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On Spiral Waves Arising in Natural Systems

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Abstract. Spiral waves appear in many different natural contexts: excitable biological tissues, fungi and amoebae colonies, chemical reactions, growing crystals, fluids and gas eddies as well as in galaxies. While the existing theories explain the presence of spirals in terms of nonlinear parabolic equations, it is explored here the fact that self-sustained spiral wave regime is already present in the linear heat operator, in terms of integer Bessel functions of complex argument. Such solutions, even if commonly not discussed in the literature because diverging at spatial infinity, play a central role in the understanding of the universality of spiral process. In particular, we have studied how in nonlinear reaction-diffusion models the linear part of the equations determines the wave front appearance while nonlinearities are mandatory to cancel out the blowup of solutions. The spiral wave pattern still requires however at least two cross-reacting species to be physically realized. Biological implications of such a results are discussed.

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1 Introduction

Regular geometrical patterns occur in Nature in many situations [1], a striking case being as an example the observation of practically perfect spherical objects, neutron stars, as

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a consequence of self-gravitational general relativistic effects [2]. Another pattern very common is the spiral. It is remarkable the fact that D'Arcy Thompson, around one century ago, devoted an entire chapter of his classical monograph *"On Growth and Form"* to the appearance of the spiral form in Nature discussing in particular animal horns and molluscan shells [3]. Nowadays spiral waves have been observed in many other different biological contexts: in the heart, for example, the motion of the spiral center seems to be associated with specific types of arrhythmias [4], while in neural tissues this motion can be related to epilepsy and to spreading depressions in the retina [5]. More in detail these centers specifically are known as phase singularities, i.e., points in physical or abstract spaces near which the full cycle of isochrons crowds together. It is possible then to have a line of singularities, as in the singular filament of organizing centers, or along the border of a "black hole", i.e., a region on "latency diagrams" on which timings are lost [6]. These filaments in heart and brain tissues are non static and their motion in severe pathological states usually appears to be turbulent (what is it known as "chemical turbulence" [7]). Some biological populations of fungi and amoebae, like the *Dictyostelium discoideum*, tend to organize themselves in spiralling structures while spiral waves appear spontaneously also in specific chemical reactions like the classical Zhabotinsky-Belousov one [4] and also in growing crystals [8]. Common patterns encountered in all these systems are also target patterns, i.e., circular expanding waves generated by oscillatory local behaviors or external stimulations. Even in plant morphogenesis processes both these patterns can occur (kinetic phyllotaxis) [9]. In Fig. 1, a picture taken by one of the authors as an example, the bark of a dead tree manifests a spiralling pattern. The spiral is constituted by outer bark layers, which are well known to be associated with the early stages of the tree, so the typical arboreal radially diffusive behavior (a sort of target one-wave pattern) in this very peculiar case has been replaced by a spiralling mode, probably in association with a "very singular"event (a lightning, an infection or similar).

Figure 1: This picture of spiraling bark was taken in Rome from one of the authors on February, 2009.

In all these systems just discussed, single armed spirals appear in various chiralities although many-armed spiral configurations have been found experimentally in chemical reactions [10]. Moreover, such phenomenologies develop in time with different time scales but they all manifest wave behaviors. Mathematically speaking, all these systems have a common root, i.e., they are properly modeled by nonlinear parabolic partial differential equations. A simple prototype for these equations are reaction-diffusion (RD) systems

$$\frac{\partial c_1}{\partial t} = D_1 \nabla^2 c_1 + f(c_1,c_2), \qquad \frac{\partial c_2}{\partial t} = D_2 \nabla^2 c_2 + g(c_1,c_2), \tag{1.1}$$

(spatial homogeneity and isotropy are here assumed in the diffusion tensors for the sake of simplicity, so that D_1 and D_2 are single diffusion coefficients), where we have two possible nonlinearly interacting concentrations c_1 and c_2. It is important to stress here that not only spiral waves but also multiform stationary pattern formation can occur in this system of PDEs as a consequence of the well known Turing diffusion-driven instability mechanism [10]. In the context of phyllotaxis kinetic in plants [9] as well as in cardiac dynamics [11,12], the evolution is described coupling reaction-diffusion equations to elasticity theory. We point out also that temperature effects can play an important role in many of these systems especially in biological [13,14] and chemical contexts. On the other hand, as anticipated, spirals appear also in vortices in ordinary liquids/gases (i.e., whirlpools) and in astrophysics in Galaxies [15]. These systems can be described by Navier-Stokes equations [15], which assuming for the sake of simplicity incompressibility, are given by

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} - \mu\nabla^2\vec{v} = -\frac{\nabla P}{\rho} + \frac{\vec{F}}{\rho}, \tag{1.2}$$

together with the constraint $\nabla\cdot\vec{v}=0$. This is a generalized nonlinear parabolic system (nonlinearities being present in the convection term $(\vec{v}\cdot\nabla)\vec{v}$ only) for the velocity components, while ρ, P and $\vec{F}$ are the density, the pressure and the body forces respectively. If gravitation is the only body-force present, $\vec{F}$ can be written as $\vec{F}=-\rho\nabla\Phi$, where the gravitational potential Φ satisfies a Poisson's equation leading to the classical theory of self gravitating systems [16]. On the other hand neglecting the viscous term (i.e., $\mu=0$ in Eq. (1.2)), one obtains perfect fluid which cannot physically support spirals, due to the lack of viscous dissipation [17,18], although not smooth solutions in weak form (i.e., generalized solutions as shock waves) have been found [19] while their physical realization is questionable on experimental grounds. Incidentally perfect fluids (described by Euler equation) can be directly connected by Madelung representation to quantum fluids described by Gross-Pitaevskii equation [20]. In these condensed systems, spiral radiation patterns have been observed in experiments, but they are a consequence of spiraling vortex line trajectories and not of a spiral geometry associated with the vortex itself [21]. Instead in the case of Complex Ginzburg-Landau equation

$$\frac{\partial A}{\partial t} = A + (1+ib)\nabla^2 A - (1+ic)A|A|^2, \tag{1.3}$$

for the complex function A (whose real and imaginary parts give two RD equations while b and c are real parameters), describing a vast variety of physical phenomena [22], dropping the nonlinear term and selecting $b=0$ one obtains a "complex diffusion equation"for the real and imaginary parts, so again spiral waves are expected, as it is effectively found in numerical experiments. It is then clear that nonlinear diffusion equations play a central role in the formation of this spiraling pattern, but a question arise naturally: can spiral waves belong to a linear regime?

In this article we discuss this question, showing that a spiral pattern comes specifically from the very simple linear diffusion equation for one specie only, so that the diffusion equations determine the shape of the wave. More impressively, associated with the appearance of a spiral a diffusive process is always occurring, a fact that has never received enough attention in the mathematical but especially physical Literature, although some analytical and numerical insights have shown spiral waves behaviors in linear "multispecies" reaction-diffusion systems [23,24], neglecting a discussion of physical implications. The main reason for this is the fact that spiral wave solutions of linear equations do not satisfy proper boundary conditions of the Stourm-Liouville problem. According to our point of view instead, the central point is to interpret the interplay between linearity and nonlinearity on physical grounds. Through analytical methods we find that when the frequency of these waves becomes zero (static solutions) the spiral pattern disappears, that is spirals must be waves, a fact widely confirmed by experiments. Nonlinearities, however result to be necessary in order to correct pathologies of the linear theory, i.e., eliminating blowups of the solution and leading to circular fronted waves (target patterns). Spiral and target patterns in fact can be approximated with linear regimes only close to the center. This requirement however is still not sufficient to physically realize spiral waves: for instance, the context of reaction-diffusion processes at least two cross-reacting species are needed. We can discuss now all these points in detail, giving a unique thread of many results scattered in the Literature by using quite simple mathematical arguments.

2 Spiral waves and the linear diffusion equation

Let's write the diffusion equation in Cartesian coordinates (x,y,z), i.e., $\partial c/\partial t = D\Delta c$, where c is the concentration, D is the diffusion coefficient associated to a homogeneous and isotropic diffusion tensor, t is time and Δ is the Laplace operator. Using dimensionless parameters [25], i.e., $T=Dt/l^2$, and $X=x/l$ (l is an arbitrary length scale), similarly for y and z, with $c=\tilde{c}(C+C_0)$, where $\tilde{c}$ is a constant (concentration), C represents the dimensionless concentration and C_0 is an arbitrary dimensionless shift, we make the diffusion coefficient disappear, i.e.,

$$\frac{\partial C}{\partial T}=\nabla^2 C, \tag{2.1}$$

where ∇^2 denotes here the Laplacian in dimensionless Cartesian coordinates.

It is convenient to write the diffusion equation above in dimensionless cylindrical coordinates (R,ϕ,Z) with R and ϕ defined so that

$$(X,Y)=R(\cos\phi,\sin\phi). \tag{2.2}$$

We use then the following separation of variables *ansatz*

$$C(R,\phi,Z,T)=P(R)e^{i\omega T+ikZ+im\phi}. \tag{2.3}$$

The linearity of the problem ensures us that the real and imaginary parts of this quantity both are solutions of Eq. (2.1). Moreover, m must be an integer in order to avoid problems of polidromy and cusps. We insert this functional form in Eq. (2.1), using in addition a rescaled dimensionless radius $R=q\zeta$, with

$$q^2=\frac{(i\omega-k^2)}{(k^4+\omega^2)}. \tag{2.4}$$

Such complex coordinate transformation brings the equation for P into the form

$$\zeta^2\frac{d^2P}{d\zeta^2}+\zeta\frac{dP}{d\zeta}+(\zeta^2-m^2)P=0, \tag{2.5}$$

which is a complex Bessel equation, whose solution is

$$P=a_1J_m(\zeta)+a_2Y_m(\zeta),$$

where a_1 and a_2 are generic constants. The Bessel functions Y_m must be disregarded being pathological on the origin of the complex plane (although nonlinear corrections, discussed in the following, may correct this pathology); hence we choose the values $a_1=1$ and $a_2=0$. The complex holomorphic function $J_m(\zeta)$ has a power series representation of the form [26,27]

$$J_m(\zeta)=\sum_{h=0}^{+\infty}\frac{(-1)^h\left(\frac{\zeta}{2}\right)^{2h+m}}{h!(h+m)!},\qquad |\arg\zeta|<\pi,\qquad m\geq 0, \tag{2.6}$$

whose convergence radius is infinite. In the case $m<0$, the relation $J_{-m}(\zeta)=(-1)^mJ_m(\zeta)$ is valid. The power series can be separated into real and imaginary parts by using Euler-De Moivre formulas, i.e., $\zeta=\rho e^{i\theta}=\rho[\cos(\theta)+i\sin(\theta)]$. In particular, $\rho=R/|q|\equiv R(k^4+\omega^2)^{1/4}$ and $\theta=1/2\arctan(\omega k^{-2})$. The final result, after a little algebra, is

$$\mathrm{Re}[J_m(\zeta)]=\sum_{h=0}^{+\infty}\frac{(-1)^h\rho^{2h+m}\cos[(2h+m)\theta]}{2^{2h+m}h!(h+m)!}, \tag{2.7}$$

$$\mathrm{Im}[J_m(\zeta)]=\sum_{h=0}^{+\infty}\frac{(-1)^h\rho^{2h+m}\sin[(2h+m)\theta]}{2^{2h+m}h!(h+m)!}, \tag{2.8}$$

both quantities depending on k, ω and m. We can now easily recompose backwards our solution whose real and imaginary parts satisfy both separately the dimensionless real diffusion equation (2.3) in cylindrical coordinates

$$C=\Big(\mathrm{Re}[J_m(\zeta)]+i\mathrm{Im}[J_m(\zeta)]\Big)e^{i\omega T+ikZ+im\phi}\equiv\mathrm{Re}[C]+i\mathrm{Im}[C], \tag{2.9}$$

where

$$\mathrm{Re}[C]=\mathrm{Re}[J_m(\zeta)]\cos(\omega T+kZ+m\phi)-\mathrm{Im}[J_m(\zeta)]\sin(\omega T+kZ+m\phi), \tag{2.10a}$$
$$\mathrm{Im}[C]=\mathrm{Re}[J_m(\zeta)]\sin(\omega T+kZ+m\phi)-\mathrm{Im}[J_m(\zeta)]\cos(\omega T+kZ+m\phi). \tag{2.10b}$$

When $k=0$, i.e., an infinite cylinder solution, the real and imaginary parts of C give moving target patterns and rotating spirals of various chiralities and numbers of arms as shown at a fixed time in Fig. 2. Fig. 3 instead shows the behavior when $k\neq 0$.

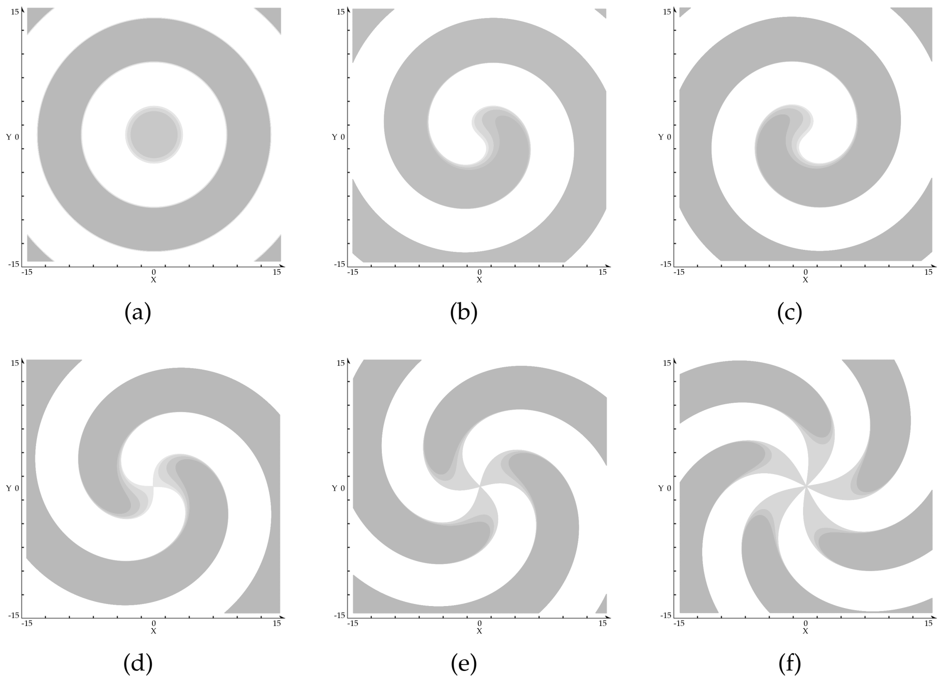


Figure 2: Real part of C (solution of the diffusion problem (2.1)), at $T=0$ assuming moreover $k=0$ (cylindrical symmetry). Surface levels $C=(0,0.15,0.5,1)$ are shown (grey color means high values while white is the opposite). For different m one obtains the following patterns: a) for $m=0$ which is reminiscent of target patterns, b) $m=1$ which is a spiral, c) $m=-1$ is a spiral with opposite chirality, d) $m=2$ a two armed spiral, e) $m=3$ a three armed spiral, f) $m=-5$ a five armed spiral with opposite chirality.

Except for selected values of θ, these real and imaginary parts of C diverge at infinity (while standard real Bessel functions J_m are well behaved everywhere). Such result is not unexpected: using the identity $J_m(i\xi)=iI_m(\xi)$, with $\xi\in\mathcal{R}$, connecting standard Bessel

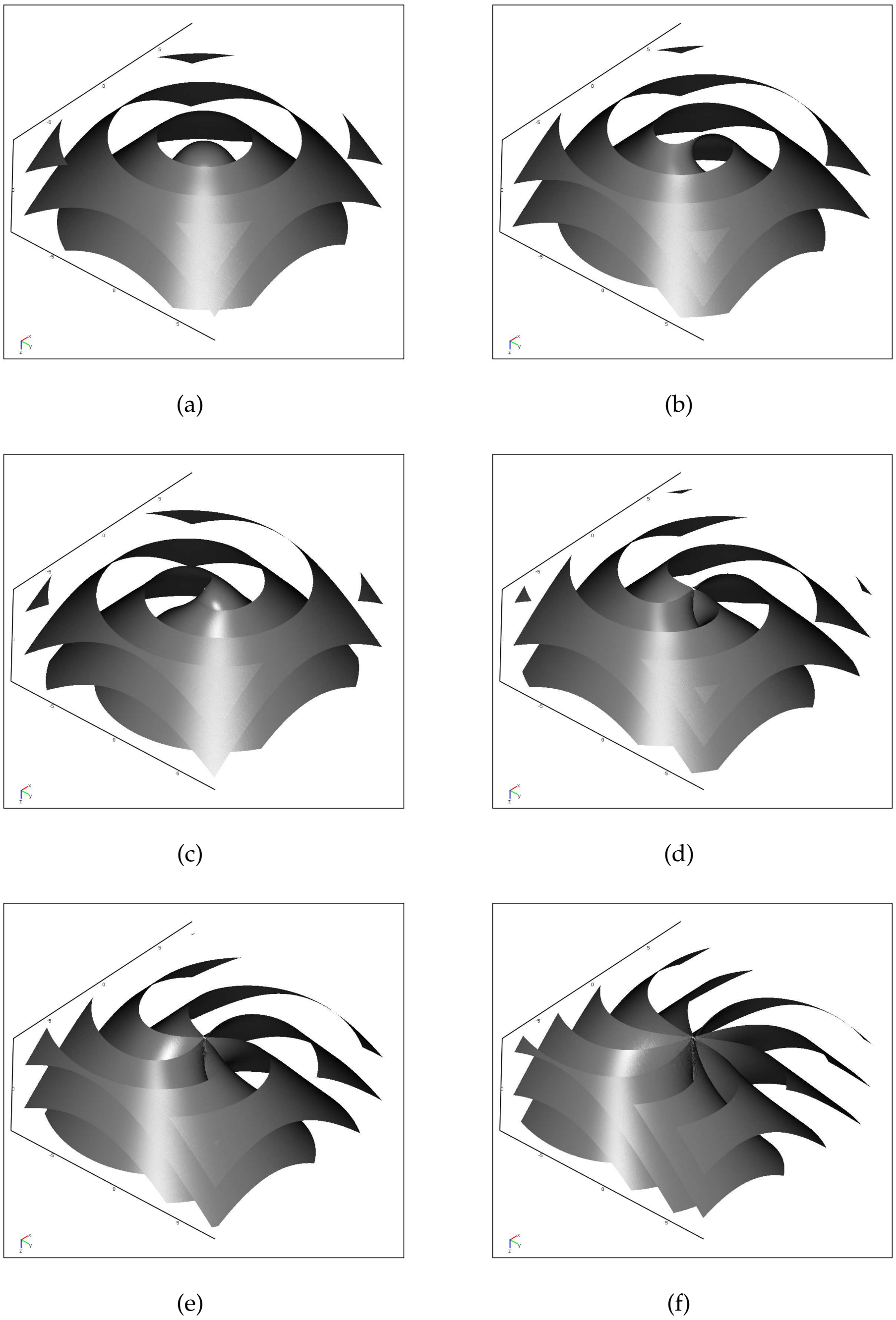


Figure 3: Real part of C (3D solution of the diffusion problem), at $T=1$ assuming $-\pi<z\leq\pi$. Surface gray level $C=0$. For different m one obtains these patterns: a) for $m=0$ which is reminiscent of target patterns, b) $m=1$ which is a spiral, c) $m=-1$ is a spiral with opposite chirality, d) $m=2$ a two armed spiral, e) $m=3$ a three armed spiral, f) $m=-5$ a five armed spiral with opposite chirality.

functions with their modified versions which diverge at large distances, we can easily see that our solutions must diverge too. We may have initially added to diffusion equation a linear term proportional to c but this step would have not changed the behavior of the divergence, i.e., we would have obtained again integer Bessel functions of complex argument.

When $\omega = 0$, coming back to the initial manipulations, we see that the Bessel functions have a real argument and spiral pattern is lost: hence, if spiral do exist they must be waves. We point out that the family of eigenfunctions found for the linear diffusion equation do not satisfy proper Sturm-Liouville problems with regularity both at the origin and at infinity. This is the main reason why we do not observe in nature in general, spirals which are described by linear parabolic equations. Such a particular pattern, belonging to the linear regime, in order to be physically realized needs nonlinear terms typical of reaction-diffusion systems in order to limit the blow up asymptotically. However in diffusion problems, when we have one diffusing species only, the mechanism above described would not produce rotating spiral waves or periodic target patterns, as discussed now.

3 The role of nonlinearity

Suppose to start from equation

$$\frac{\partial C}{\partial T} = \nabla^2 C + F(C). \tag{3.1}$$

Locally, i.e., neglecting the spatial variations, we obtain the ordinary differential equation $dC/dT = F(C)$, which represents the flow on a line, and cannot have periodic solutions (oscillations) unless the domain is topologically bent to form a circle [28]. Let's take as an example the Zeldovich's bistable equation, which has relevance both for gas dynamics and for nerve signal propagation [29,30], i.e., a normal form for many different dynamical systems with

$$F(C) = aC(1-C)(C-\alpha), \quad \text{with} \quad 0 < \alpha < 1, \tag{3.2}$$

(we assume in the following $a = 1$ to simplify relations) with $C = 0$ and $C = 1$ being sinks and $C = \alpha$ being a source. The word *bistable* comes from the fact that this type of equation has two possible stable solutions which are asymptotically reached once one starts at right or left of value α. It's clear that periodic spirals cannot exist because such a dynamics is not "excitable", i.e., the system does not explore a large portion of phase-space (which here, due to the first order derivative in time is trivially one dimensional) before coming back to the stable point [29]. A nonlinear spiral should in fact repeatedly stimulate the various portion of the domain, which is not possible looking at Fig. 4. In fact as anticipated the system is bistable, which means that every point of the domain must be flow-dragged towards one of the two sinks. Once such a point arrives at value $C = 1$, a possible spiral front (having at maximum value $C \sim 1$, because bistable waves act as

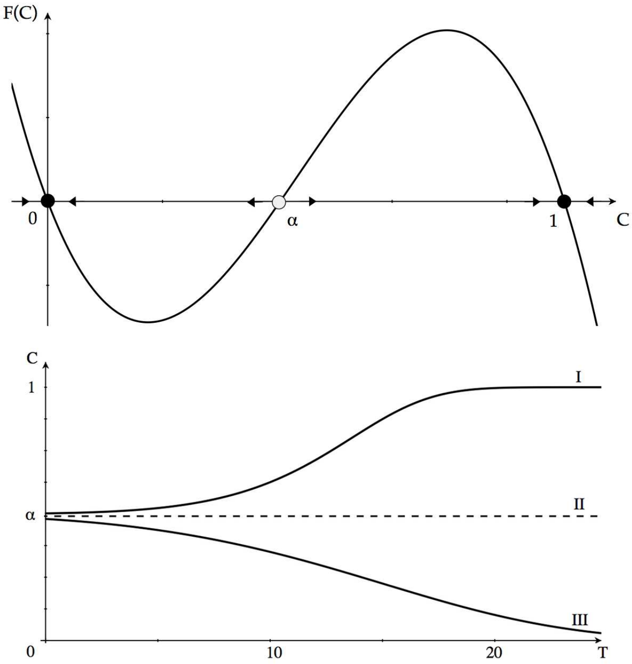


Figure 4: Upper panel: Phase space of the zero dimensional bistable equation (diffusion suppressed). There are two sinks and one repeller. Lower panel: Behavior of possible solutions of zeroth dimensional bistable equation with initial conditions over the repeller (curve I), on the repeller (curve II) and below the repeller (curve III).

a sort of shock waves connecting two asymptotic states) should move this point away from this stable fixed point. But this behavior is not possible because of the definition of stable point; so once the front arrives it extinguishes itself, as easily confirmed by simple numerical simulations. Manipulations in the one-dimensional case show that a traveling wave solution for bistable equation has analytical form [29]

$$C(T,X)=\frac{1}{2}+\frac{1}{2}\tanh\left[\frac{\sqrt{a}(X+VT)}{2\sqrt{2}}\right], \qquad V=\frac{\sqrt{a}}{\sqrt{2}}(1-2\alpha), \tag{3.3}$$

which manifest the limitation of blowups at infinite distance due to the presence of non-linearities. In the linear case

$$\frac{\partial C}{\partial T}=\frac{\partial^2 C}{\partial X^2}, \tag{3.4}$$

in fact the basic travelling wave solution is given by

$$C(T,X)=C_1+C_2\exp[v(X+vT)], \tag{3.5}$$

where C_1 and C_2 are arbitrary constants and v is the non dimensional velocity. This solution shows a divergence in space and time corrected in the nonlinear case previously discussed. In higher dimensions numerical solutions only are possible but the result is

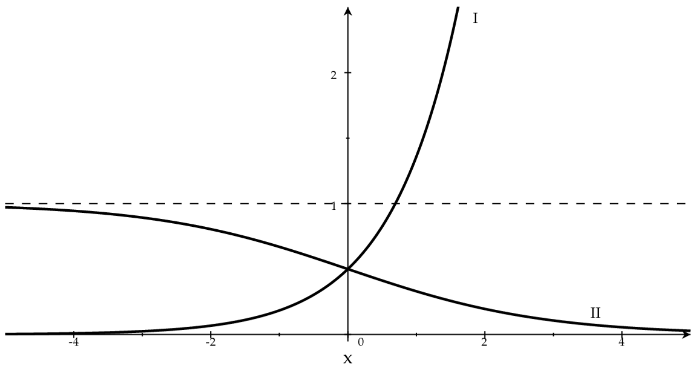


Figure 5: Graphics of the two solutions in space: I, linear case (3.5) with $C_1=0$ and $C_2=\frac{1}{2}$, II, nonlinear case (3.3).

not changed: a unique radial pattern evolving in time brings the entire domain at value $C=1$ asymptotically. More complicated functional forms $F(C)$ shall not modify such a behavior because the phase space will be a collection of fixed points on a line. Even a collection of semi-stable points described by the non-polynomial dynamics

$$F(C)=\frac{1}{2}(1+\sin C)$$

has a trivial phase-space which cannot support repeated waves. A possible way out exists however: we may consider in fact for the nonlinear diffusion equation above an extension with a spatial dependent term, as in happens in crystal growth problems as an example [31], i.e., in cylindrical coordinates we may write

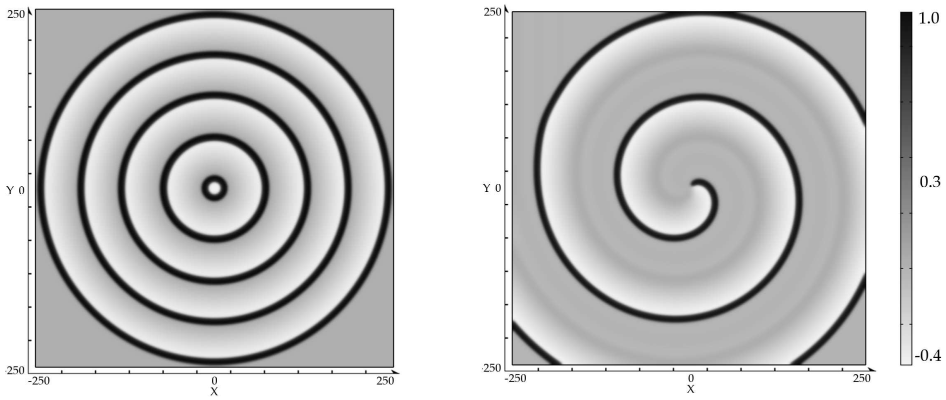
$$F(C)=\beta\sin(C-\gamma\phi)+\sigma,$$

with β, γ and σ real numbers. This choice produces rotating spiral waves but this nonlinear term breaks the isotropy and homogeneity of space (external magnetic or gravitational fields may give similar effects). Here, on the other hand, we are interested in genuine self-sustained spiral behaviors embedded in homogeneous and isotropic domains. The examples discussed above show that in order to have spirals, a two-dimensional phase-space at least is needed.

Consequently, again, in order to have natural spiral waves and circular waves, we are forced to require at least two species nonlinearly reacting, i.e., a proper nonlinear reaction-diffusion system. As an example we show in Fig. 6 the typical patterns obtained numerically integrating a very simple two variables reaction diffusion system as in Eq. (1.1) of FitzHugh-Nagumo type (FHN) [4,32], which implies

$$f(c_1,c_2)=c_1(1-c_1)(c_1-\alpha)-c_2, \quad \text{and} \quad g(c_1,c_2)=\epsilon(c_1-\gamma c_2).$$

Here the variable c_1 could be associated with a dimensionless action potential while the quantity c_2 with a gating variable (an electrophysiological problem), but because of the



(a) (b)

Figure 6: The model is a FitzHugh-Nagumo type with parameters $D_1=0.5$, $D_2=0.1$, $\alpha=0.01$, $\epsilon=0.01$ and $\gamma=0.05$ and the value of variable c_1 is shown at a given time of the evolution for target patterns (a) and a spiral wave (b).

generality and simplicity of the FHN model, they could represent differently chemical quantities or other. While a negative parameter α leads to a self-oscillatory behavior, a positive one leads to excitable dynamics [29]. The numerical integrations of these equations have been performed using finite elements techniques, modifying an existing code described in [13]. The mesh is made of squared elements sufficiently small to ensure stability and convergence of the simulations. More in detail we have adopted several discretizations of the domain starting from 25 up to 100 equally spaced points per side and selected in the different simulations second, third and fourth order Lagrange elements in order to ensure convergence and stability. The numerical integrations have been performed adopting Comsol Multiphysics® software running on a 64 bit dual core Xeon® Hewlett-Packard workstation with 6Gb of RAM. Specifically we have adopted a direct solver (UMFPACK) while time steps have been optimally chosen by the software (although in order to have additional checks, the best meshed simulations have been performed also adopting user-constrained time steps with $\Delta t = 5\cdot 10^{-4}$). Finally the relative and absolute errors thresholds have been selected at 10^{-7}. Our FHN model with $D_2=0$ can be easily obtained as a simplification of the Hodgkin-Huxley theory of the action potential in the giant squid axon, which is described by four variables, while the RD system here numerically studied is governed by c_1 and c_2 only, leading to a two dimensional phase-space (suppressing space variations) which simplifies noticeably the physics of excitability. We stress that it is well known the presence of spiral waves even if one of the two species does not diffuse (the case of the electrophysiological FHN just discussed) so that in order to have spirals it is necessary at least to have cross-reaction with one diffusing species and not necessarily cross-diffusion (both diffusion coefficients nonvanishing). The production of circular or spiral patterns in our simulations strongly depends on specific initial conditions and external actions (especially for the circular fronted ones which here required external periodic stimuli). In fact if the cylindrical symmetry is maintained circular waves but not spirals occur. An external perturbation of non axisymmetric type on the other hand breaks the symmetry and drives the system towards more complicated regimes, possibly attractor configurations containing one-armed spiral waves only.

4 Concluding remarks

In this paper, we have explored the fact that self-sustained spirals patterns, commonly encountered in many natural systems, have their geometrical roots already in the single linear diffusion equation. Physically realized spirals however are determined by a nonlinearly corrected diffusion equation with two —at least— cross-reacting species are involved. In fact, while in linear diffusion problems spiral solutions are mostly associated with a blow up and non regularities, in nonlinear problems those solutions become mostly regular instead. Moreover, as soon as time dependence becomes negligible, diffusion equation predicts the disappearance of such a pattern, a fact observed in experiments. The diffusion process must be seen as a theoretical model of possible discrete complex systems (i.e., cells, molecules) whose dynamics can be described in a first approximation with a nonlinear continuum field theory, similarly to what happens to the Boltzmann equation for the statistical description of particles in a fluid, where the real molecular theory can be approximated with Euler and Navier-Stokes limits [33]. In any case, it is important to stress that diffusion processes and spiral waves in natural systems (biological or not) are associated phenomena, therefore, if there are more than one species cross-reacting and one of them at least diffuses, a spiral wave pattern may be expected to arise. This point of view can be an extremely important instrument in order to hypothize on mathematical and physical grounds other biological or more general physical situations in which the ubiquous spirals can appear as a byproduct of chemical reaction-diffusion processes, and perform then *ad hoc* experiments.

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References

[1] H. Weyl, Symmetry, Princeton University Press, 1952.
[2] R. Belusevic, Relativity, Astrophysics and Cosmology, Wiley CH, 2008.
[3] W. T. D'Arcy, On Growth and Form, Cambridge University Press, 1992.
[4] A. T. Winfree, The Geometry of Biological Time, Springer, 2000.
[5] M. A. Dahlem and S. C. Muller, Image processing techniques applied to excitation waves in chicken retina, Methods., 21 (2000), 317–323.

[6] A. T. Winfree, When Time Breaks Down: The Three-Dimensional Dynamics of Electrochemical Waves and Cardiac Arrhythmias, Princeton University Press, 1987.
[7] Y. Kuramoto, Chemical Oscillations, Waves, and Turbulence, Dover, 2003.
[8] C. Kittel, Introduction to Solid State Physics, 8th ed., John Wiley & Sons, 2004.
[9] R. V. Jean, Phyllotaxis: A Systemic Study in Plant Morphogenesis, Cambridge University Press, 1994.
[10] J. D. Murray, Mathematical Biology vol. II, 3rd ed., Springer, 2008.
[11] C. Cherubini, S. Filippi, P. Nardinocchi and L. Teresi, An electromechanical model of cardiac tissue: constitutive issues and electrophysiological effects, Prog. Biophys. Mol. Bio., 97 (2008), 562–573.
[12] D. Bini, C. Cherubini and S. Filippi, Viscoelastic FitzHugh-Nagumo models, Phys. Rev. E, 72 (2005), 041929–1/041929–9.
[13] D. Bini, C. Cherubini and S. Filippi, Heat transfer in. FitzHugh-Nagumo models, Phys. Rev. E, 74 (2006), 041905–1/041905–12.
[14] D. Bini, C. Cherubini and S. Filippi, On vortices heating biological excitable media, Chaos. Soliton. Fract., 42 (2009), 2057–2066.
[15] J. Binney and S. Tremaine, Galactic Dynamics, Princeton University Press, 1987.
[16] S. Chandrasekhar, Ellipsoidal Figures of Equilibrium, New Haven Conn, Yale University Press, 1969.
[17] J. Jeans, Astronomy and Cosmogony, Cambridge University Press, 2009.
[18] R. A. Granger, Fluid Mechanics, Dover, 1995.
[19] T. Chang, G. Q. Chen and S. Yang, Nonlinear Problems in Engineering and Science-Numerical and Analytical Approach, Science Press, Beijing, 1991.
[20] T. Kambe, Elementary Fluid Mechanics, World Scientific, 2007.
[21] N. G. Parker, N. P. Proukakis, C. F. Barenghi and C. S. Adams, Controlled vortex-sound interactions in atomic Bose-Einstein condensates, Phys. Rev. Lett., 92 (2004), 160403–1/160403–4.
[22] I. S. Aranson and L. Kramer, The world of the complex Ginzburg-Landau equation, Rev. Mod. Phys., 74 (2002), 99–144.
[23] J. Gomatam, Pattern synthesis from singular solutions in the Debye limit: helical waves and twisted toroidal scroll structures, J. Phys. A, 15 (1982), 1463–1476.
[24] M. Golubitsky and I. Stewart, The Symmetry Perspective: From Equilibrium to Chaos in Phase Space and Physical Space, Birkhäuser, Basel, 2004.
[25] J. Crank, The Mathematics of Diffusion, 2nd ed., Oxford University Press, 1975.
[26] M. Abramowitz and I. Stegun, Handbook of Mathematical Functions, Dover, 1964.
[27] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series and Products, 6th ed., Academic Press, 2000.
[28] S. H. Strogatz, Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry and Engineering, Westview Press, 2001.
[29] J. Keener and J. Sneyd, Mathematical Physiology, Springer, 2001.
[30] Y. B. Zeldovich and D. A. Frank-Kamenetsky, A theory of thermal propagation of flame, Acta. Physicochim., 9 (1938), 341–350.
[31] T. Ogiwara and K. Nakamura, Spiral traveling wave solutions of nonlinear diffusion equations related to a model of spiral crystal growth, Publ. Res. Inst. Math. Sci., 39 (2003), 767–783.
[32] V. S. Zykov, A. S. Mikhailov and S. C. Muller, Wave instabilities in excitable media with fast inhibitor diffusion, Phys. Rev. Lett., 81 (1998), 2811–2814.
[33] K. Huang, Statistical Mechanics, 2nd ed., Wiley, 1987.