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The Stability and Convergence of Fully Discrete Galerkin-Galerkin FEMs for Porous Medium Flows

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Abstract. The paper is concerned with the unconditional stability and error estimates of fully discrete Galerkin-Galerkin FEMs for the equations of incompressible miscible flows in porous media. We prove that the optimal L^2 error estimates hold without any time-step (convergence) conditions, while all previous works require certain time-step restrictions. Theoretical analysis is based on a splitting of the error into two parts: the error from the time discretization of the PDEs and the error from the finite element discretization of the corresponding time-discrete PDEs, which was proposed in our previous work [26, 27]. Numerical results for both two and three-dimensional flow models are presented to confirm our theoretical analysis.

AMS subject classifications: 65N12, 65N30, 35K61

Key words: Unconditional stability, optimal error estimate, Galerkin FEMs, incompressible miscible flows.

1 Introduction

We consider incompressible miscible flow in porous media, which is governed by the following system of equations:

$$\Phi\frac{\partial c}{\partial t}-\nabla\cdot(D(\mathbf{u})\nabla c)+\mathbf{u}\cdot\nabla c=\hat{c}q^I-cq^P, \tag{1.1}$$

$$\nabla\cdot\mathbf{u}=q^I-q^P, \tag{1.2}$$

$$\mathbf{u}=-\frac{k(x)}{\mu(c)}\nabla p, \tag{1.3}$$

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where p is the pressure of the fluid mixture, $\mathbf{u}$ is the velocity and c is the concentration; $k(x)$ is the permeability of the medium, $\mu(c)$ is the concentration-dependent viscosity, Φ is the porosity of the medium, q^I and q^P are the given injection and production sources, $\hat{c}$ is the concentration in the injection source, and $D(\mathbf{u}) = [D_{ij}(\mathbf{u})]_{d\times d}$ is the diffusion-dispersion tensor which may be given in different forms (see [5,6] for details). We assume that the system is defined in a bounded smooth domain Ω in $\mathbb{R}^d$ $(d=2,3)$, for $t\in[0,T]$, coupled with the initial and boundary conditions:

$$\mathbf{u}\cdot\mathbf{n}=0, \quad D(\mathbf{u})\nabla c\cdot\mathbf{n}=0 \qquad \text{for } x\in\partial\Omega,\ t\in[0,T], \tag{1.4a}$$

$$c(x,0)=c_0(x) \qquad \text{for } x\in\Omega. \tag{1.4b}$$

The system (1.1)-(1.4) has been studied extensively in the last several decades, see [11, 35] and the references therein. Existence of weak solutions of the system was obtained by Feng [20] for the 2D model and by Chen and Ewing [9] for the 3D problem. Existence of semi-classical/classical solutions is unknown. Numerical simulations have been done with various applications [4, 7, 13, 17, 40, 41]. Optimal error estimates of a Galerkin-Galerkin method for the system in two-dimensional space was given first by Ewing and Wheeler [18] roughly under the time-step condition $\tau = o(h)$, in which a linearized semi-implicit Euler scheme was used in the time direction and a standard Galerkin FE approximation was used for both the concentration and the pressure. Later, a Galerkin-mixed finite element method was proposed by Douglas et al. [12] for this system, where a Galerkin approximation was applied for the concentration equation and a mixed approximation in the Raviart-Thomas finite element space [38] was used for the pressure equation. A linearized semi-implicit Euler scheme, the same as one used in [18], was applied for the time discretization. Optimal error estimates were obtained under a similar time-step condition $\tau=o(h)$. There are many other numerical methods in the literature for solving the equations of incompressible miscible flows in porous media, such as see [46] for an ELLAM in two-dimensional space, [47] for an MMOC-MFEM approximation for the 2D problem, [14, 43] for a characteristic-mixed method in two and three dimensional spaces, respectively, and [30, 31] for a collocation-mixed method and a characteristic-collocation method, respectively. In all those works, error estimates were established under certain time-step conditions. Moreover, it has been noted that linearized semi-implicit schemes have been analyzed for many other nonlinear parabolic-type systems, such as the Navier-Stokes equations [2, 19, 21, 24, 28], nonlinear thermistor problems [15, 51], viscoelastic fluid flow [8, 16, 48], KdV equations [33, 50], nonlinear Schrödinger equation [3, 39, 45], Ginzburg-Landau equations [10, 29] and some other equations [22, 42]. A time-step condition was always imposed to get suitable error estimates. A key issue in analysis of FEMs is the boundedness of the numerical solution in L^∞ norm or a stronger norm, which in a routine way can be estimated by mathematical induction with an inverse inequality, such as,

$$\|u_h^n - R_h u(\cdot,t_n)\|_{L^\infty} \le Ch^{-d/2}\|u_h^n - R_h u(\cdot,t_n)\|_{L^2} \le Ch^{-d/2}(\tau^m + h^{r+1}), \tag{1.5}$$

where u_h^n is the finite element solution, u is the exact solution, and R_h is certain projection operator. A time-step restriction arises immediately from the above inequality. Such a time-step restriction may result in the use of an unnecessary small time step and extremely time-consuming in practical computations. Clearly the time-step condition arises mainly due to the limitation of the technical tools used in analysis. A new theoretical analysis was presented in our recent works [26, 27], also see [25]. A nonlinear Joule heating model was studied in [26] with a linearized backward Euler Galerkin FEM and the incompressible miscible flow model was investigated in [27] with a linearized backward Euler Galerkin-mixed FEM. Optimal error estimates were obtained unconditionally. The basic concept in [26,27] is the error splitting

$$\|u_h^n-u(\cdot,t_n)\|\le\|U^n-u(\cdot,t_n)\|+\|u_h^n-U^n\|, \tag{1.6}$$

where U^n is the solution of a corresponding time-discrete system, an elliptic system. It was proved in [26, 27] that the last term in the above equation is τ-independent. The boundedness of numerical solution can be obtained by applying mathematical induction and inverse inequalities for the last term if suitable regularity of the solution U^n can be proved.

Numerical analysis for the time-discrete equations from some other time-dependent problems was made by several authors [23, 32, 34, 36, 49] for different purposes. Pani et al. [34] studied the linearized backward Euler approximation (time-discrete) to the Oldroyd model of viscoelastic fluid with more realistic initial data. A first-order time-discrete viscosity-splitting scheme was studied in [23] for the three-dimensional Navier-Stokes equations, and the optimal error estimate for the pressure (in the time direction) was obtained. Fully discrete schemes were not investigated in both [23,34]. In [36], a degenerate parabolic equation was studied with the fully implicit backward Euler scheme and a mixed finite element approximation, in which a Newton-type iterative algorithm was used for solving the nonlinear system arising at each time step. The convergence order for both time-discrete system and fully discrete system were estimated. In [32], authors studied miscible displacements in two-dimensional porous media by the linearized backward Euler scheme and a streamline-upwind-Petrov-Galerkin method combined with a post-process technique on the velocity. Both time discrete system and fully discrete system were investigated. Error estimates with quasi-optimal rates were derived for the fully discrete system by assuming that the solution of the corresponding time-discrete equations in W_∞^{r+1}-norm is bounded and under certain conditions for both time stepsize and spatial stepsize.

In this paper, we present two linearized semi-implicit Euler schemes with a standard Galerkin-Galerkin finite element approximation in the spatial direction for the system (1.1)-(1.4). One is semi-decoupled and one is fully decoupled. We establish optimal L^2 error estimates almost without any time-step restriction (or when h and τ are smaller than some positive constants). The theoretical analysis is based on the splitting technique proposed in [26, 27]. Numerical simulations for the system in both two and three dimensional spaces are provided. Our numerical results show clearly that no time step restriction is needed for both schemes.

The rest of the paper is organized as follows. In Section 2, we introduce two linearized semi-implicit Euler schemes with a Galerkin approximation in the spatial direction for the system (1.1)-(1.4) and present our main results. In Section 3, we present a priori estimates of the solution to the corresponding time-discrete system and the optimal error estimates of the semi-decoupled discrete scheme in L^2-norm are given. Analysis presented in this paper can be extended easily to the fully decoupled scheme. Finally we present numerical results in Section 4 to illustrate the convergence rate and the unconditional stability of schemes.

2 The Galerkin FEMs and the main results

For any integer $m\geq 0$, $1\leq p\leq\infty$ and $0<\alpha<1$, let $W^{m,p}$ and $C^{m+\alpha}$ be the usual Sobolev and Hölder spaces [1], respectively. Let π_h be a quasi-uniform division of Ω into triangles T_j, $j=1,\cdots,M$, in $\mathbb{R}^2$ or tetrahedrons in $\mathbb{R}^3$, and let $h=\max_{1\leq j\leq M}\{\mathrm{diam}\, T_j\}$ denote the mesh size. For a triangle T_j with two nodes (or a tetrahedron with three nodes) on the boundary, we denote by $\overline{T}_j$ the triangle with one curved edge (or a tetrahedron with one curved face) with the same nodes as T_j. For an interior element, $\overline{T}_j=T_j$. We define the standard finite element space on $\Omega_h=\cup_1^M T_j$ by

$$\widehat{V}_h^r=\{w_h\in C^0(\overline{\Omega}_h): w_h|_{T_j} \text{ is a polynomial of degree } r \text{ for each } T_j\in\pi_h\}.$$

Let $x=G(\hat{x})$ denote a map from Ω_h to Ω such that for each triangle T_j, G maps T_j one-to-one onto $\overline{T}_j$ [52]. And we define an operator $\mathcal{G}_V$ on $\widehat{V}_h^r$ by $\mathcal{G}_V w(x):=w(G^{-1}(x))$ for $x\in\Omega$. Then, the finite element space is defined by

$$V_h^r=\{\mathcal{G}_V w_h: w_h\in\widehat{V}_h^r\},$$

and $\widetilde{V}_h^r=V_h^r/\{\text{constant}\}$. Let $\widehat{I}_h: L^2(\Omega_h)\to\widehat{V}_h^r$ be the Lagrange interpolation operator of degree r. We define $I_h v=\mathcal{G}_V\widehat{I}_h\mathcal{G}_V^{-1}v$ for any $v\in H^1(\Omega)$. By classical interpolation theory, it is easy to see that

$$\|I_h v-v\|_{L^2}+h\|\nabla(I_h v-v)\|_{L^2}\leq Ch^{r+1}\|v\|_{H^{r+1}}. \tag{2.1}$$

In the rest part of this paper, we assume that the solution to the initial-boundary value problem (1.1)-(1.4) exists and satisfies

$$\begin{aligned}\|p\|_{L^\infty(I;H^3)}+\|\mathbf{u}\|_{L^\infty(I;H^2)}+\|\mathbf{u}_t\|_{L^2(I;W^{1,3/2})}+\|c\|_{L^\infty(I;W^{2,4})}\\ +\|c_t\|_{L^\infty(I;H^2)}+\|c_t\|_{L^4(I;W^{1,4})}+\|c_{tt}\|_{L^4(I;L^4)}\leq C\end{aligned} \tag{2.2}$$

and

$$\|q^I\|_{H^1},\|q^P\|_{H^1}\leq C, \tag{2.3}$$

where $I := [0,T]$.

Correspondingly, we assume that the permeability $k \in W^{2,\infty}(\Omega)$ and satisfies

$$k_0^{-1} \le k(x) \le k_0 \quad \text{for} \;\; x \in \Omega, \tag{2.4}$$

and the concentration-dependent viscosity $\mu \in W^{2,\infty}(\mathbb{R})$ satisfies

$$\mu_0^{-1} \le \mu(s) \le \mu_0 \quad \text{for} \;\; s \in \mathbb{R}, \tag{2.5}$$

for some positive constants k_0 and μ_0. Moreover, we assume that the diffusion-dispersion tensor is given by $D(\mathbf{u}) = \Phi d_m I + D^*(\mathbf{u})$, where $d_m > 0$, $D^*(\mathbf{u}) = d_1(\mathbf{u}) I + d_2(\mathbf{u})(\mathbf{u} \otimes \mathbf{u})$ is symmetric and positive definite and $\partial_{u_i} D \in L^\infty(\Omega)$, $\partial^2_{u_i u_j} D \in L^\infty(\Omega)$ [6]. For the initial-boundary value problem (1.1)-(1.4) to be well-posed, we require

$$\int_\Omega q^I \,\mathrm{d}x = \int_\Omega q^P \,\mathrm{d}x. \tag{2.6}$$

Let $\{t_n\}_{n=0}^N$ be a uniform partition of the time interval $[0,T]$ with $\tau = T/N$ and denote

$$p^n = p(x,t_n), \quad \mathbf{u}^n = \mathbf{u}(x,t_n), \quad c^n = c(x,t_n).$$

For any sequence of functions $\{f^n\}_{n=0}^N$, we define

$$D_\tau f^{n+1} = \frac{f^{n+1} - f^n}{\tau}.$$

A semi-decoupled time-discrete Galerkin finite element scheme is to find $P_h^n \in \widetilde{V}_h^{r+1}$ and $\mathcal{C}_h^n \in V_h^r$, $n = 0,1,\cdots,N$, such that for all $(\varphi_h, \phi_h) \in V_h^{r+1} \times V_h^r$,

$$\Big(\Phi D_\tau \mathcal{C}_h^{n+1}, \phi_h\Big) + \Big(D(\mathbf{U}_h^n)\nabla \mathcal{C}_h^{n+1}, \nabla \phi_h\Big) + \Big(\mathbf{U}_h^n \cdot \nabla \mathcal{C}_h^{n+1}, \phi_h\Big) = \Big(\hat{c} q^I - \mathcal{C}_h^{n+1} q^P, \phi_h\Big), \tag{2.7}$$

$$\left(\frac{k(x)}{\mu(\mathcal{C}_h^{n+1})} \nabla P_h^{n+1}, \nabla \varphi_h\right) = \Big(q^I - q^P, \varphi_h\Big), \tag{2.8}$$

where

$$\mathbf{U}_h^n = -\frac{k(x)}{\mu(\mathcal{C}_h^n)} \nabla P_h^n$$

and the initial data $\mathcal{C}_h^0 = I_h c_0$.

With an explicit treatment of the nonlinear convection, source and concentration-dependent viscosity, a slightly different semi-implicit Galerkin scheme is defined by

$$\Big(\Phi D_\tau \mathcal{C}_h^{n+1}, \phi_h\Big) + \Big(D(\mathbf{U}_h^n)\nabla \mathcal{C}_h^{n+1}, \nabla \phi_h\Big) + \Big(\mathbf{U}_h^n \cdot \nabla \mathcal{C}_h^n, \phi_h\Big) = \Big(\hat{c} q^I - \mathcal{C}_h^n q^P, \phi_h\Big), \tag{2.9}$$

$$\left(\frac{k(x)}{\mu(\mathcal{C}_h^n)} \nabla P_h^{n+1}, \nabla \varphi_h\right) = \Big(q^I - q^P, \varphi_h\Big), \tag{2.10}$$

where P_h^0 can be obtained from (2.8) with $n=-1$. Clearly, the second linearized scheme is fully decoupled. At each time step, one can solve these two linear systems in the second scheme for (P_h^{n+1},C_h^{n+1}) in parallel, while for the first scheme, one has to solve the system (2.7) for C_h^{n+1} and then, the system (2.8) for P_h^{n+1}. Here, we present our theoretical analysis only for the linearized scheme (2.7)-(2.8). The analysis presented in this paper can be easily extended to the second linearized scheme. Numerical results given in section 4 will show clearly that both linearized schemes are of the optimal accuracy and unconditional stability.

In this paper, we denote by C a generic positive constant and by ϵ a generic small positive constant, which are independent of n, h and τ. We present our main results in the following theorem.

Theorem 2.1. *Suppose that the initial-boundary value problem* (1.1)-(1.4) *has a unique solution* (p,c) *which satisfies* (2.2). *Then there exist positive constants h_0 and τ_0 such that when $h<h_0$ and $\tau<\tau_0$, the finite element system* (2.7)-(2.8) *admits a unique solution* $\{(P_h^n,C_h^n)\}_{n=1}^N \in (\widetilde{V}_h^2,V_h^1)$, *which satisfies*

$$\|P_h^n-p^n\|_{H^1}+\|C_h^n-c^n\|_{L^2}\le C(\tau+h^2), \tag{2.11}$$

$$\|C_h^n-c^n\|_{H^1}\le C(\tau+h). \tag{2.12}$$

3 The proof of Theorem 2.1

We define a time-discrete solution $(P^n,\mathcal{C}^n)$ by the following elliptic system:

$$\Phi D_\tau \mathcal{C}^{n+1}-\nabla\cdot(D(\mathbf{U}^n)\nabla\mathcal{C}^{n+1})+\mathbf{U}^n\cdot\nabla\mathcal{C}^{n+1}=\hat{c}q^I-\mathcal{C}^{n+1}q^P, \tag{3.1}$$

$$-\nabla\cdot\left(\frac{k(x)}{\mu(\mathcal{C}^{n+1})}\nabla P^{n+1}\right)=q^I-q^P, \tag{3.2}$$

for $x\in\Omega$ and $t\in[0,T]$, with the initial and boundary conditions

$$D(\mathbf{U}^n)\nabla\mathcal{C}^{n+1}\cdot\mathbf{n}=0,\quad \frac{k(x)}{\mu(\mathcal{C}^{n+1})}\nabla P^{n+1}\cdot\mathbf{n}=0,\qquad \text{for } x\in\partial\Omega,\ t\in[0,T], \tag{3.3a}$$

$$\mathcal{C}^0(x)=c_0(x)\qquad \text{for } x\in\Omega, \tag{3.3b}$$

where

$$\mathbf{U}^n=-\frac{k(x)}{\mu(\mathcal{C}^n)}\nabla P^n$$

and the condition $\int_\Omega P^{n+1}dx=0$ is enforced for the uniqueness of solution. With the solution of the time-discrete system $(P^n,\mathcal{C}^n)$, the error functions can be split into

$$\|p^n-P_h^n\|\le\|p^n-P^n\|+\|P^n-P_h^n\|, \tag{3.4}$$

$$\|c^n-C_h^n\|\le\|c^n-\mathcal{C}^n\|+\|\mathcal{C}^n-C_h^n\|. \tag{3.5}$$

The estimate for the first part of the error splitting in (3.4)-(3.5) and the regularity of the solution of the time-discrete system (3.1)-(3.3) were given by Theorem 3.1 of [27]. We present the results in the following lemma.

Lemma 3.1. *Suppose that the initial-boundary value problem* (1.1)-(1.4) *has a unique solution* $(p,\mathbf{u},c)$ *which satisfies* (2.2). *Then there exists a positive constant* τ_1 *such that when* $\tau<\tau_1$, *the time-discrete system* (3.1)-(3.3) *admits a unique solution* $(P^n,\mathcal{C}^n)$, $n=1,\cdots,N$, *which satisfies*

$$\|P^n\|_{H^3}^2+\|D_\tau\mathcal{C}^n\|_{L^4}^2+\|\mathcal{C}^n\|_{W^{2,4}}^2+\sum_{n=1}^N\tau\|D_\tau P^n\|_{H^2}^2+\sum_{n=1}^N\tau\|D_\tau\mathcal{C}^n\|_{H^2}^2\le C,$$

and

$$\|P^n-p^n\|_{H^1}+\|\mathcal{C}^n-c^n\|_{L^2}\le C\tau. \tag{3.6}$$

The following Sobolev embedding inequality will be used in our proof.

$$\|u_h\|_{L^p}\le C\|u_h\|_{H^1},\quad 1\le p\le 6,\quad u_h\in V_h^1. \tag{3.7}$$

To present a τ-independent estimate for the second part of the error splitting in (3.4)-(3.5), we define two projections below.

Let $\Pi_h:L^2(\Omega)\to V_h^2$ be the L^2 projection defined by

$$(\Pi_h\phi,\chi)=(\phi,\chi),\quad \text{for all } \phi\in L^2 \text{ and } \chi\in V_h^2.$$

For any fixed integer $n\ge 0$, let $\Pi_h^{n+1}:H^1(\Omega)\to V_h^1$ be a projection defined by the following elliptic problem,

$$\Big(D(\mathbf{U}^n)\nabla(v-\Pi_h^{n+1}v),\nabla\phi_h\Big)=0,\quad \text{for all } \phi_h\in V_h^1,\ v\in H^1(\Omega) \tag{3.8}$$

with $\int_\Omega(v-\Pi_h^{n+1}v)dx=0$.

By the classical theory of finite element methods for linear elliptic problems [37,44], with the regularity $\mathbf{U}^n\in H^2(\Omega)$, we have

$$\|v-\Pi_h v\|_{L^2}+h\|v-\Pi_h v\|_{H^1}\le Ch^3\|v\|_{H^3},\qquad \text{for all } v\in H^3(\Omega), \tag{3.9}$$

$$\|v-\Pi_h^n v\|_{L^2}+h\|v-\Pi_h^n v\|_{H^1}\le Ch^2\|v\|_{H^2},\qquad \text{for all } v\in H^2(\Omega), \tag{3.10}$$

$$\|\nabla\Pi_h^n v\|_{L^p}\le\|v\|_{W^{1,p}},\quad 2\le p\le\infty,\qquad \text{for all } v\in W^{1,p}(\Omega), \tag{3.11}$$

and

$$\left(\sum_{n=0}^{N-1}\tau\|D_\tau(\mathcal{C}^{n+1}-\Pi_h^{n+1}\mathcal{C}^{n+1})\|_{H^{-1}}^2\right)^{1/2}\le Ch^2, \tag{3.12}$$

Now we start to prove Theorem 2.1. Since the coefficient matrices of the finite element systems (2.7) and (2.8) at each time step are symmetric positive definite, the existence and uniqueness of the numerical solution follow immediately. Let

$$e_h^n = P_h^n - \Pi_h P^n \quad \text{and} \quad \theta_h^n = \mathcal{C}_h^n - \Pi_h^n \mathcal{C}^n,$$

where $\{(P^n,\mathcal{C}^n)\}_{n=1}^N$ is the solution of the time-discrete system (3.1)-(3.3). First we prove the following estimate

$$\max_{1\le n\le N}\|e_h^n\|_{H^1} + \max_{1\le n\le N}\|\theta_h^n\|_{L^2} \le Ch^2. \tag{3.13}$$

We rewrite the time-discrete system (3.1)-(3.3) in a weak form by

$$\Big(\Phi D_\tau \mathcal{C}^{n+1},\phi_h\Big) + \Big(D(\mathbf{U}^n)\nabla\mathcal{C}^{n+1},\nabla\phi_h\Big) + \Big(\mathbf{U}^n\cdot\nabla\mathcal{C}^{n+1},\phi_h\Big) = \Big(\hat{c}q^I - \mathcal{C}^{n+1}q^P,\phi_h\Big), \tag{3.14}$$

$$\Big(\frac{k(x)}{\mu(\mathcal{C}^{n+1})}\nabla P^{n+1},\nabla\varphi_h\Big) = \Big(q^I - q^P,\varphi_h\Big), \tag{3.15}$$

for any $\phi_h \in V_h^1$ and $\varphi_h \in V_h^2$. From the finite element system (2.7)-(2.8) and the above equations, we see that the error functions $(e_h^{n+1},\theta_h^{n+1})$ satisfy the equations

$$\begin{aligned}
&\Big(\Phi D_\tau \theta_h^{n+1},\phi_h\Big) + \Big(D(\mathbf{U}_h^n)\nabla\theta_h^{n+1},\nabla\phi_h\Big)\\
=&\Big(\Phi D_\tau(\mathcal{C}^{n+1} - \Pi_h^{n+1}\mathcal{C}^{n+1}),\phi_h\Big) - \Big(\mathbf{U}^n\cdot\nabla(\mathcal{C}_h^{n+1} - \mathcal{C}^{n+1}),\phi_h\Big)\\
&-\Big((\mathbf{U}_h^n - \mathbf{U}^n)\cdot\nabla\mathcal{C}_h^{n+1},\phi_h\Big) - \Big((\mathcal{C}_h^{n+1} - \mathcal{C}^{n+1})q^P,\phi_h\Big)\\
&+\Big((D(\mathbf{U}^n) - D(\mathbf{U}_h^n))\nabla\Pi_h^{n+1}\mathcal{C}^{n+1},\nabla\phi_h\Big)\\
:=&J_1(\phi_h) + J_2(\phi_h) + J_3(\phi_h) + J_4(\phi_h) + J_5(\phi_h),
\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}
&\Big(\frac{k(x)}{\mu(\mathcal{C}_h^{n+1})}\nabla e_h^{n+1},\nabla\varphi_h\Big)\\
=&\Big(\frac{k(x)}{\mu(\mathcal{C}_h^{n+1})}\nabla(P^{n+1} - \Pi_h P^{n+1}),\nabla\varphi_h\Big) - \Big(\Big(\frac{k(x)}{\mu(\mathcal{C}_h^{n+1})} - \frac{k(x)}{\mu(\mathcal{C}^{n+1})}\Big)\nabla P^{n+1},\nabla\varphi_h\Big),
\end{aligned} \tag{3.17}$$

for any $\phi_h \in V_h^1$ and $\varphi_h \in V_h^2$.

We substitute $\varphi_h = e_h^{n+1}$ into (3.17) to obtain

$$\begin{aligned}
\|e_h^{n+1}\|_{H^1} &\le C(\|P^{n+1} - \Pi_h P^{n+1}\|_{H^1} + \|\mathcal{C}_h^{n+1} - \mathcal{C}^{n+1}\|_{L^2})\\
&\le Ch^2 + C\|\theta_h^{n+1}\|_{L^2}.
\end{aligned} \tag{3.18}$$

Taking $\phi_h = \theta_h^{n+1}$ in (3.16), we get

$$\begin{aligned}
|J_1(\theta_h^{n+1})| &\le C\|D_\tau(\mathcal{C}^{n+1}-\Pi_h^{n+1}\mathcal{C}^{n+1})\|_{H^{-1}}\|\theta_h^{n+1}\|_{H^1}\\
&\le \epsilon\|\theta_h^{n+1}\|_{H^1}^2 + C\epsilon^{-1}\|D_\tau(\mathcal{C}^{n+1}-\Pi_h^{n+1}\mathcal{C}^{n+1})\|_{H^{-1}}^2,\\
|J_4(\theta_h^{n+1})| &\le C\|q^P\|_{L^3}(\|\theta_h^{n+1}\|_{L^3}^2 + \|\mathcal{C}^{n+1}-\Pi_h^{n+1}\mathcal{C}^{n+1}\|_{L^2}\|\theta_h^{n+1}\|_{L^6})\\
&\le \epsilon\|\nabla\theta_h^{n+1}\|_{L^2}^2 + C\epsilon^{-1}\|\theta_h^{n+1}\|_{L^2}^2 + C\epsilon^{-1}h^4,\\
|J_5(\theta_h^{n+1})| &\le C\|\nabla\Pi_h^{n+1}\mathcal{C}^{n+1}\|_{L^\infty}\|\mathbf{U}_h^n-\mathbf{U}^n\|_{L^2}\|\nabla\theta_h^{n+1}\|_{L^2}\\
&\le C\epsilon^{-1}\|\mathbf{U}_h^n-\mathbf{U}^n\|_{L^2}^2 + \epsilon\|\nabla\theta_h^{n+1}\|_{L^2}^2\\
&\le \epsilon\|\nabla\theta_h^{n+1}\|_{L^2}^2 + C\epsilon^{-1}\|\theta_h^n\|_{L^2}^2 + C\epsilon^{-1}h^4,
\end{aligned}$$

and

$$\begin{aligned}
|J_3(\theta_h^{n+1})| &= \big|\big((\mathbf{U}_h^n-\mathbf{U}^n)\cdot\nabla\theta_h^{n+1},\theta_h^{n+1}\big) + \big((\mathbf{U}_h^n-\mathbf{U}^n)\cdot\nabla\Pi_h^{n+1}\mathcal{C}^{n+1},\theta_h^{n+1}\big)\big|\\
&\le C\|\mathbf{U}_h^n-\mathbf{U}^n\|_{L^2}\|\theta_h^{n+1}\|_{L^6}(\|\nabla\theta_h^{n+1}\|_{L^3} + \|\nabla\Pi_h^{n+1}\mathcal{C}^{n+1}\|_{L^3})\\
&\le (Ch^{-d/6}\|\theta_h^n\|_{L^2} + Ch^{2-d/6} + \epsilon)\|\theta_h^{n+1}\|_{H^1}^2 + C\epsilon^{-1}\|\theta_h^n\|_{L^2}^2 + C\epsilon^{-1}h^4,
\end{aligned}$$

where we have noted (3.7) and used the inverse inequality

$$\|\nabla\theta_h^{n+1}\|_{L^3} \le Ch^{-d/6}\|\nabla\theta_h^{n+1}\|_{L^2}. \tag{3.19}$$

Moreover, by noting the fact that $\nabla\cdot\mathbf{U}^n = q^I - q^P$ and using integration by part, we have

$$\begin{aligned}
J_2(\theta_h^{n+1}) &= -(\mathbf{U}^n\cdot\nabla(\theta_h^{n+1}+\Pi^{n+1}\mathcal{C}^{n+1}-\mathcal{C}^{n+1}),\theta_h^{n+1})\\
&= ((q^I-q^P)(\theta_h^{n+1}+\Pi_h^{n+1}\mathcal{C}^{n+1}-\mathcal{C}^{n+1}),\theta_h^{n+1})\\
&\quad + (\theta_h^{n+1}+\Pi^{n+1}\mathcal{C}^{n+1}-\mathcal{C}^{n+1},\mathbf{U}^n\cdot\nabla\theta_h^{n+1})
\end{aligned}$$

and therefore,

$$\begin{aligned}
|J_2(\theta_h^{n+1})| &\le C\|q^I-q^P\|_{L^3}\|\theta_h^{n+1}\|_{L^6}(\|\mathcal{C}^{n+1}-\Pi_h^{n+1}\mathcal{C}^{n+1}\|_{L^2} + \|\theta_h^{n+1}\|_{L^2})\\
&\quad + C\|\mathbf{U}^n\|_{L^\infty}\|\nabla\theta_h^{n+1}\|_{L^2}(\|\mathcal{C}^{n+1}-\Pi_h^{n+1}\mathcal{C}^{n+1}\|_{L^2} + \|\theta_h^{n+1}\|_{L^2})\\
&\le \epsilon\|\theta_h^{n+1}\|_{H^1}^2 + C\epsilon^{-1}\|\theta_h^{n+1}\|_{L^2}^2 + C\epsilon^{-1}h^4.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{1}{2}D_\tau\Phi\|\theta_h^{n+1}\|_{L^2}^2 + \big\|\sqrt{D(\mathbf{U}_h^n)}\nabla\theta_h^{n+1}\big\|_{L^2}^2\\
&\le (\epsilon + Ch^{-d/6}\|\theta_h^n\|_{L^2} + Ch^{2-d/6})\|\nabla\theta_h^{n+1}\|_{L^2}^2 + C\epsilon^{-1}(\|\theta_h^{n+1}\|_{L^2}^2 + \|\theta_h^n\|_{L^2}^2) + C\epsilon^{-1}h^4\\
&\quad + C\epsilon^{-1}\big\|D_\tau(\mathcal{C}^{n+1}-\Pi_h^{n+1}\mathcal{C}^{n+1})\big\|_{H^{-1}}^2.
\end{aligned} \tag{3.20}$$

Now we prove the τ-independent estimate

$$\|\theta_h^n\|_{L^2} \le h \tag{3.21}$$

by mathematical induction. Since $\|\theta_h^0\|_{L^2} \le Ch^2$, there exists a positive constant h_1 such that $\|\theta_h^0\|_{L^2} < h$ when $h < h_1$. We assume that the inequality (3.21) holds for $0 \le n \le k$. Then there exists a positive constant h_2 such that when $h < h_2$, (3.20) reduces to

$$\begin{aligned} &D_\tau \Phi \|\theta_h^{n+1}\|_{L^2}^2 + \frac{1}{2}\|\sqrt{D(\mathbf{U}_h^n)}\nabla\theta_h^{n+1})\|_{L^2}^2 \\ &\le Ch^4 + C(\|\theta_h^{n+1}\|_{L^2}^2 + \|\theta_h^n\|_{L^2}^2) + C\|D_\tau(\mathcal{C}^{n+1} - \Pi_h^{n+1}\mathcal{C}^{n+1})\|_{H^{-1}}^2 \end{aligned}$$

for $0 \le n \le k$. By applying Gronwall's inequality and (3.12), there exits $\tau_2 > 0$ such that,

$$\|\theta_h^{n+1}\|_{L^2} \le Ch^2 \tag{3.22}$$

for $0 \le n \le k$, when $\tau < \tau_2$.

Therefore, there exists $h_3 > 0$ such that when $h < h_3$ we have

$$\|\theta_h^{k+1}\|_{L^2} < h.$$

Taking $\tau_0 \le \min\{\tau_1, \tau_2\}$ and $h_0 \le \min\{h_1, h_2, h_3\}$, the mathematical induction is closed. We see that the inequality (3.21) holds and therefore, (3.22) holds for all $0 \le n \le N-1$.

By (3.18) and (3.22) we further derive that

$$\|e_h^{n+1}\|_{H^1} + \|\theta_h^{n+1}\|_{L^2} \le Ch^2. \tag{3.23}$$

Secondly, by Lemma 3.1, the projection error estimates (3.9)-(3.10) and the above inequality, we derive (2.11). Moreover, (2.12) follows from Lemma 3.1 and (3.23) together with the inverse inequality. The proof of Theorem 2.1 is complete. □

Remark 3.1. The above analysis for unconditional stability relies on the τ-independent estimate (3.21) or (3.22), and the inverse inequality (3.19), while the previous analysis was based on an τ-dependent estimate $\|C_h^n - \Pi_h^n c^n\|_{L^2} \le C(\tau + h^2)$.

4 Numerical examples

In this section, we present some numerical results for incompressible miscible flows in both two and three-dimensional porous media. We focus on the unconditional stability of the linearized semi-implicit Galerkin FEMs. All computations are performed by using the software FreeFem++.

We rewrite the system (1.1)-(1.4) by

$$\frac{\partial c}{\partial t} - \nabla\cdot(D(\mathbf{u})\nabla c) + \mathbf{u}\cdot\nabla c = g, \tag{4.1}$$

$$\nabla\cdot\mathbf{u} = f, \tag{4.2}$$

$$\mathbf{u} = -\frac{1}{\mu(c)}\nabla p, \tag{4.3}$$

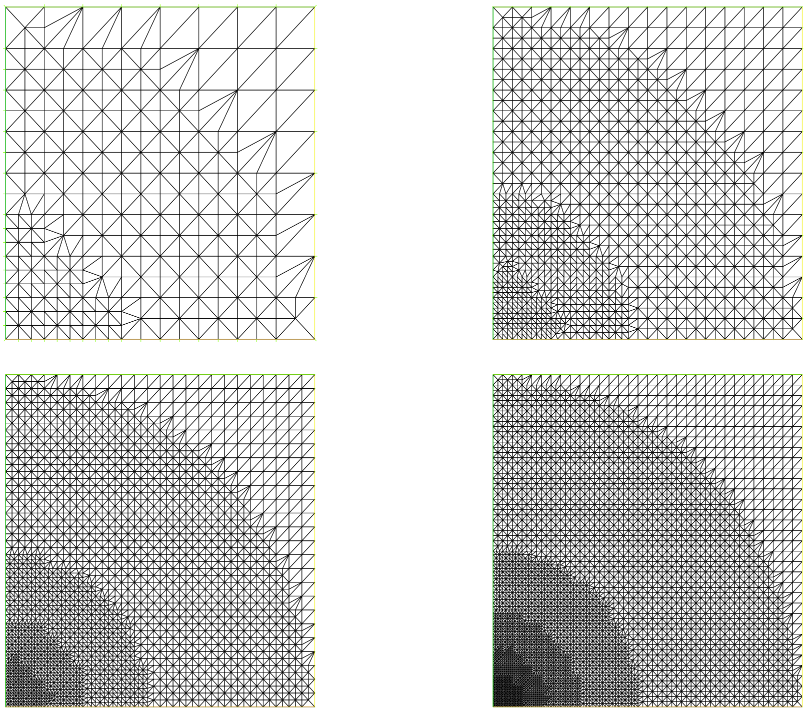

Figure 1: Two-dimensional non-uniform meshes.

where $\Omega=(0,1)\times(0,1)$ and $D(\mathbf{u})=1+|\mathbf{u}|^2/(1+|\mathbf{u}|)$ and $\mu(c)=1+c$.

First, we consider a two-dimensional model where $\Omega=[0,1]\times[0,1]$. The functions f and g are chosen corresponding to the exact solution

$$p=1000x^2(1-x)^3y^2(1-y)^3t^2e^{-t}, \tag{4.4}$$

$$c=0.1+50x^2(1-x)^2y^2(1-y)^2te^t, \tag{4.5}$$

which satisfies the boundary condition (1.4).

Two types of triangular meshes, a uniform mesh and a locally refined mesh, are used in our numerical tests. The uniform one is generated by a triangular partition with $M+1$ nodes of uniform distribution in each direction. Four refined ones are given in Fig. 1 with 474, 2202, 5392 and 10136 triangular elements, respectively. Clearly, previous analyses enforce a stronger time-step restriction when a non-uniform mesh is used. We solve the system (4.1)-(4.3) by the proposed two schemes presented in (2.7)-(2.8) and (2.9)-(2.10), respectively, up to the time $t=1$, while theoretical analysis was given only for the first linearized scheme. We apply the FE approximation in $\widetilde{V}_h^{r+1}\times V_h^r$, $r=1,2$, in the spatial direction, with which the optimal error in L^2-norm is $\mathcal{O}(\tau+h^{r+1})$. A restarted GMRES algorithm is applied for solving the linear systems at each time step. To illustrate our error estimates, we take $\tau=\mathcal{O}(h^{r+1})$ and present numerical errors in Table 1 for the

Table 1: Errors of the semi-decoupled scheme (2.7)-(2.8) with the uniform mesh.

$\tau=8h^2$	$r=1$	
h	$\|P_h^N-p(\cdot,t_N)\|_{H^1}$	$\|C_h^N-c(\cdot,t_N)\|_{L^2}$
1/8	4.970E-02	2.141E-02
1/16	1.291E-02	4.937E-03
1/32	3.264E-03	1.209E-03
convergence rate	1.98	2.03
$\tau=64h^3$	$r=2$	
h	$\|P_h^N-p(\cdot,t_N)\|_{H^1}$	$\|C_h^N-c(\cdot,t_N)\|_{L^2}$
1/8	7.492E-03	9.913E-03
1/16	7.476E-04	9.059E-04
1/32	9.182E-05	1.098E-04
convergence rate	3.03	3.04

Table 2: Errors of the fully decoupled scheme (2.9)-(2.10) with the uniform mesh.

$\tau=8h^2$	$r=1$	
h	$\|P_h^N-p(\cdot,t_N)\|_{H^1}$	$\|C_h^N-c(\cdot,t_N)\|_{L^2}$
1/8	6.269E-02	5.912E-02
1/16	1.641E-02	1.535E-03
1/32	4.155E-03	3.877E-03
convergence rate	1.98	1.99
$\tau=64h^3$	$r=2$	
h	$\|P_h^N-p(\cdot,t_N)\|_{H^1}$	$\|C_h^N-c(\cdot,t_N)\|_{L^2}$
1/8	3.519E-02	5.026E-02
1/16	4.437E-03	6.306E-03
1/32	5.561E-04	7.898E-04
convergence rate	3.00	3.00

semi-decoupled scheme (2.7)-(2.8) and in Table 2 for the fully decoupled scheme (2.9)-(2.10). We can observe from Tables 1-2 that the L^2 errors for both the semi-decoupled and the fully decoupled schemes are proportional to $\mathcal{O}(h^{r+1})$. Compared with the fully decoupled scheme, the semi-decoupled scheme shows better accuracy. However, the fully decoupled scheme is more efficient in computation.

To demonstrate the unconditional stability of the schemes, we take several different spatial meshes with $M=8,16,32,64$ for each fixed τ and we plot the error functions $\|P_h^N-p(\cdot,t_N)\|_{H^1}$ and $\|C_h^N-c(\cdot,t_N)\|_{L^2}$ in Figs. 2-3 for the uniform mesh and in Figs. 4-5 for the non-uniform mesh. Based on our theoretical analysis, in this case $r=1$,

$$\|P_h^N-p(\cdot,t_N)\|_{H^1},\ \|C_h^N-c(\cdot,t_N)\|_{L^2}=\mathcal{O}(\tau+h^2)$$

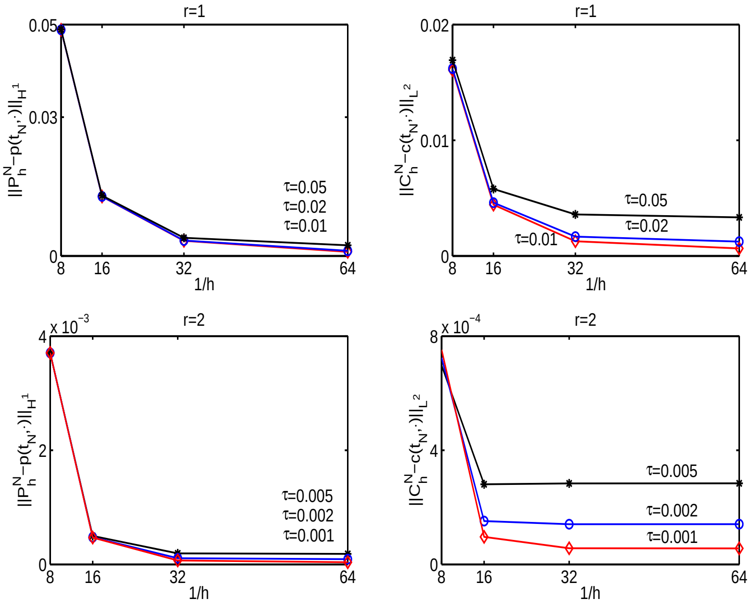

Figure 2: Errors of the semi-decoupled scheme (2.7)-(2.8) with the 2D uniform mesh where $h=1/M$.

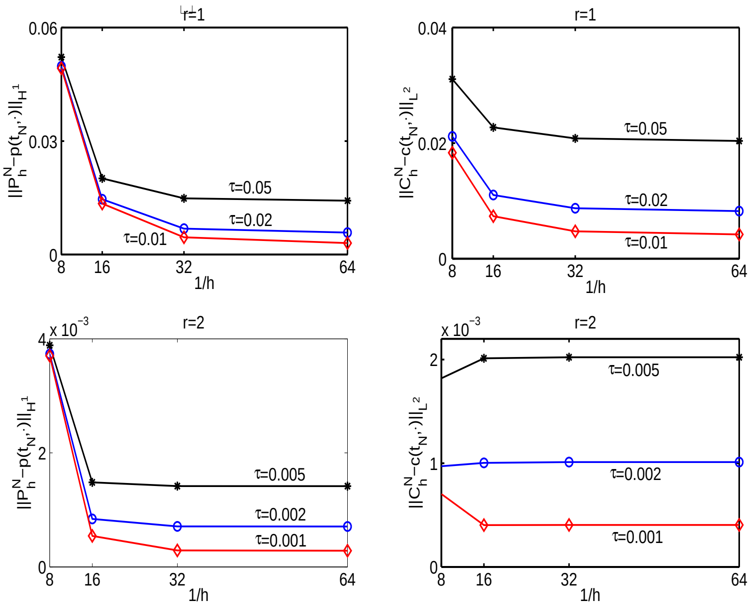

Figure 3: Errors of the fully decoupled scheme (2.9)-(2.10) with the 2D uniform mesh where $h=1/M$.

which tends to $\mathcal{O}(\tau)$ as $h\to 0$. We can see clearly from Figs. 2-3 that the numerical errors behave like $\mathcal{O}(\tau)$ as $h\to 0$ (while as $\tau/h\to\infty$), which shows that no time step condition is needed.

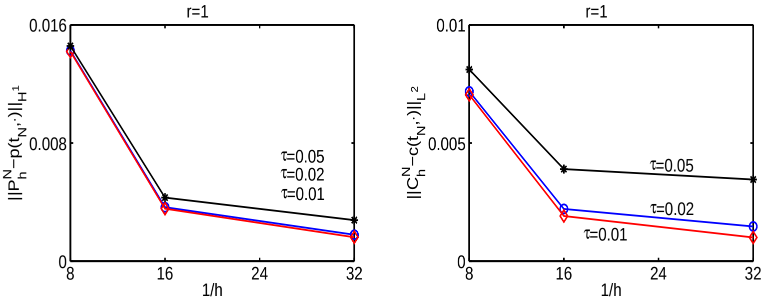

Figure 4: Errors of the semi-decoupled scheme (2.7)-(2.8) with the 2D non-uniform mesh where $h=1/M$.

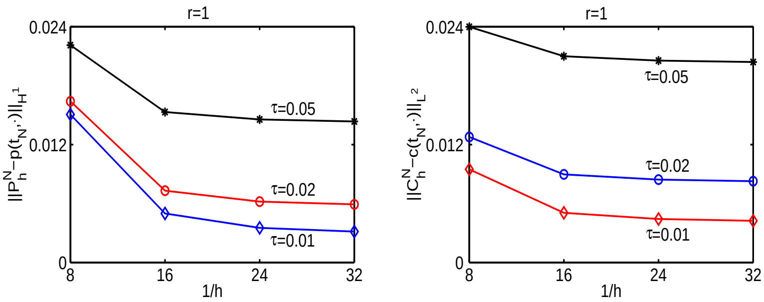

Figure 5: Errors of the fully decoupled scheme (2.9)-(2.10) with the 2D non-uniform mesh where $h=1/M$.

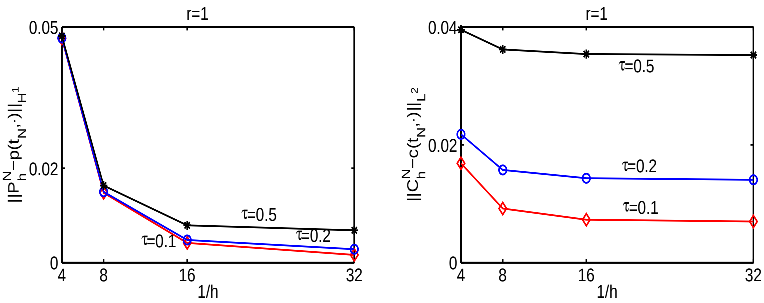

Figure 6: Errors of the semi-decoupled scheme (2.9)-(2.10) with the 3D uniform mesh where $h=1/M$.

Secondly, we solve the equations (4.1)-(4.3) in a three-dimensional cube $[0,1]\times[0,1]\times[0,1]$. The functions f and g are chosen corresponding to the exact solution

$$p=10000x^2(1-x)^3y^2(1-y)^3z^2(1-z)^3t^2e^{-t}, \tag{4.6}$$

$$c=0.1+2000x^2(1-x)^2y^2(1-y)^2z^2(1-z)^3te^{-t}. \tag{4.7}$$

A uniform tetrahedral partition of the cube with $M+1$ mesh points at each spatial direction is used. We solve the equations by the semi-decoupled scheme (2.7)-(2.8) with $r=1$ up to time $t=1$ and we present errors of the numerical solution in Fig. 6. Again, the numerical results show that the scheme is unconditionally stable.

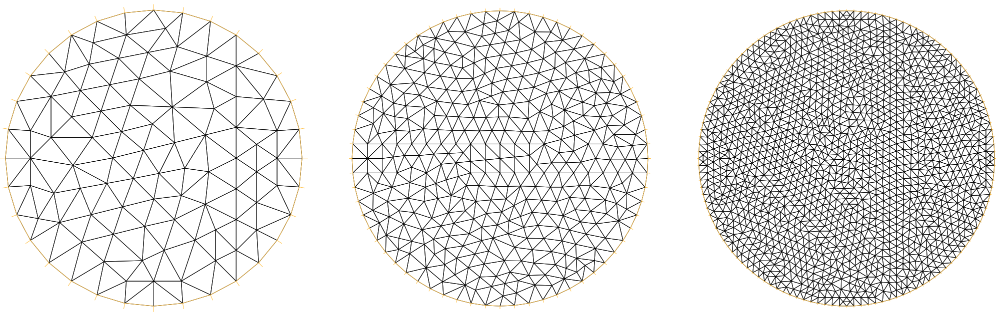

Figure 7: The FEM meshes with $M=32$, $M=64$ and $M=128$, respectively.

Finally we solve the equations (4.1)-(4.3) in a unit circle by the semi-decoupled scheme (2.7)-(2.8) with $r=1$ up to time $t=1$ and the same physical parameters as above. The functions f and g are chosen corresponding to the exact solution

$$p=1+40e^{2t}(1+t^2)x^2y^2(1-x)^3(1-y)^3, \tag{4.8}$$

$$c=1+50e^{t}(1+t^3)\sin(x^2)\sin(y^2)(1-x)^3(1-y)^3. \tag{4.9}$$

The meshes are generated by the software with M boundary points, where we take $M=16,32,64,128$, respectively. Three typical meshes are shown in Fig. 7. We present the L^2 errors of the pressure and the concentration in Fig. 8. The same observations can be made here.

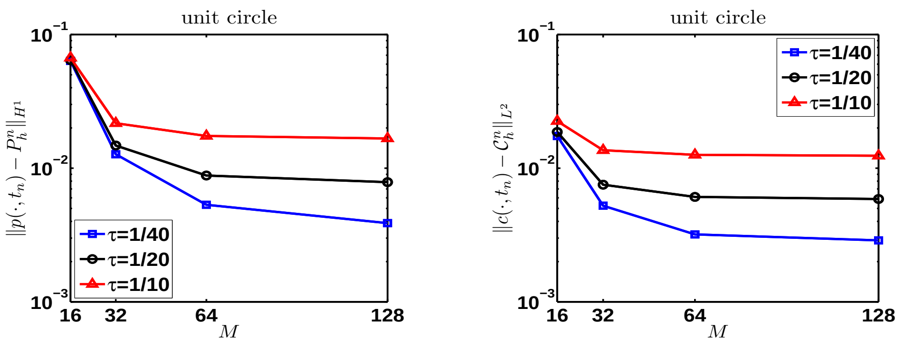

Figure 8: Errors of the semi-decoupled scheme (2.9)-(2.10) in a unit circle.

5 Conclusions

The time stepsize condition is always a key issue for linearized schemes. In this paper, we have proved unconditional stability of a commonly-used linearized semi-implicit Euler scheme with standard Galerkin FEM for a nonlinear and strongly coupled parabolic system from incompressible miscible flow in porous media. With the stability analysis, optimal L^2 error estimates of the linearized Galerkin FEMs are obtained also unconditionally, while all previous works have imposed certain restriction on the time-step size. Our numerical results confirm our analysis. The approach presented in this paper can be extended to many other nonlinear equations and other linearized semi-implicit methods.

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