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SHORT NOTE

A Hybrid Numerical Method to Cure Numerical Shock Instability

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Abstract. In this note, we propose a new method to cure numerical shock instability by hybriding different numerical fluxes in the two-dimensional Euler equations. The idea of this method is to combine a "full-wave" Riemann solver and a "less-wave" Riemann solver, which uses a special modified weight based on the difference in velocity vectors. It is also found that such blending does not need to be implemented in all equations of the Euler system. We point out that the proposed method is easily extended to other "full-wave" fluxes that suffer from shock instability. Some benchmark problems are presented to validate the proposed method.

AMS subject classifications: 52B10, 65D18, 68U05, 68U07

Key words: Godunov methods, numerical shock instability, carbuncle phenomenon.

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1 Introduction

In the last several decades, Godunov [1] schemes based on Riemann solvers are among the most successful methods in computational fluid dynamics (CFD), which exhibit strong robustness in most situations. However, there may have some problems in extending Godunov methods to two-dimensional cases, for example, Roe solver [2] and HLLC solver [3] for the Euler equations may suffer from "carbuncle" and "odd-even decoupling" phenomena that are called numerical shock instability [4]. Some Flux-Vector-Splitting (FVS) methods such as AUSMD [5] are also found to suffer from the same problems.

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Quirk [4] suggested a framework to deal with shock instability problems by employing two different types of flux functions: one is sharp in capturing discontinuities ("full-wave" flux) and known to induce shock instability, and the other is dissipative but stable for multidimensional shocks. Quirk's approach is very useful but involves a user-defined parameter which is to determine when and where to use the Riemann solver. From then on some correction routines [6–9] have been proposed to cure the multidimensional numerical shock instability. These corrections involve the detection for grid faces deemed as susceptible to the shock instability. At grid faces, the original numerical flux functions are either modified with some special procedures resulting from multidimensional considerations, or replaced by more dissipative flux functions. These remedies have been proved to be useful, but may fail when the underlying problem involves interactions of complex flow features. Ren [10] presented a rotated Roe Riemann solver to eliminate the shock instability, where the upwind direction is determined by the velocity-difference vector. However, this method requires that the numerical flux is computed two times at each grid face, which means less efficiency in computations. Nishikawa and Kitamura [11] proposed a method which uses a weight based on the difference in velocity vector in the form of rotated fluxes. However, their method can only be applied to the Roe solver.

In this paper, we propose a new method combining the Roe solver and the HLL scheme. At first, our approach is to blend a full-wave flux "Roe" and a less-wave flux "HLL". The combined coefficients are related to velocity difference in neighbor cells and grid interface norm vector. Furthermore we find that such combination is required only for the first and the third equations in one-dimensional extended Euler system. Through the above elaborate procedure, the new method has higher resolution while keeping its robustness.

2 The hybrid method

Consider the Euler equations in two dimensions,

$$\frac{\partial U}{\partial t}+\frac{\partial F(U)}{\partial x}+\frac{\partial G(U)}{\partial y}=0, \tag{2.1}$$

with

$$U=\begin{bmatrix}\rho\\ \rho u\\ \rho v\\ E\end{bmatrix},\qquad F(U)=\begin{bmatrix}\rho u\\ \rho u^2+p\\ \rho uv\\ u(E+p)\end{bmatrix},\qquad G(U)=\begin{bmatrix}\rho v\\ \rho uv\\ \rho v^2+p\\ v(E+p)\end{bmatrix},$$

where ρ is density, u and v are the velocities in x-direction and y-direction respectively, $E=\frac{1}{2}\rho(u^2+v^2)+\rho e$ is the total energy and e is the specific inner energy. Here, we only consider the ideal gas, which has a specifically equation of state: $p=(\gamma-1)\rho e$, with p the pressure and γ the specific heat ratio.

Discretize system (2.1) with a finite volume method over a computational domain dissected into structured quadrilateral control grid $\Omega_{i,j}$:

$$\frac{d}{dt}\int_{\Omega_{i,j}} U dxdy + \int_{\partial\Omega_{i,j}} H\cdot ndl = 0, \tag{2.2}$$

where $\partial\Omega_{i,j}$ is the boundary of $\Omega_{i,j}$, $H=F\mathbf{i}+G\mathbf{j}$ is the tensor of the fluxes, $n=n_x\mathbf{i}+n_y\mathbf{j}$ is the outward unit vector normal to the surface $\partial\Omega_{i,j}$. Each grid $\Omega_{i,j}$ has four faces $I_{i,j,k}$, $k=1,\cdots,4$, and

$$\int_{\partial\Omega_{i,j}} H\cdot ndl = \sum_{k=1}^{4}\int_{I_{i,j,k}} H\cdot ndl. \tag{2.3}$$

The numerical scheme to fully discretize (2.2) is

$$U_{i,j}^{n+1} = U_{i,j}^{n} - \frac{\Delta t}{|\Omega_{i,j}|}\sum_{k=1}^{4}\Phi_{i,j,k}|I_{i,j,k}|, \tag{2.4}$$

where $|\Omega_{i,j}|$ is the area of the cell $\Omega_{i,j}$, $|I_{i,j,k}|$ is the length of grid interface $I_{i,j,k}$. In grid-aligned finite volume method, the numerical flux $\Phi_{i,j,k}$ is determined by solving a Riemann problem of the one-dimensional Euler equations in the normal direction of the interface.

It is found that the numerical shock instability phenomenon is related to insufficient dissipation of the contact and shear surfaces, see, e.g., [6–8]. To cure shock instability for a "full-wave" flux, a more dissipative term is added to the flux which resolve contact surface in the domain of shock instability. In our method, the Roe flux and the HLL flux are combined in grid-aligned framework. The numerical flux can be expressed by the following formula:

$$\Phi_{new}(n) = \beta_1\Phi_{Roe}(n) + \beta_2\Phi_{HLL}(n), \tag{2.5}$$

where $\Phi_{Roe}(n)$ and $\Phi_{HLL}(n)$ represent Roe flux and HLL flux respectively, β_1,β_2 are positive coefficients satisfying $\beta_1+\beta_2=1$. The key problem is how to choose the coefficients β_1 and β_2.

Numerical shock instability usually occurs when the shock crosses the computing grid at an oblique angle. Denote shock direction as n_s and $\alpha_1=|n_s\cdot n|$. Then an adaptive weight can be chosen as

$$\beta_1 = \frac{1}{2}+\frac{1}{2}\frac{\alpha_1}{\alpha_1+\alpha_2}, \qquad \beta_2 = \frac{1}{2}\frac{\alpha_2}{\alpha_1+\alpha_2}, \tag{2.6}$$

where $\alpha_2=\sqrt{1-\alpha_1^2}$.

The shock direction is unknown in numerical calculation, but we can approximate it with the velocity-difference vector between two adjacent grids [10]:

$$n_s = \begin{cases} \frac{\Delta u\mathbf{i}+\Delta v\mathbf{j}}{\|\Delta u\mathbf{i}+\Delta v\mathbf{j}\|}, & \text{if } \|\Delta u\mathbf{i}+\Delta v\mathbf{j}\|>\epsilon, \\ n, & \text{otherwise,} \end{cases} \tag{2.7}$$

where ϵ is determined by

$$\epsilon = 10^{-4} U_*.$$

where U_* is a reference velocity at inlet of the flow field.

Remark 2.1. If the shock direction is the same as the interface norm direction (i.e., $n_s = n$), the hybrid scheme flux is degenerated to the Roe flux. If the shock direction is perpendicular to the interface norm direction, the hybrid scheme flux is degenerated to the HLL flux. So the dissipation in the proposed scheme is implemented mainly in shear discontinuity. The dissipation in contact discontinuity can be added by entropy fix approach [12].

After many numerical tests, it is found that it is enough to eliminate the numerical shock instability if the hybrid flux (2.5) is only implemented in the "first" and the "third" equations of the one-dimensional Euler system. In fact, such procedure adds dissipation on contact discontinuities and shear waves adaptively. The final hybrid Roe-HLL scheme is expressed by

$$\begin{cases} \Phi_{new}(n)(1) = \beta_1 \Phi_{Roe}(n) + \beta_2 \Phi_{HLL}(n), \\ \Phi_{new}(n)(2) = \Phi_{Roe}(n), \\ \Phi_{new}(n)(3) = \beta_1 \Phi_{Roe}(n) + \beta_2 \Phi_{HLL}(n), \\ \Phi_{new}(n)(4) = \Phi_{Roe}(n). \end{cases} \tag{2.8}$$

Remark 2.2. If $\Phi_{Roe}(n)$ in (2.8) is replaced by other full wave fluxes, e.g., AUSMD flux, then the hybrid AUSMD-HLL method can also cancel the numerical shock instability.

3 Numerical results

In this section, we verify our method by using several numerical test problems. Numerical results of the first three benchmark problems show its ability of the hybrid method for eliminating shock instability. The last two examples show the robustness of our method.

3.1 Odd-even decoupling problem

This test is well-known and first reported by Quirk in [4]. A plane shock travels right at the speed of Mach 6. This problem is computed in a domain cover by 800×20 square unit grids, except those on the centerline where the grid is perturbed in the following manner:

$$y(i,10) = \begin{cases} y_{mid} + 0.001, & \text{for } i \text{ even}, \\ y_{mid} - 0.001, & \text{for } i \text{ odd}, \end{cases} \tag{3.1}$$

where $y(i,10)$ is the y coordinate of a cell vertex $(i,10)$, y_{mid} is the y coordinate of the halfway line. Other condition can be found in [10].

Fig. 1 shows the density contour computed by Roe solver, AUSMD scheme, Roe-HLL scheme and AUSMD-HLL scheme. It can be seen that the two new schemes do not suffer shock instability and keep the shock all the way through.

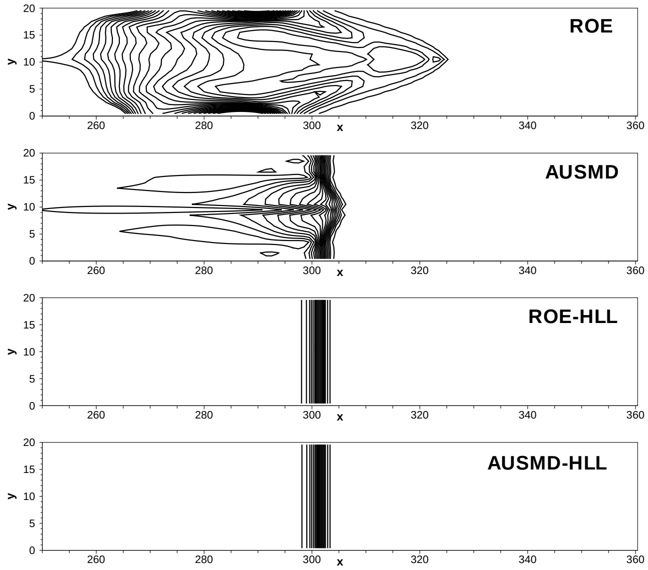

Figure 1: The contour of the density of odd-even grid perturbation problem.

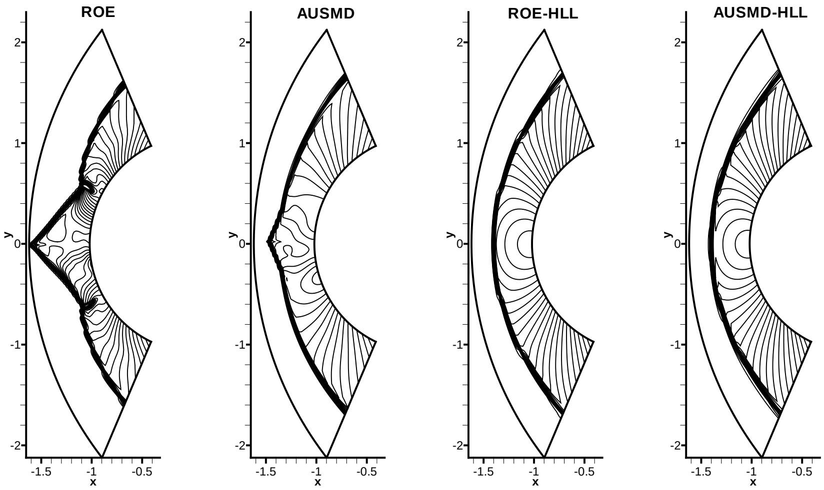

Figure 2: The contour of the pressure of Mach 20 hypersonic flow over a cylinder problem.

3.2 Mach 20 hypersonic flow over a cylinder

This is another well-known test to examine the catastrophic carbuncle failings of upwind schemes. The numerical simulation of a Mach number 20 inviscid flow around a circular cylinder, is considered a normal routine in CFD. In this test problem, 20×160 grids are used and the pressure contour by Roe solver, AUSMD scheme, Roe-HLL scheme and AUSMD-HLL scheme. From Fig. 2, we can see that the results by Roe and AUSMD scheme are not correct, and both the hybrid schemes have good shock pictures.

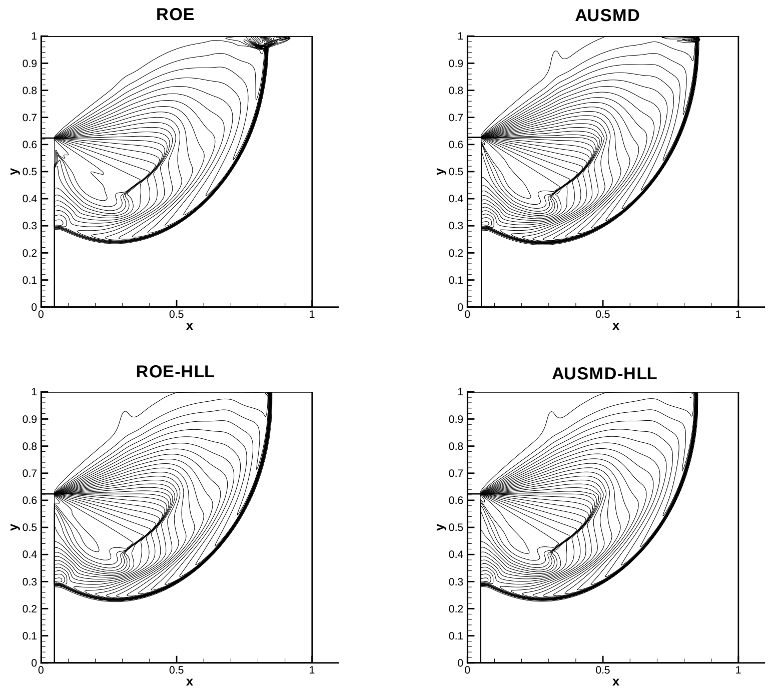

Figure 3: The contour of the density of diffraction of a supersonic shock moving over a 90° corner problem.

3.3 The diffraction of a supersonic shock moving over a 90° corner

The shock diffraction problem is another test for which many Godunov-type schemes are known to fail. The shock Mach number is 5.09 in this problem. The computational domain is a unit square $[0,1]\times[0,1]$ that is discretized into a 400×400 uniform grids. The corner is at $(x,y)=(0.05,0.625)$. The shock is initially at $x=0.05$. Other conditions can be found in [10]. Fig. 3 shows the results computed by Roe solver with entropy fix, AUSMD scheme, Roe-HLL scheme and AUSMD-HLL scheme respectively. It is obvious that Roe solver and AUSMD scheme suffer from shock instability. Our hybrid schemes do not have such flaws.

3.4 Double mach reflection problem

The formulation of this problem, computational setup and detailed discussion of the flow physics can be found in [14]. In computation, we chose $\Delta x=\Delta y=\frac{1}{100}$. Fig. 4 shows the density contour computed by Roe solver and Roe-HLL scheme. The Mach stem in Roe solver is inexplicably kinked giving rise to a spurious triple point (D). It should be noted that this kinking is not related to the slight bulging that is often observed in experiments. However, our method produced the flow pattern generally accepted in the present literature as correct one and there is not kinking.

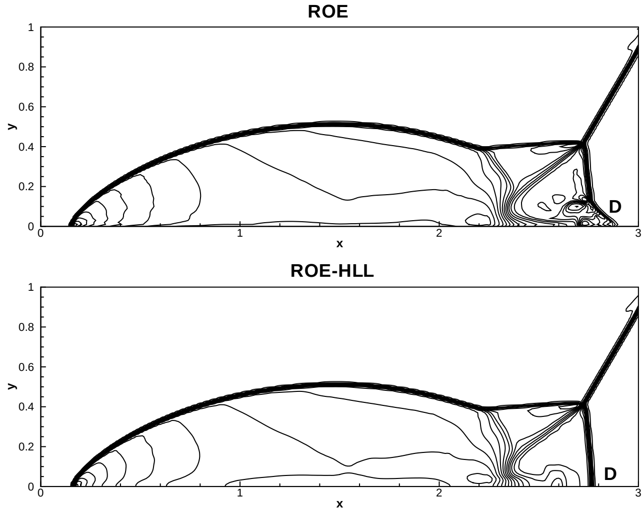

Figure 4: The contour of the density of double mach reflection problem.

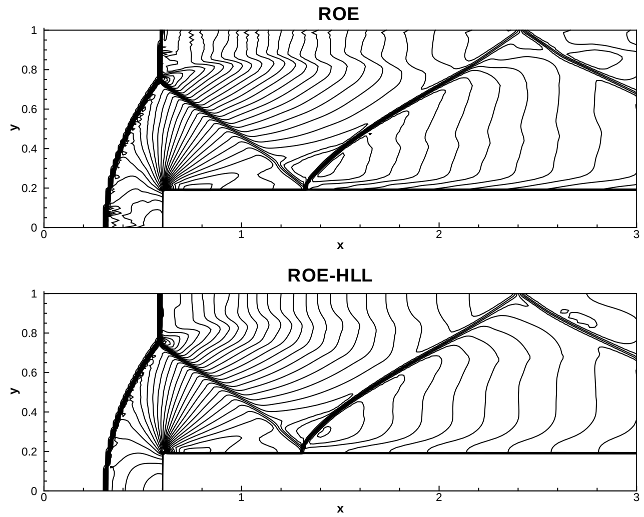

Figure 5: The contour of the density of forward facing step problem.

3.5 Forward facing step problem

This problem was also studied in details by Woodward and Colella and all computational conditions are also found in [14]. In computation, the grid size is $\Delta x = \Delta y = \frac{1}{80}$. The computed density contour corresponding to the grid-aligned Roe solver and Roe-HLL scheme in Fig. 5. Even in this case, the grid-aligned Roe solver exhibits instability in regions after the normal shock. However, our scheme works well.

4 Concluding remarks

In this note, a new hybrid method to combine "full-wave" fluxes and HLL flux is proposed. The main feature of our method is that the hybrid flux is only implemented in the "first" and "third" equations in the original system. Although the mechanism of such procedure is still unclear, the new method does cure numerical shock instability. Its robustness are verified in computing in double Mach reflection problem and forward facing step problem.

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