1}^{1}\frac{\mathrm{d}}{\mathrm{d}\eta}h_i^x\Big(\frac{(\xi_k-a)\eta+a+\xi_k}{2}\Big)\frac{\mathrm{d}\eta}{(1-\eta)^{\frac{\beta}{2}}}\\&=\frac{\xi_k-a}{\Gamma(1-\frac{\beta}{2})2^{1-\frac{\beta}{2}}}\sum_{l=0}^{M+1}\frac{\mathrm{d}}{\mathrm{d}\eta}h_i^x\Big(\frac{(\xi_k-a)\eta_l^{-\frac{\beta}{2},0}+a+\xi_k}{2}\Big)\rho_l^{-\frac{\beta}{2},0},\\\psi_m(\xi_k)&=\frac{b-\xi_k}{\Gamma(1-\frac{\beta}{2})2^{1-\frac{\beta}{2}}}\int_{-1}^{1}\frac{\mathrm{d}}{\mathrm{d}\eta}h_m^x\Big(\frac{(b-\xi_k)\eta+b+\xi_k}{2}\Big)\frac{\mathrm{d}\eta}{(1+\eta)^{\frac{\beta}{2}}}\\&=\frac{b-\xi_k}{\Gamma(1-\frac{\beta}{2})2^{1-\frac{\beta}{2}}}\sum_{l=0}^{M+1}\frac{\mathrm{d}}{\mathrm{d}\eta}h_m^x\Big(\frac{(b-\xi_k)\eta_l^{0,-\frac{\beta}{2}}+b+\xi_k}{2}\Big)\rho_l^{0,-\frac{\beta}{2}}.\end{aligned}$$

Finally, the integral

$$\int_0^T {}^R D_t^{\frac{\alpha}{2}} h_j^t(t)\ {}_t^R D^{\frac{\alpha}{2}} h_n^t(t)\mathrm{d}t,$$

can be computed in a similar way.

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