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CONVERGENCE OF A RELAXATION SCHEME FOR A 2×2 TRIANGULAR SYSTEM OF CONSERVATION LAWS

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Abstract. We study relaxation approximations to solutions of a 2×2 triangular system of conservation laws. We show that smooth relaxation approximations exist for all time. A finite difference approximation of the relaxation system gives rise to a relaxation scheme of the Jin and Xin type. In both cases we show that a sequence of approximate solutions is produced where the limit is a weak solution of the triangular system. Compensated compactness is used to establish convergence.

Key words. triangular systems of conservation laws, relaxation, compensated compactness

1. Introduction

The aim of this paper is to prove convergence of two sequences of functions approximating a weak solution (u, v) of the 2×2 hyperbolic system

$$\begin{cases} u_t + f(u)_x = 0, \\ v_t + g(u,v)_x = 0, \end{cases} \qquad x \in \mathbb{R},\ t > 0, \qquad (u,v)(x,0) = (u_0(x), v_0(x)), \quad x \in \mathbb{R}, \tag{1}$$

where the flux functions f and g, and the initial data are known. This type of system arises in models of three-phase flow in porous media, see [9]. Systems of this kind are called *triangular*, since the first equation is independent of the second. An interesting class occurs if $f \equiv 0$, so that u acts as a coefficient which may be discontinuous. In recent years, conservation laws with discontinuous coefficients has received considerable attention, see e.g. [1, 10] and the references therein. For scalar conservation laws with a discontinuous coefficient, i.e., $f \equiv 0$ in (1), [1] outlines a theory of well-posedness. We emphasise that no such theory exists if $f \not\equiv 0$, and while a corollary of the convergence proved in this paper is the existence of weak solutions of (1), our methods yield no information regarding uniqueness or continuous dependence on the initial data for this weak solution.

The first sequence $\{(u^\varepsilon, v^\varepsilon)\}$ for which we prove convergence consists of the solutions of the weakly coupled strictly hyperbolic relaxation system:

$$\begin{cases} u^\varepsilon_t + w^\varepsilon_x = 0, \\ w^\varepsilon_t + a^2 u^\varepsilon_x = \frac{1}{\varepsilon}\left(f(u^\varepsilon) - w^\varepsilon\right), \\ v^\varepsilon_t + z^\varepsilon_x = 0, \\ z^\varepsilon_t + b^2 v^\varepsilon_x = \frac{1}{\varepsilon}\left(g(u^\varepsilon, v^\varepsilon) - z^\varepsilon\right), \end{cases} \qquad x \in \mathbb{R},\ t > 0, \tag{2}$$

with initial condition

$$(u^\varepsilon, w^\varepsilon, v^\varepsilon, z^\varepsilon)(x,0) = (u^\varepsilon_0, w^\varepsilon_0, v^\varepsilon_0, z^\varepsilon_0)(x) = (u^\varepsilon_0, f(u^\varepsilon_0), v^\varepsilon_0, g(u^\varepsilon_0, v^\varepsilon_0))(x), \tag{3}$$

for $x \in \mathbb{R}$. We consider the triangular system (1) as an equilibrium for the Cauchy problem (2) - (3) as first introduced by Liu in [13]. The second sequence for which we prove convergence is made by applying a finite difference scheme to the relaxation system (2). The main advantage in construction a numerical scheme in this manner

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is that one does not rely on solving local Riemann problems when approximating solutions of (1). Moreover, the scheme is explicit leaving it easy to implement.

The relaxation approximation $(u^\varepsilon, w^\varepsilon)$ given by (2) has been shown to converge strongly to $(u, f(u))$ where u is the entropy solution of the single conservation law

$$u_t + f(u)_x = 0,\ u(x,0) = u_0(x),\ x \in \mathbb{R},\ t > 0. \tag{4}$$

See for instance [15, 16]. In [7] a numerical scheme was constructed based on the relaxation approximations, and convergence to the entropy solution of (4) was proved in [2]. In this paper we extend these results to hold for the relaxation approximations and a relaxation scheme for the complete 2×2 system (1). In particular, we show that a subsequence $\{(v^{\varepsilon_n}, z^{\varepsilon_n})\}_{n\in\mathbb{N}}$ of solutions of (2) converges in L^p_{loc}, $1 \le p < \infty$, to a weak solution of

$$v_t + g(u,v)_x = 0,\ v(x,0) = v_0(x), \tag{5}$$

where u is the entropy solution of (4), and similarly in the numerical case. In [9] finite volume schemes was used to construct approximate solutions of (1), and convergence of a subsequence to a weak solution was shown following the compensated compactness approach of [11] where convergence of the Lax-Friedrichs scheme was established for conservation laws with a discontinuous space-time dependent flux. Finite volume schemes has also been used to approximate solutions to the relaxation approximations (2) for a general $n \times n$ system, see [3].

If $f \equiv 0$ and $u(x) = u_0(x)$ is some BV function, our numerical scheme will reduce to the relaxation scheme in [8]. In both works [8, 9] convergence to a weak solution of (1) of some sequences of approximate solutions given by the respective schemes, was proved under some strong CFL conditions depending on the flux functions. In order to prove convergence of the relaxation scheme we will also need to introduce such a strengthened CFL condition. The fact that the approximations of the entropy solution of (4) is of bounded variation in space uniformly in the approximation parameters is crucial when proving convergence. A major difficulty in extending the results to hold for relaxation approximations to solutions of $n \times n$ triangular systems, is that we have no BV estimates on the approximations of the function v in (1). This lack of regularity also seems an obstacle in obtaining existence results for such triangular systems.

We will consider the system (1) under a set of assumptions, presented in the following section, which are needed to assure that solutions are bounded. Moreover, the main stability criterion is motivated by a Chapman-Enskog expansion, which in the case of system (2) reads within an $\mathcal{O}(\varepsilon^2)$ term

$$\begin{aligned} u^\varepsilon_t + f(u^\varepsilon)_x &= \varepsilon\left[\left(a^2 - f'(u^\varepsilon)^2\right) u^\varepsilon_x\right]_x, \\ v^\varepsilon_t + g(u^\varepsilon, v^\varepsilon)_x &= \varepsilon\left[\left(b^2 - g_v(u^\varepsilon, v^\varepsilon)^2\right) v^\varepsilon_x\right]_x \\ &\quad - \varepsilon\left[\left(f'(u^\varepsilon) g_u(u^\varepsilon, v^\varepsilon) + g_u(u^\varepsilon, v^\varepsilon) g_v(u^\varepsilon, v^\varepsilon)\right) u^\varepsilon_x\right]_x. \end{aligned} \tag{6}$$

Equation (6) gives us a first order correction to (1). For the equations to be parabolic we need $\left(a^2 - f'(u^\varepsilon)\right) \ge 0$ and $\left(b^2 - g_v(u^\varepsilon, u^\varepsilon)\right) \ge 0$. The condition

$$|f'(u)| < a \text{ and } |\partial_u g(u,v)|,\ |\partial_v g(u,v)| < b,$$

is called the subcharacteristic condition and is due to Whitham [17], Liu [13] and Chen, Levermore and Liu [5]. Note also that the variables w^ε and z^ε in (2) can be eliminated. The result is a system of two conservation laws that has been regularized by a second order term

$$\begin{aligned} u_t^\varepsilon + f(u^\varepsilon)_x &= -\varepsilon\left(u_{tt}^\varepsilon - a^2 u_{xx}^\varepsilon\right),\\ v_t^\varepsilon + g(u^\varepsilon, v^\varepsilon)_x &= -\varepsilon\left(v_{tt}^\varepsilon - b^2 v_{xx}^\varepsilon\right).\end{aligned}$$

Hence we expect (2) to be a first order approximation of (1) as $\varepsilon \downarrow 0$.

The remaining part of this paper is organized as follows. In Section 2 we present some preliminary results. The main convergence results can be found in Section 3 and Section 4 for the smooth relaxation approximations and the relaxation scheme respectively. Finally, a numerical experiment is presented in Section 5.

2. Mathematical framework

We assume that the initial data in (1) and the flux functions f and g satisfy the following assumptions:

$$(u_0, v_0) \in \left(L^1(\mathbb{R}) \cap BV(\mathbb{R})\right)^2, \; (u_0, v_0)(x) \in [0,1]^2 \text{ for all } x \in \mathbb{R}, \tag{A.1}$$

$$f \in C^2([0,1];\mathbb{R}), \;\; g \in C^2\left([0,1]^2;\mathbb{R}\right), \tag{A.2}$$

$$|f'(u)| < a \text{ and } |\partial_u g(u,v)|, \; |\partial_v g(u,v)| < b \text{ for all } (u,v) \in [0,1]^2, \tag{A.3}$$

$$g(\cdot,0) = g(\cdot,1) = f(0) = f(1) = 0, \tag{A.4}$$

(A.5) The map $v \to g(u,v)$ is genuinely nonlinear for each $0 \le u \le 1$.

An example of a set of flux functions satisfying the above criteria is considered in Section 5.

Remark 2.1. *The assumptions that the initial data take values in* $[0,1]$*, together with the assumption A.4, is used to obtain supnorm bounds on the approximate solutions. If such bounds are available by other means, these assumptions can be relaxed. Furthermore, the assumption on* f*;* $f(0) = f(1) = 0$ *is not really necessary, but is only used so that a straightforward modification of the proof of boundedness of* v^ε *can be used to show boundedness for* u^ε*.*

Furthermore, the initial data for the relaxed system (2) satisfies for all $\varepsilon > 0$

$$\begin{aligned} &(u_0^\varepsilon, v_0^\varepsilon) \in C_0^2(\mathbb{R}) \times C_0^2(\mathbb{R}), && (u_0^\varepsilon, v_0^\varepsilon)(x) \in [0,1]^2 \text{ for all } x \in \mathbb{R},\\ &w_0^\varepsilon = f(u_0^\varepsilon),\; z_0^\varepsilon = g(u_0^\varepsilon, v_0^\varepsilon), && \sup_{\varepsilon>0}\left\{\|\partial_x u_0^\varepsilon\|_{L^1(\mathbb{R})}, \|\partial_x v_0^\varepsilon\|_{L^1(\mathbb{R})}\right\} < \infty,\end{aligned}$$

$$u_0^\varepsilon \to u_0,\; v_0^\varepsilon \to v_0 \text{ in } L^1(\mathbb{R}) \text{ as } \varepsilon \downarrow 0.$$

We want to show that there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0,\infty)$, $\varepsilon_n \to 0$, such that the solution $(u^{\varepsilon_n}, v^{\varepsilon_n})$ of (2) converges to a weak solution (u,v) of (1), defined as the entropy solution of (4) together with a distributional solution of (5).

Definition 2.1. *Let* $u, v : \mathbb{R} \times (0,\infty) \to \mathbb{R}$ *be two functions. We say that the pair* (u,v) *is a weak solution of the Cauchy problem (1) if* $u, v \in L^\infty(\mathbb{R} \times (0,\infty))$ *satisfy* (1) *in the sense of distributions on* $\mathbb{R} \times [0,\infty)$*, and* u *is the entropy solution of (4) in the Kružkov sense.*

Lemma 2.1. *There exists a global classical solution* $(u^\varepsilon, w^\varepsilon)$ *of the Cauchy problem*

$$\begin{cases} u_t^\varepsilon + w_x^\varepsilon = 0, & x \in \mathbb{R},\; t > 0,\\ w_t^\varepsilon + a^2 u_x^\varepsilon = \frac{1}{\varepsilon}\left(f(u^\varepsilon) - w^\varepsilon\right), & x \in \mathbb{R},\; t > 0 \end{cases} \tag{7}$$

with $(u^\varepsilon, w^\varepsilon)(x,0) = (u_0^\varepsilon, u_0^\varepsilon)(x)$, $x \in \mathbb{R}$. *Furthermore,* $T.V._x(u^\varepsilon(\cdot,t)), T.V._x(w^\varepsilon(\cdot,t)) \leq C$ *for some constant* C *that does not depend on* ε *or* t, *and* $u^\varepsilon \to u, w^\varepsilon \to f(u)$ *in* $L^p_{\text{loc}}(\mathbb{R}\times(0,\infty))$ *where* u *is the unique entropy solution of* (4).

For a proof see [15]. In order to show that the set $\{v^\varepsilon\}_{\varepsilon>0}$ is compact in $L^p_{\text{loc}}(\mathbb{R}\times(0,\infty))$, we will use the following result from [6] (Lemma 2.1) based on the Murat-Tartar compensated compactness method:

Lemma 2.2. *Let* u *be the unique entropy solution of the single conservation law* (4), *and let* $\{v^\varepsilon\}_{\varepsilon>0}$ *be a family of functions defined on* $\mathbb{R}\times(0,\infty)$. *If* $\{v^\varepsilon\}_{\varepsilon>0}$ *is in a bounded set of* $L^1_{\text{loc}}(\mathbb{R}\times(0,\infty))$, *and for every constant* $c \in \mathbb{R}$ *the family*

$$\left\{\partial_t |v^\varepsilon - c| + \partial_x \left(\text{sign}(v^\varepsilon - c)(g(u,v^\varepsilon) - g(u,c))\right)\right\}_{\varepsilon>0}$$

is in a compact set of $H^{-1}_{\text{loc}}(\mathbb{R}\times(0,\infty))$, *then there exists a sequence* $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0,\infty)$, $\varepsilon_n \to 0$, *and a map* $v \in L^\infty_{\text{loc}}(\mathbb{R}\times(0,\infty))$ *such that*

$$v^{\varepsilon_n} \to v \text{ in } L^p_{\text{loc}}(\mathbb{R}\times(0,\infty)),\ 1 \leq p < \infty.$$

We will also need the following technical lemma:

Lemma 2.3. *Let* Ω *be a bounded open subset of* $\mathbb{R}^N$, $N \geq 2$. *Suppose the sequence* $\{\mathcal{L}_n\}_{n\in\mathbb{N}}$ *of distributions is bounded in* $W^{-1,\infty}(\Omega)$. *Suppose also that*

$$\mathcal{L}_n = \mathcal{L}_{1,n} + \mathcal{L}_{2,n}$$

where $\{\mathcal{L}_{1,n}\}$ *lies in a compact subset of* $H^{-1}_{\text{loc}}(\Omega)$ *and* $\{\mathcal{L}_{2,n}\}$ *lies in a bounded subset of* $\mathcal{M}_{\text{loc}}(\Omega)$. *Then* $\{\mathcal{L}_n\}$ *lies in a compact subset of* $H^{-1}_{\text{loc}}(\Omega)$.

3. The relaxation approximations

Define

$$r^\varepsilon = v^\varepsilon + \frac{z^\varepsilon}{b},\quad s^\varepsilon = v^\varepsilon - \frac{z^\varepsilon}{b},\quad \widetilde{r}^\varepsilon = u^\varepsilon + \frac{w^\varepsilon}{a},\ \text{and}\ \widetilde{s}^\varepsilon = u^\varepsilon - \frac{w^\varepsilon}{a}. \tag{8}$$

If $(u^\varepsilon, w^\varepsilon, v^\varepsilon, z^\varepsilon)$ solves (2), then

$$\begin{cases} r^\varepsilon_t + b r^\varepsilon_x = \frac{1}{b\varepsilon}\left(g\left(u^\varepsilon, (r^\varepsilon + s^\varepsilon)/2\right) - \left((r^\varepsilon - s^\varepsilon)/2\right)b\right), \\ s^\varepsilon_t - b s^\varepsilon_x = \frac{-1}{b\varepsilon}\left(g\left(u^\varepsilon, (r^\varepsilon + s^\varepsilon)/2\right) - \left((r^\varepsilon - s^\varepsilon)/2\right)b\right), \end{cases} \qquad x \in \mathbb{R},\ t > 0, \tag{9}$$

and

$$\begin{cases} \widetilde{r}^\varepsilon_t + a\widetilde{r}^\varepsilon_x = \frac{1}{a\varepsilon}\left(f\left((\widetilde{r}^\varepsilon + \widetilde{s}^\varepsilon)/2\right) - \left((\widetilde{r}^\varepsilon - \widetilde{s}^\varepsilon)/2\right)\right), \\ \widetilde{s}^\varepsilon_t - a\widetilde{s}^\varepsilon_x = \frac{-1}{a\varepsilon}\left(f\left((\widetilde{r}^\varepsilon + \widetilde{s}^\varepsilon)/2\right) - \left((\widetilde{r}^\varepsilon - \widetilde{s}^\varepsilon)/2\right)a\right), \end{cases} \qquad x \in \mathbb{R},\ t > 0. \tag{10}$$

Existence of a local (in time) classical solution $R^\varepsilon = (r^\varepsilon, s^\varepsilon, \widetilde{r}^\varepsilon, \widetilde{s}^\varepsilon)$ to the system consisting of (2) or (9) is known (see for instance [14]). Moreover, the following result is classical: Either there exists a solution for all $t > 0$, or there is some finite blow-up time T such that $\lim_{t\to T_-} \|R^\varepsilon\|_{L^\infty(\mathbb{R}\times(0,t))} = \infty$.

Lemma 3.1. *Assume* $u \in [0,1]$ *and that* (r,s) *solves* (9) *with initial data* $(r,s)(x,0) = (r_0,s_0)(x)$ *where* $(r_0,s_0) \in C^2_0(\mathbb{R})$. *If*

$$0 \leq r_0(x), s_0(x) \leq 1 \quad \textit{for all } x \in \mathbb{R}$$

then

$$0 \leq r(x,t), s(x,t) \leq 1 \quad \textit{for all } x \in \mathbb{R},\ t > 0.$$

Proof. We will show that $[0,1]^2$ is an invariant region for solutions to (9). For $t=0$ we have $(r,s)\in[0,1]^2$ by the requirement on the initial data. Let

$$\hat{t}=\inf\left\{t>0 \;\middle|\; r\left(\hat{x},t\right)=0 \text{ for some } \hat{x}\in\mathbb{R}\right\}.$$

We will show that if $r\left(\hat{x},\hat{t}\right)=0$ then $r_t\left(\hat{x},\hat{t}\right)\geq 0$. This will imply that $r(x,t)\geq 0$ for all $x\in\mathbb{R}$, $t>0$. First we observe that since $r(x,\hat{t})\geq 0$ for all $x\in\mathbb{R}$ and $r(\hat{x},\hat{t})=0$, $r(x,\hat{t})$ has a local minimum in the point $x=\hat{x}$. Hence $r_x(\hat{x},\hat{t})=0$. At the point $(x,t)=(\hat{x},\hat{t})$ we have

$$r_t=\frac{1}{b\varepsilon}\left(g\left(u,\frac{s}{2}\right)+\frac{s}{2}b\right).$$

In order to estimate $r_t(\hat{x},\hat{t})$ we define $h:[0,1]^2\mapsto\mathbb{R}$ by

$$h(r,s)=g\left(u,\frac{r+s}{2}\right)-\left(\frac{r-s}{2}\right)b.$$

By the subcharacteristic condition (A.3), h is decreasing in r and increasing in s. In particular $h(r,0)$ and $h(r,1)$ are decreasing. Hence

$$h(r,0)\leq h(0,0)=0 \quad\text{and}\quad h(r,1)\geq h(1,1)=0 \quad\text{for all}\quad r\in[0,1].$$

And since $h(0,s)$ and $h(1,s)$ are increasing functions of s, we get

$$h(0,s)\geq h(0,0)=0 \quad\text{and}\quad h(1,s)\ \leq h(1,1)=0 \text{ for all}\quad s\in[0,1].$$

Now at the point $(\hat{x},\hat{t})$ where $r=0$ we have $r_t(\hat{x},\hat{t})=\frac{1}{b\varepsilon}h\left(0,s(\hat{x},\hat{t})\right)\geq 0$. Hence $r(x,t)\geq 0$ for all $x\in\mathbb{R}$ and $t>0$. Similarly we see that if $\hat{t}$ is the minimal time such that $r(\hat{x},\hat{t})=1$ for some $\hat{x}\in\mathbb{R}$, then $r_t(\hat{x},\hat{t})=\frac{1}{b\varepsilon}h\left(1,s(\hat{x},\hat{t})\right)\leq 0$, so $r(x,t)\leq 1$ for all $x\in\mathbb{R}$ and $t>0$. Analogously for $s(x,t)$ we get

$$s(\hat{x},\hat{t}\,)=0 \;\Rightarrow\; s_t(\hat{x},\hat{t})=\frac{-1}{b\varepsilon}h\left(r(\hat{x},\hat{t}\,),0\right)\geq 0,$$
$$s(\hat{x},\hat{t}\,)=1 \;\Rightarrow\; s_t(\hat{x},\hat{t}\,)=\frac{-1}{b\varepsilon}h\left(r(\hat{x},\hat{t}\,),1\right)\leq 0,$$

so we must have that $0\leq s(x,t)\leq 1$ for all $x\in\mathbb{R}$, $t>0$. □

Lemma 3.2. *Let $(u^\varepsilon,w^\varepsilon,v^\varepsilon,z^\varepsilon)$ solve the Cauchy problem (2)–(3) with initial u_0^ε and v_0^ε taking values in $[0,1]$. Then*

$$0\leq u^\varepsilon,\, v^\varepsilon\leq 1,\quad -\frac{a}{2}\leq w^\varepsilon\leq\frac{a}{2},\quad\text{and}\quad -\frac{b}{2}\leq z^\varepsilon\leq\frac{b}{2},$$

for all $x\in\mathbb{R}$, $t\geq 0$.

Proof. We will prove the estimates on v^ε and z^ε under the assumption that $u^\varepsilon\in[0,1]$. Then the estimates on u^ε and w^ε will follow immediately. Assume (u,w,v,z) solves (2) with $u,u_0,v_0\in[0,1]$ and $z_0=g(u_0,v_0)$. Let(r,s) be the Riemann invariants defined in (8). Then (r,s) solves (9) with initial condition

$$r_0=v_0+\frac{g(u_0,v_0)}{b},\quad s_0=v_0-\frac{g(u_0,v_0)}{b}.$$

For some fixed $u\in[0,1]$, define $h^\pm:[0,1]\to\mathbb{R}$ by

$$h^\pm(v)=v\pm\frac{g(u,v)}{b}.$$

Now $\partial_v h^\pm(v)=1\pm\frac{g_v(u,v)}{b}>0$, so $h^\pm$ are increasing functions on $[0,1]$. Thus since $h^\pm(0)=0$ and $h^\pm(1)=1$ we have that $0\leq h^\pm(v)\leq 1$ for $v\in[0,1]$. Hence

$$0\leq r_0,s_0\leq 1.$$

From Lemma 3.1 we then have that $0 \le r(x,t), s(x,t) \le 1$ for all $x \in \mathbb{R}$, $t > 0$, and so

$$v = \frac{r+s}{2} \in [0,1], \quad z = \left(\frac{r-s}{2}\right) \in \left[-\frac{b}{2}, \frac{b}{2}\right].$$

□

Definition 3.1. *If u is a smooth function, a weak solution (v,z) of*

$$\begin{cases} v_t + z_x = 0 \\ z_t + b^2 v_x = \frac{1}{\varepsilon}\left(g(u,v) - z\right) \end{cases} \tag{11}$$

is said to satisfy the entropy condition if

$$E(u,v,z)_t + Q(u,v,z)_x - E_u(u,v,z)u_t - Q_u(u,v,z)u_x \le \frac{1}{\varepsilon} E_z(u,v,z)\left(g(u,v) - z\right) \text{ in } \mathcal{D}' \tag{12}$$

for all functions $E, Q : [0,1]^3 \to \mathbb{R}$ satisfying the compatibility conditions

$$Q_v(u,v,z) = b^2 E_z(u,v,z), \qquad Q_z(u,v,z) = E_v(u,v,z).$$

An entropy/entropy-flux pair for (5) is a pair of functions $\eta \in C^2([0,1];\mathbb{R})$, $q \in C^2([0,1]^2;\mathbb{R})$ such that $\partial_v q(u,v) = \partial_v g(u,v)\eta'(v)$. Define

$$h^{\pm}(u,v) := v \pm \frac{g(u,v)}{b}.$$

Then by the subcharacteristic condition we have that $\partial_v h^{\pm}(u,v) = 1 \pm \frac{g_v(u,v)}{b} > 0$ for $(u,v) \in [0,1]^2$, so we can define their inverses $k^{\pm}(\cdot,\cdot)$ such that

$$k^{\pm}(u, h^{\pm}(u,v)) = v \text{ for all } (u,v) \in [0,1]^2.$$

In the following Lemma we see how we can make use of the functions η, q and $h^{\pm}$ to extend an arbitrary entropy/entropy-flux pair (η, q) for the conservation law (5), to an entropy/entropy-flux pair (E,Q) for the relaxation system (11).

Lemma 3.3. *Let (η, q) be a C^2 entropy/entropy-flux pair for (5). Then there exists a C^2 entropy/entropy-flux pair (E,Q) for (11), and functions $E, Q : [0,1]^3 \to \mathbb{R}$ are given explicitly as*

$$\begin{cases} E(u,v,z) = e^{+}\left(u, v + \frac{z}{b}\right) + e^{-}\left(u, v - \frac{z}{b}\right) = e^{+}(u,r) + e^{-}(u,s), \\ Q(u,v,z) = be^{+}\left(u, v + \frac{z}{b}\right) - be^{-}\left(u, v - \frac{z}{b}\right) = be^{+}(u,r) - be^{-}(u,s), \end{cases}$$

where e^{+} and e^{-} take the form

$$\begin{cases} e^{+}(u,r) = \frac{1}{2}\left(\eta\left(k^{+}(u,r)\right) + \frac{1}{b} q\left(u, k^{+}(u,r)\right)\right), \\ e^{-}(u,s) = \frac{1}{2}\left(\eta\left(k^{-}(u,s)\right) - \frac{1}{b} q\left(u, k^{-}(u,s)\right)\right), \end{cases}$$

here r and s are the Riemann invariants defined in (8). Moreover, the following properties hold for all $(u,v,z) \in [0,1]^3$:

- (a) *$E(u,v,g(u,v)) = \eta(v)$ and $Q(u,v,g(u,v)) = q(u,v)$.*
- (b) *$\eta''(v) \ge (>)\, 0$ for all $v \in [0,1] \;\Rightarrow\; E_{zz}(u,v,z) \ge (>)\, 0$.*
- (c) *If η is convex, $E_z(u,v,z)(g(u,v) - z) \le 0$.*
- (d) *If η is strictly convex, $E_z(u,v,z)(g(u,v) - z) \le -\frac{\alpha}{2}(g(u,v) - z)^2$, where $\alpha > 0$ depends only on η and g.*
- (e) *$|E_z(u,v,z)| \le C\,|(g(u,v) - z)|$ for some constant C depending on η and g.*

This is shown [8, Lemma 3.2] for the case when $u = u(x)$ is a smooth bounded function. Extending the proof to hold for $u = u(x,t)$ is straightforward.

Lemma 3.4. *The following estimate holds for $\eta \in C^2, \eta \geq 0$ and $\eta'' \geq 0$:*

$$\frac{1}{\varepsilon}\int_0^T \int \left(g(u^\varepsilon, v^\varepsilon) - z^\varepsilon\right)^2 dx\, dt \leq C \int \eta(v_0^\varepsilon)\, dx + CT$$

for some constant C that only depends on η.

Proof. Given $\eta \in C^2, \eta'' \geq 0$ we can use Lemma 3.3 to construct a C^2 entropy/entropy-flux pair (E, Q) for (11). We then have for some $\alpha > 0$

$$\begin{aligned} E(u^\varepsilon, v^\varepsilon, z^\varepsilon)_t + Q(u^\varepsilon, v^\varepsilon, z^\varepsilon)_x - E_u(u^\varepsilon, v^\varepsilon, z^\varepsilon)u_t^\varepsilon - Q_u(u^\varepsilon, v^\varepsilon, z^\varepsilon)u_x^\varepsilon \\ \leq \frac{1}{\varepsilon}E_z(u^\varepsilon, v^\varepsilon, z^\varepsilon)\left(g(u^\varepsilon, v^\varepsilon) - z^\varepsilon\right) \leq -\frac{\alpha}{2\varepsilon}\left(g(u^\varepsilon, v^\varepsilon) - z^\varepsilon\right)^2, \end{aligned}$$

which yields

$$\begin{aligned} &\frac{\alpha}{2\varepsilon}\int_0^T \int \left(g(u^\varepsilon, v^\varepsilon) - z^\varepsilon\right)^2\, dxdt \\ &\quad \leq \int \eta(v_0^\varepsilon)\, dx - \int E(u^\varepsilon, v^\varepsilon, z^\varepsilon)|^{t=T}\, dx \\ &\qquad + \int_0^T \int E_u(u^\varepsilon, v^\varepsilon, z^\varepsilon)u_t^\varepsilon + Q_u(u^\varepsilon, v^\varepsilon, z^\varepsilon)u_x^\varepsilon\, dxdt \\ &\quad \leq \int \eta(v_0^\varepsilon)\, dx - \int E(u^\varepsilon, v^\varepsilon, z^\varepsilon)|^{t=T}\, dx \\ &\qquad + T\left(\|E_u\|_{L^\infty}\|u_t^\varepsilon\|_{L^1(\mathbb{R})} + \|Q_u\|_{L^\infty}\|u_x^\varepsilon\|_{L^1(\mathbb{R})}\right). \end{aligned}$$

We know that E has a unique minimum for $z^\varepsilon = g(u^\varepsilon, v^\varepsilon)$. Hence $E(u^\varepsilon, v^\varepsilon, z^\varepsilon) \geq E(u^\varepsilon, v^\varepsilon, g(u^\varepsilon, v^\varepsilon) = \eta(v^\varepsilon) \geq 0$. Using this, this we get

$$\begin{aligned} &\frac{\alpha}{2\varepsilon}\int_0^T \int \left(g(u^\varepsilon, v^\varepsilon) - z^\varepsilon\right)^2\, dxdt \\ &\qquad \leq \int \eta(v_0^\varepsilon)\, dx + T\left(\|E_u\|_{L^\infty}\|u_t^\varepsilon\|_{L^1(\mathbb{R})} + \|Q_u\|_{L^\infty}\|u_x^\varepsilon\|_{L^1(\mathbb{R})}\right), \end{aligned}$$

where the L^1 norms of u_x^ε and $u_t^\varepsilon = -w_x^\varepsilon$ are bounded uniformly in ε by Lemma 2.1. ∎

3.1. Convergence of the relaxation approximations.

Theorem 3.1. *There exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ $\varepsilon_n \downarrow 0$, and a weak solution (u, v) of (1) such that*

$$u^{\varepsilon_n} \to u,\ v^{\varepsilon_n} \to v \text{ in } L^p_{\text{loc}}\left(\mathbb{R}\times(0,\infty)\right),\ 1 \leq p < \infty.$$

Proof. We know that $u^\varepsilon \to u$ in $L^p_{\text{loc}}(\mathbb{R}\times(0,\infty))$, $1 \leq p < \infty$, where u is the entropy solution of (4). To prove compactness of the set $\{v^\varepsilon\}_{\varepsilon>0}$ in L^p_{loc} we will show that the set

$$\left\{\partial_t\left|v^\varepsilon - c\right| + \partial_x\left(\text{sign}(v^\varepsilon - c)(g(u, v^\varepsilon) - g(u, c))\right)\right\}_{\varepsilon>0} \tag{13}$$

lies in a compact set of $H^{-1}_{\text{loc}}(\mathbb{R}\times(0,\infty))$. Let

$$\begin{aligned} \eta_0(v) &= |v - c| - c, \\ q_0(u, v) &= \text{sign}(v - c)\left(g(u, v) - g(u, c)\right) + \text{sign}(-c)g(u, c), \end{aligned} \tag{14}$$

and let $\{(\eta_\varepsilon, q_\varepsilon)\}_{\varepsilon>0}$ be a smooth convex entropy/entropy-flux pair approximating (η_0, q_0) such that

$$\|\eta_\varepsilon - \eta_0\|_{L^\infty([0,1])} \le \varepsilon, \quad \|\eta'_\varepsilon - \eta'_0\|_{L^1([0,1])} \le \varepsilon, \quad \|\eta'_\varepsilon\|_{L^\infty([0,1])} \le 1, \\ \eta_\varepsilon(0) = q_\varepsilon(u,0) = 0, \tag{15}$$

for each $\varepsilon > 0$. We now have that

$$\partial_t |v^\varepsilon - c| + \partial_x (\mathrm{sign}(v^\varepsilon - c)(g(u, v^\varepsilon) - g(u,c))) \\ = \partial_t \eta_\varepsilon(v^\varepsilon) + \partial_x q_\varepsilon(u^\varepsilon, v^\varepsilon) + I_{1,\varepsilon} + I_{2,\varepsilon} + I_{3,\varepsilon} + I_{4,\varepsilon}$$

in $\mathcal{D}'(\mathbb{R}\times(0,T))$, where

$$I_{1,\varepsilon} = \partial_t (\eta_0(v^\varepsilon) - \eta_\varepsilon(v^\varepsilon)), \qquad I_{2,\varepsilon} = \partial_x (q_0(u, v^\varepsilon) - q_\varepsilon(u, v^\varepsilon)), \\ I_{3,\varepsilon} = \partial_x (q_\varepsilon(u, v^\varepsilon) - q_\varepsilon(u^\varepsilon, v^\varepsilon)), \qquad I_{4,\varepsilon} = \partial_x \,\mathrm{sign}(-c) g(u,c).$$

Let $\varphi \in C_0^\infty(\mathbb{R}\times(0,T))$. Using (15) we get

$$|\langle I_{1,\varepsilon}, \varphi\rangle| = \Big|\int_0^T \int (\eta_0(v^\varepsilon) - \eta_\varepsilon(v^\varepsilon))\, \varphi_t \, dxdt\Big| \le \varepsilon \|\varphi\|_{H^1(\mathbb{R}\times(0,T))}.$$

Since $C_0^\infty(\mathbb{R}\times(0,T))$ is dense in $H_0^1(\mathbb{R}\times(0,T))$ by definition we have that

$$I_{1,\varepsilon} \to 0 \text{ in } H^{-1}_{\mathrm{loc}}(\mathbb{R}\times(0,\infty)) \text{ when } \varepsilon \to 0.$$

Similarly $|\langle I_{2,\varepsilon}, \varphi\rangle| \le \varepsilon \|\partial_u g\|_{L^\infty([0,1]^2)} \|\varphi\|_{H^1(\mathbb{R}\times(0,T))}$. Hence $I_{2,\varepsilon} \to 0$ in $H^{-1}_{\mathrm{loc}}(\mathbb{R}\times(0,\infty))$. As for $I_{3,\varepsilon}$ we use the that $u^\varepsilon \to u$ in $L^2_{\mathrm{loc}}(\mathbb{R}\times(0,T))$ to get

$$|\langle I_{3,\varepsilon}, \varphi\rangle| = \Big|\iint_\Omega (q_\varepsilon(u, v^\varepsilon) - q_\varepsilon(u^\varepsilon, v^\varepsilon))\, \varphi_x \, dxdt\Big| \\ \le \left\|\partial^2_{uv} g\right\|_{L^\infty} \|u^\varepsilon - u\|_{L^2(\Omega)} \|\varphi\|_{H^1(\Omega)}$$

for $\Omega \subset \mathbb{R}\times(0,\infty)$, Ω bounded, which gives us $I_{3,\varepsilon} \to 0$ in $H^{-1}_{\mathrm{loc}}(\mathbb{R}\times(0,\infty))$. Finally we have that $I_{4,\varepsilon} \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}\times(0,T))$ since

$$\begin{aligned} |\langle I_{4,\varepsilon}, \varphi\rangle| &\le \|\varphi\|_{L^\infty(\mathbb{R}\times(0,T))} \int_0^T \int |\partial_x g(u,c)| \, dxdt \\ &= \|\varphi\|_{L^\infty(\mathbb{R}\times(0,T))} \int_0^T T.V._x(g(u,c))\, dt \\ &\le \|\varphi\|_{L^\infty(\mathbb{R}\times(0,T))} \|g_u\|_{L^\infty([0,1]^2)} \int_0^T T.V._x(u)\, dt \\ &\le T \|\varphi\|_{L^\infty(\mathbb{R}\times(0,T))} \|g_u\|_{L^\infty} T.V._x(u_0). \end{aligned}$$

Here we have used the TVD property of u. Now, by Lemma 2.3 we see that to prove compactness of (13) in $H^{-1}_{\mathrm{loc}}(\mathbb{R}\times(0,\infty))$ it is enough to show this for $\{\partial_t \eta_\varepsilon(v^\varepsilon) + \partial_x q_\varepsilon(u^\varepsilon, v^\varepsilon)\}_{\varepsilon>0}$. We start by using $(\eta_\varepsilon, q_\varepsilon)$ and Lemma 3.3 to create a C^2 entropy/entropy-flux pair $(E_\varepsilon, Q_\varepsilon)$ for (11). We then have

$$\partial_t \eta_\varepsilon(v^\varepsilon) + \partial_x q_\varepsilon(u^\varepsilon, v^\varepsilon) = I_{5,\varepsilon} + I_{6,\varepsilon} + I_{7,\varepsilon}$$

in $\mathcal{D}'(\mathbb{R}\times(0,T))$, where

$$\begin{aligned} I_{5,\varepsilon} &= \partial_t (E_\varepsilon(u^\varepsilon, v^\varepsilon, g(u^\varepsilon, v^\varepsilon)) - E_\varepsilon(u^\varepsilon, v^\varepsilon, z^\varepsilon)) \\ I_{6,\varepsilon} &= \partial_x (Q_\varepsilon(u^\varepsilon, v^\varepsilon, g(u^\varepsilon, v^\varepsilon)) - Q_\varepsilon(u^\varepsilon, v^\varepsilon, z^\varepsilon)) \\ I_{7,\varepsilon} &= \partial_t E_\varepsilon(u^\varepsilon, v^\varepsilon, z^\varepsilon) + \partial_x Q_\varepsilon(u^\varepsilon, v^\varepsilon, z^\varepsilon). \end{aligned}$$

By Lemma 3.4 we get

$$\begin{aligned}|\langle I_{5,\varepsilon},\varphi\rangle| &= \Big|\int_0^T\int_{\mathbb{R}} \left(E_\varepsilon(u^\varepsilon,v^\varepsilon,g(u^\varepsilon,v^\varepsilon)) - E_\varepsilon(u^\varepsilon,v^\varepsilon,z^\varepsilon)\right)\varphi_t\,dxdt\Big|\\ &\le C\int_0^T\int |g(u^\varepsilon,v^\varepsilon)-z^\varepsilon|\,|\varphi_t|\,dxdt\\ &\le C\left\|g(u^\varepsilon,v^\varepsilon)-z^\varepsilon\right\|_{L^2(\mathbb{R}\times(0,T))}\left\|\varphi\right\|_{L^2(\mathbb{R}\times(0,T))}\\ &\le \sqrt{\varepsilon}C(T)\left\|\varphi\right\|_{H^1(\mathbb{R}\times(0,T))},\end{aligned}$$

for some constant that only depends on T. Hence $I_{5,\varepsilon}\to 0$ in $H^{-1}_{\mathrm{loc}}(\mathbb{R}\times(0,\infty))$. By the same calculations we find that $I_{6,\varepsilon}\to 0$ in $H^{-1}_{\mathrm{loc}}(\mathbb{R}\times(0,\infty))$. As for $I_{7,\varepsilon}$ we have that

$$I_{7,\varepsilon} = \frac{1}{\varepsilon}\partial_z E_\varepsilon(u^\varepsilon,v^\varepsilon,z^\varepsilon)\left(g(u^\varepsilon,v^\varepsilon)-z^\varepsilon\right) + \partial_u E_\varepsilon(u^\varepsilon,v^\varepsilon,z^\varepsilon)u^\varepsilon_t + \partial_u E_\varepsilon(u^\varepsilon,v^\varepsilon,z^\varepsilon)u^\varepsilon_x.$$

Here $\partial_u E_\varepsilon(u^\varepsilon,v^\varepsilon,z^\varepsilon)u^\varepsilon_t$ and $\partial_u E_\varepsilon(u^\varepsilon,v^\varepsilon,z^\varepsilon)u^\varepsilon_x$ are in $L^1(\mathbb{R}\times(0,T))$, and by part (e) of Lemma 3.3 and Lemma 3.4 we have

$$\frac{1}{\varepsilon}\left|\partial_z E_\varepsilon(u^\varepsilon,v^\varepsilon,z^\varepsilon)\right|\left|g(u^\varepsilon,v^\varepsilon)-z^\varepsilon\right| \le \frac{C}{\varepsilon}\left(g(u^\varepsilon,v^\varepsilon)-z^\varepsilon\right)^2 \in L^1(\mathbb{R}\times(0,T)).$$

Hence $I_{7,\varepsilon}\in L^1(\mathbb{R}\times(0,T))$. Summing up, we have shown

$$\begin{aligned}\partial_t\left|v^\varepsilon-c\right|| + \partial_x\left(\mathrm{sign}(v^\varepsilon-c)(g(u,v^\varepsilon-g(u,c))\right)&\\ = I_{1,\varepsilon}+I_{2,\varepsilon}+I_{3,\varepsilon}+I_{4,\varepsilon}+I_{5,\varepsilon}+I_{6,\varepsilon}+I_{7,\varepsilon}&\end{aligned}$$

where $I_{j,\varepsilon}$ is bounded in $W^{-1,\infty}_{\mathrm{loc}}$ for $j=1,\dots,7$, and

$$\begin{aligned}&\{I_{1,\varepsilon}\},\ \{I_{2,\varepsilon}\},\ \{I_{3,\varepsilon}\},\ \{I_{5,\varepsilon}\}\text{ and }\{I_{6,\varepsilon}\}\text{ converge in } H^{-1}_{\mathrm{loc}}(\mathbb{R}\times(0,\infty)),\\ &I_{4,\varepsilon}\in\mathcal{M}_{\mathrm{loc}}(\mathbb{R}\times(0,\infty)),\\ &\{I_{7,\varepsilon}\}\text{ is bounded in } L^1(\mathbb{R}\times(0,T))\text{ for each } T>0.\end{aligned}$$

Hence from Lemma 2.3 and Lemma 2.2 there are functions u, v in $L^\infty_{\mathrm{loc}}(\mathbb{R}\times(0,\infty))$ and a subsequence $(u^{\varepsilon_n},v^{\varepsilon_n})$ of $(u^\varepsilon,v^\varepsilon)$ such that

$$v^{\varepsilon_n}\to v \quad\text{and}\quad u^{\varepsilon_n}\to u \quad\text{in } L^p_{\mathrm{loc}}(\mathbb{R}\times(0,\infty)),\ 1\le p<\infty.$$

What remains is then to show that (u,v) is in fact a weak solution of the Cauchy problem (1) according to Definition 2.1. To simplify the notation we assume that $(u^\varepsilon,v^\varepsilon)$ is the converging sequence. We know that u is the entropy solution of (4), so to complete the proof we only have to show that v solves $v_t+g(u,v)_x=0$ in the sense of distributions. Let Ω be a bounded set in $\mathbb{R}\times(0,\infty)$. By Lemma 3.4 we have that $(z^\varepsilon-g(u^\varepsilon,v^\varepsilon))\to 0$ in $L^2(\Omega)$. Hence,

$$\begin{aligned}&\left\|z^\varepsilon-g(u,v)\right\|_{L^2(\Omega)}\\ &\quad\le\left\|z^\varepsilon-g(u^\varepsilon,v^\varepsilon)\right\|_{L^2(\Omega)} + C\left(\left\|u^\varepsilon-u\right\|_{L^2(\Omega)}+\left\|v^\varepsilon-v\right\|_{L^2(\Omega)}\right)\to 0 \text{ as } \varepsilon\downarrow 0,\end{aligned}$$

and so $z^\varepsilon\to g(u,v)$ in $L^2(\Omega)$. We know that z^ε converges in L^p_{loc} for $1\le p<\infty$, so we must have that $z^\varepsilon\to g(u,v)$ in $L^p_{\mathrm{loc}}(\mathbb{R}\times(0,\infty))$. Then the result follows from the dominated convergence theorem, since for any $\varphi\in C_0^\infty(\mathbb{R}\times[0,\infty))$ we have that

$$\begin{aligned}\int_0^\infty \int v\varphi_t + g(u,v)\varphi_x \, dxdt + \int v_0\varphi(x,0)\, dx \\ = \lim_{\varepsilon\to 0}\int_0^\infty \int v^\varepsilon\varphi_t + z^\varepsilon\varphi_x \, dxdt + \int v_0^\varepsilon\varphi(x,0)\, dx \\ = \lim_{\varepsilon\to 0} -\int_0^\infty \int \left(v_t^\varepsilon + z_x^\varepsilon\right)\varphi \, dxdt = 0.\end{aligned}$$

□

4. The relaxation scheme

For some $h>0$ and $\Delta t>0$, let $x_k = kh$ for $k=0,\pm\frac{1}{2},\pm 1,\pm\frac{3}{2},\dots$ and set

$$I_j = \left(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}\right), \quad j\in\mathbb{Z}.$$

Furthermore, set $t_n = n\Delta t$ for $n=0,1,\dots,N$ where $N\in\mathbb{N}$ is chosen such that $N\Delta t = T$, resulting in the time strips (t_n, t_{n+1}), $n=0,1,\dots,N-1$. We approximate $v^\varepsilon(x_j,t_n)$ by v_j^n and $z^\varepsilon(x_j,t_n)$ with z_j^n. Consider a semi-implicit upwind scheme discretization of (9),

$$\begin{cases}\frac{1}{\Delta t}\left(r_j^{n+1}-r_j^n\right)+\frac{b}{h}\left(r_j^n - r_{j-1}^n\right) = \frac{1}{b\varepsilon}\left(g(u_j^{n+1},v_j^{n+1}) - z_j^{n+1}\right)\\ \frac{1}{\Delta t}\left(s_j^{n+1}-s_j^n\right)-\frac{b}{h}\left(s_{j+1}^n - s_j^n\right) = \frac{-1}{b\varepsilon}\left(g(u_j^{n+1},v_j^{n+1}) - z_j^{n+1}\right)\end{cases} \tag{16}$$

In the original variables, this scheme reads

$$\begin{cases}\frac{1}{\Delta t}\left(v_j^{n+1}-v_j^n\right)+\frac{1}{2h}\left(z_{j+1}^n - z_{j-1}^n\right) - \frac{b}{2h}\left(v_{j-1}^n - 2v_j^n + v_{j+1}^n\right) = 0\\ \dfrac{1}{\Delta t}\left(z_j^{n+1}-z_j^n\right)+\dfrac{b^2}{2h}\left(v_{j+1}^n - v_{j-1}^n\right) - \dfrac{b}{2h}\left(z_{j-1}^n - 2z_j^n + z_{j+1}^n\right)\\ \qquad\qquad\qquad = \dfrac{1}{\varepsilon}\left(g(u_j^{n+1},v_j^{n+1}) - z_j^{n+1}\right)\end{cases} \tag{17}$$

The approximation u_j^n is defined via an analogous scheme, with $f(u)$ replacing $g(u,v)$ and w_j^n replacing z_j^n in (17). Note that the scheme is explicit since we first update v_j^{n+1} using the first equation in (17), and then z_j^{n+1} using the second. In order to start the iterations we specify the initial data for $n=0$,

$$v_j^0 = \frac{1}{h}\int_{I_j} v_0^\varepsilon(x)\,dx, \quad z_j^0 = \frac{1}{h}\int_{I_j} g(u_0^\varepsilon(x), v_0^\varepsilon(x))\,dx, \quad r_j^0 = v_j^0 + \frac{z_j^0}{b}, \quad s_j^0 = v_j^0 - \frac{z_j^0}{b}.$$

We will also assume that the following CFL condition holds:

$$\max\{a,b\}\lambda \le 1, \quad \lambda = \frac{\Delta t}{h}. \tag{18}$$

Lemma 4.1. *Let $u^{\varepsilon,h}(x,t) = u_j^n$ and $w^{\varepsilon,h}(x,t) = w_j^n$ for $(x,t)\in I_j\times[t_n,t_{n+1})$. Then $u^{\varepsilon,h}\to u$ in $L^p_{\mathrm{loc}}(\mathbb{R}\times(0,\infty))$ as $\varepsilon, h\downarrow 0$, where u is the entropy solution of (4). Moreover, $u^{\varepsilon,h}$ and $w^{\varepsilon,h}$ satisfy*

$$\begin{gathered}(u^{\varepsilon,h}, w^{\varepsilon,h}) \in [0,1]\times[-a/2,a/2] \text{ for all } \varepsilon,h>0, \text{ and}\\ T.V._x(u^{\varepsilon,h}(\cdot,t)), T.V._x(w^{\varepsilon,h}(\cdot,t)) \le C\end{gathered}$$

for some constant C that does not depend on h,ε or t.

This is shown in [12, 4]. Since $(v_0^\varepsilon, z_0^\varepsilon) \in [1,0]\times[-b/2,b/2]$ we have that $(v_j^0, z_j^0) \in [1,0]\times[-b/2,b/2]$ for all j. Let $h^\pm(u,v) = v \pm g(u,v)/b$. In the proof of Lemma 3.2 we have shown that $0 \le h^\pm(u,v) \le 1$ for all $(u,v) \in [0,1]^2$. Hence $r_j^0 = v_j^0 + z_j^0/b = (1/h)\int_{I_j} h^+(u_0^\varepsilon, v_0^\varepsilon)dx \in [0,1]$, and similarly $s_j^0 = v_j^0 - z_j^0/b = (1/h)\int_{I_j} h^-(u_0^\varepsilon, v_0^\varepsilon)dx \in [0,1]$.

Lemma 4.2 (Discrete L^∞ estimate)**.** *Assume that the CFL condition (18) holds. If*

$$(v_j^0, z_j^0) \in [1,0]\times[-b/2,b/2] \;\; \textit{and}\; (r_j^0, s_j^0) \in [0,1]^2 \;\textit{for all}\; j,$$

then

$$(v_j^n, z_j^n) \in [1,0]\times[-b/2,b/2] \;\; \textit{and}\; (r_j^n, s_j^n) \in [0,1]^2 \;\textit{for all}\; j.$$

for $n = 1, 2, \dots, N$.

In [8, Lemma 3.1] this is shown if $f \equiv 0$, this proof is easily modified to cover the case where $u = u(x,t)$.

Lemma 4.3 (Discrete L^2_{loc} estimate)**.** *Assume that the strengthened CFL condition (35) found in the proof below holds. Then for any J,*

$$\frac{h\Delta t}{\varepsilon}\sum_{n=0}^{N-1}\sum_{j=-J}^{J}\left(g(u_j^{n+1}, v_j^{n+1}) - z_j^{n+1}\right)^2 \le C,$$

$$h\Delta t\sum_{n=0}^{N-1}\sum_{j=-J}^{J}\left[\left(r_j^{n+1} - r_j^n\right)^2 + \left(s_j^{n+1} - s_j^n\right)^2\right] \le C\Delta t,$$

$$h\Delta t\sum_{n=0}^{N-1}\sum_{j=-J}^{J}\left[\left(r_j^{n} - r_{j-1}^n\right)^2 + \left(s_{j+1}^{n} - s_j^n\right)^2\right] \le Ch,$$

for some constant C that depends on Jh and $N\Delta t$.

Proof. Let (η, q) be an entropy/entropy-flux pair for (5) with η strictly convex, and let (E, Q) be the corresponding entropy/entropy-flux pair for (11) given by Lemma 3.3. We will make use of the following Taylor expansion: For a twice differentiable function h and for b_1, $b_2 \in \mathbb{R}$

$$(b_2 - b_1)h'(b_2) = h(b_2) - h(b_1) + \frac{1}{2}h''(\xi)(b_2 - b_1)^2$$

for some ξ between b_1 and b_2.

To get expressions for $E(v_j^n, z_j^n)$ and $Q(v_j^n, z_j^n)$ we multiply the first equation in (16) with $e_r^+(u_j^{n+1}, r_j^{n+1})$ to obtain

$$\begin{aligned}
&e^+(u_j^{n+1}, r_j^{n+1}) - e^+(u_j^{n+1}, r_j^n) \\
&\qquad + \frac{1}{2}e_{rr}^+(u_j^{n+1}, \xi_j^{n+\frac{1}{2}})\left(r_j^{n+1} - r_j^n\right)^2 + b\lambda\left(r_j^n - r_{j-1}^n\right)e_r^+(u_j^{n+1}, r_j^{n+1}) \\
&\qquad\qquad = \frac{\Delta t}{b\varepsilon}\left(g(u_j^{n+1}, v_j^{n+1}) - z_j^{n+1}\right)e_r^+(u_j^{n+1}, r_j^{n+1}).
\end{aligned}$$

We then use that

$$\begin{aligned}
&e^+(u_j^{n+1}, r_j^{n+1}) - e^+(u_j^{n+1}, r_j^n) \\
&\quad = e^+(u_j^{n+1}, r_j^{n+1}) - e^+(u_j^n, r_j^n) - \left(e^+(u_j^{n+1}, r_j^n) - e^+(u_j^n, r_j^n)\right),
\end{aligned}$$

and

$$
\begin{aligned}
&\left(r_j^n - r_{j-1}^n\right) e_r^+(u_j^{n+1}, r_j^{n+1}) \\
&\quad = \left(r_j^n - r_{j-1}^n\right) e_r^+(u_j^n, r_j^n) + \left(r_j^n - r_{j-1}^n\right)\left(e_r^+(u_j^{n+1}, r_j^{n+1}) - e_r^+(u_j^n, r_j^n)\right) \\
&\quad = e^+(u_j^n, r_j^n) - e^+(u_j^n, r_{j-1}^n) + \frac{1}{2} e_{rr}^+(u_j^n, \xi_{j-\frac{1}{2}}^n)\left(r_j^n - r_{j-1}^n\right)^2 \\
&\qquad + \left(r_j^n - r_{j-1}^n\right)\left(e_r^+(u_j^{n+1}, r_j^{n+1}) - e_r^+(u_j^n, r_j^n)\right) \\
&\quad = e^+(u_j^n, r_j^n) - e^+(u_{j-1}^n, r_{j-1}^n) + \frac{1}{2} e_{rr}^+(u_j^n, \xi_{j-\frac{1}{2}}^n)\left(r_j^n - r_{j-1}^n\right)^2 \\
&\qquad + \left(r_j^n - r_{j-1}^n\right)\left(e_r^+(u_j^{n+1}, r_j^{n+1}) - e_r^+(u_j^n, r_j^n)\right) \\
&\qquad\quad - \left(e^+(u_j^n, r_{j-1}^n) - e^+(u_{j-1}^n, r_{j-1}^n)\right),
\end{aligned}
$$

to get

$$
\begin{aligned}
&e^+(u_j^{n+1}, r_j^{n+1}) - e^+(u_j^n, r_j^n) + \frac{1}{2} e_{rr}^+(u_j^{n+1}, \xi_j^{n+\frac{1}{2}})\left(r_j^{n+1} - r_j^n\right)^2 \\
&\qquad + b\lambda\left(e^+(u_j^n, r_j^n) - e^+(u_{j-1}^n, r_{j-1}^n)\right) + \frac{b\lambda}{2} e_{rr}^+(u_j^n, \xi_{j-\frac{1}{2}}^n)\left(r_j^n - r_{j-1}^n\right)^2 \\
&\qquad + b\lambda\left(r_j^n - r_{j-1}^n\right)\left(e_r^+(u_j^{n+1}, r_j^{n+1}) - e_r^+(u_j^n, r_j^n)\right) \\
&\qquad - b\lambda\left(e^+(u_j^n, r_{j-1}^n) - e^+(u_{j-1}^n, r_{j-1}^n)\right) - \left(e^+(u_j^{n+1}, r_j^n) - e^+(u_j^n, r_j^n)\right) \\
&\quad = \frac{\Delta t}{b\varepsilon}\left(g(u_j^{n+1}, v_j^{n+1}) - z_j^{n+1}\right) e_r^+(u_j^{n+1}, r_j^{n+1}).
\end{aligned}
$$

Setting

$$
\begin{aligned}
e_j^{+,n} &= e^+(u_j^n, r_j^n), \qquad e_j^{-,n} = e^-(u_j^n, r_j^n), \\
A_j^{+,n} &= -b\lambda\left(e^+(u_j^n, r_{j-1}^n) - e^+(u_{j-1}^n, r_{j-1}^n)\right) - \left(e^+(u_j^{n+1}, r_j^n) - e^+(u_j^n, r_j^n)\right), \\
B_{j+\frac{1}{2}}^{+,n} &= b\lambda\left(r_j^n - r_{j-1}^n\right)\left(e_r^+(u_j^{n+1}, r_j^{n+1}) - e_r^+(u_j^n, r_j^n)\right),
\end{aligned}
$$

we can rewrite the above as

$$
\begin{aligned}
e_j^{+,n+1} - e_j^{+,n} + b\lambda\left(e_j^{+,n} - e_{j-1}^{+,n}\right) &+ \frac{1}{2} e_{rr}^+(u_j^{n+1}, \xi_j^{n+\frac{1}{2}})\left(r_j^{n+1} - r_j^n\right)^2 \\
&+ \frac{b\lambda}{2} e_{rr}^+(u_j^n, \xi_{j-\frac{1}{2}}^n)\left(r_j^n - r_{j-1}^n\right)^2 + A_j^{+,n} \\
&\quad = \frac{\Delta t}{b\varepsilon}\left(g(u_j^{n+1}, v_j^{n+1}) - z_j^{n+1}\right) e_r^+(u_j^{n+1}, r_j^{n+1}) - B_{j+\frac{1}{2}}^{+,n}.
\end{aligned} \tag{19}
$$

By the mean value theorem we can find a $\boldsymbol{\zeta}_j^{n+1/2} = (\zeta_j^{u,n+1/2}, \zeta_j^{r,n+1/2})$ on the line between (u_j^{n+1}, r_j^{n+1}) and (u_j^n, r_j^n) such that

$$
e_r^+(u_j^{n+1}, r_j^{n+1}) - e_r^+(u_j^n, r_j^n) = e_{ru}^+\left(\boldsymbol{\zeta}_j^{n+\frac{1}{2}}\right)\left(u_j^{n+1} - u_j^n\right) + e_{rr}^+\left(\boldsymbol{\zeta}_j^{n+\frac{1}{2}}\right)\left(r_j^{n+1} - r_j^n\right),
$$

so

$$
\begin{aligned}
&B_{j+\frac{1}{2}}^{+,n} \\
&= b\lambda e_{ru}^+\left(\boldsymbol{\zeta}_j^{n+\frac{1}{2}}\right)\left(u_j^{n+1} - u_j^n\right)\left(r_j^n - r_{j-1}^n\right) + b\lambda e_{rr}^+\left(\boldsymbol{\zeta}_j^{n+\frac{1}{2}}\right)\left(r_j^{n+1} - r_j^n\right)\left(r_j^n - r_{j-1}^n\right).
\end{aligned}
$$

By Cauchy's inequality with τ we have that

$$B^{+,n}_{j+\frac{1}{2}} \geq -b\lambda e^{+}_{ru}\left(\boldsymbol{\zeta}_j^{n+\frac{1}{2}}\right)\left[\tau\left(u_j^{n+1}-u_j^n\right)^2+\frac{1}{4\tau}\left(r_j^n-r_{j-1}^n\right)^2\right] - b\lambda e^{+}_{rr}\left(\boldsymbol{\zeta}_j^{n+\frac{1}{2}}\right)\left[\tau\left(r_j^{n+1}-r_j^n\right)^2+\frac{1}{4\tau}\left(r_j^n-r_{j-1}^n\right)^2\right].$$

We substitute this in (19) to get the inequality

$$\begin{aligned} &e_j^{+,n+1}-e_j^{+,n}+b\lambda\left(e_j^{+,n}-e_{j-1}^{+,n}\right)+A_j^{+,n} \\ &\quad + b\lambda\left[\frac{1}{2}e^{+}_{rr}(u_j^n,\xi^n_{j-\frac{1}{2}})-\frac{1}{4\tau}\left(e^{+}_{ru}(\boldsymbol{\zeta}_j^{n+\frac{1}{2}})+e^{+}_{rr}\left(\boldsymbol{\zeta}_j^{n+\frac{1}{2}}\right)\right)\right]\left(r_j^n-r_{j-1}^n\right)^2 \\ &+\left[\frac{1}{2}e^{+}_{rr}(u_j^{n+1},\xi_j^{n+\frac{1}{2}})-b\lambda\tau e^{+}_{rr}(\boldsymbol{\zeta}_j^{n+\frac{1}{2}})\right]\left(r_j^{n+1}-r_j^n\right)^2-b\lambda\tau e^{+}_{ru}\left(\boldsymbol{\zeta}_j^{n+\frac{1}{2}}\right)\left(u_j^{n+1}-u_j^n\right)^2 \\ &\qquad \leq \frac{\Delta t}{b\varepsilon}\left(g(u_j^{n+1},v_j^{n+1})-z_j^{n+1}\right)e^{+}_{r}(u_j^{n+1},r_j^{n+1}). \end{aligned} \tag{20}$$

Differentiating e_r^+ and e_s^- with respect to r and s we find

$$e^{+}_{r}(u,r)=\frac{1}{2}\eta'(k^{+}(u,r)),\quad e^{-}_{s}(u,s)=\frac{1}{2}\eta'(k^{-}(u,s)),$$

where $k^{\pm}(u,v)$ are the inverse functions to $h^{\pm}(u,v)=v\pm g(u,v)/b$, such that $k^{\pm}(u,h^{\pm}(u,v))=v$ for all $(u,v)\in[0,1]^2$. Hence

$$\begin{aligned} e^{+}_{rr}(u,r) &= \frac{1}{2}\eta''(k^{+}(u,r))k^{+}_{r}(u,r) \\ &= \frac{1}{2}\eta''(k^{+}(u,r))\frac{1}{h^{+}_{v}(u,k^{+}(u,r))} = \frac{\frac{1}{2}\eta''(k^{+}(u,r))}{\left(1+\frac{g_v(u,k^{+}(u,r))}{b}\right)}. \end{aligned} \tag{21}$$

Since $k^{+}(u,h^{+}(u,v))=v$, we have that

$$\begin{aligned} &k^{+}_{u}(u,h^{+}(u,v))+k^{+}_{v}(u,h^{+}(u,v))h^{+}_{u}(u,v)=\partial_u\left[k^{+}(u,h^{+}(u,v))\right]=0 \\ &\quad\Rightarrow k^{+}_{u}(u,h^{+}(u,k^{+}(u,r)))+k^{+}_{v}(u,h^{+}(u,k^{+}(u,r)))h^{+}_{u}(u,k^{+}(u,r))=0 \\ &\qquad\Rightarrow k^{+}_{u}(u,r)+k^{+}_{v}(u,r)h^{+}_{u}(u,k^{+}(u,r))=0 \\ &\qquad\quad\Rightarrow k^{+}_{u}(u,r)+k^{+}_{r}(u,r)h^{+}_{u}(u,k^{+}(u,r))=0, \end{aligned}$$

and thus

$$\begin{aligned} e^{+}_{ru}(u,r)+e^{+}_{rr}(u,r) &= \frac{1}{2}\eta''(k^{+}(u,r))\left(k^{+}_{u}(u,r)+k^{+}_{r}(u,r)\right) \\ &= \frac{1}{2}\eta''(k^{+}(u,r))k^{+}_{r}(u,r)\left(1-h^{+}_{u}(u,k^{+}(u,r))\right), \end{aligned}$$

which gives us

$$e^{+}_{ru}(u,r)+e^{+}_{rr}(u,r)=\frac{\frac{1}{2}\eta''(k^{+}(u,r))\left(1-\frac{g_u(u,k^{+}(u,r))}{b}\right)}{\left(1+\frac{g_v(u,k^{+}(u,r))}{b}\right)}. \tag{22}$$

Similarly we get

$$e^{-}_{ss}(u,s)=\frac{\frac{1}{2}\eta''(k^{-}(u,s))}{\left(1-\frac{g_v(u,k^{-}(u,s))}{b}\right)}, \tag{23}$$

and

$$e_{su}^{-}(u,s)+e_{ss}^{-}(u,s)=\frac{\frac{1}{2}\eta''(k^{-}(u,s))\left(1-\frac{g_u(u,k^{-}(u,s))}{b}\right)}{\left(1-\frac{g_v(u,k^{-}(u,s))}{b}\right)}. \tag{24}$$

Using (21) - (24) we can find constants m_1 and m_2 such that

$$\begin{aligned} 0<&m_1\le e_{rr}^{+}(u,r)\le m_2,\\ &m_1\le e_{ss}^{-}(u,s)\le m_2,\end{aligned}\quad (u,s)\in[0,1]^2. \tag{25}$$

We can also choose m_1 and m_2 such that

$$\begin{aligned} m_1\le e_{ru}^{+}(u,r)+e_{rr}^{+}(u,r)\le m_2,\\ m_1\le e_{su}^{-}(u,s)+e_{ss}^{-}(u,s)\le m_2,\end{aligned}\quad (u,s)\in[0,1]^2. \tag{26}$$

Next we choose $\tau=2m_2/m_1$ in (20), and then

$$\frac{1}{2}e_{rr}^{+}(u_j^n,\xi_{j-\frac{1}{2}}^n)-\frac{1}{4\tau}\left(e_{ru}^{+}\left(\boldsymbol{\zeta}_j^{n+\frac{1}{2}}\right)+e_{rr}^{+}\left(\boldsymbol{\zeta}_j^{n+\frac{1}{2}}\right)\right)\ge\frac{m_1}{2}-\frac{m_1}{8m_2}m_2>\frac{m_1}{4}.$$

In order to bound the other quadratic term in (20) we demand that the modified CFL condition (27) holds,

$$b\lambda\le\frac{m_1^2}{8m_2^2}, \tag{27}$$

so that

$$\frac{1}{2}e_{rr}^{+}\left(u_j^{n+1},\xi_j^{n+\frac{1}{2}}\right)-b\lambda\tau e_{rr}^{+}\left(\boldsymbol{\zeta}_j^{n+\frac{1}{2}}\right)\ge\frac{m_1}{2}-b\lambda\frac{2m_2}{m_1}m_2\ge\frac{m_1}{2}-\frac{m_1}{4}=\frac{m_1}{4}$$

We can now use these bounds in (20) to obtain

$$\begin{aligned} e_j^{+,n+1}-e_j^{+,n}+b\lambda\left(e_j^{+,n}-e_{j-1}^{+,n}\right)+A_j^{+,n}+\frac{m_1}{4}\left(r_j^{n+1}-r_j^n\right)^2&\\ +b\lambda\frac{m_1}{4}\left(r_j^n-r_{j-1}^n\right)^2-2b\lambda\frac{m_1}{m_2}e_{ru}^{+}\left(\boldsymbol{\zeta}_j^{n+\frac{1}{2}}\right)\left(u_j^{n+1}-u_j^n\right)^2&\\ \le\frac{\Delta t}{b\varepsilon}\left(g(u_j^{n+1},v_j^{n+1})-z_j^{n+1}\right)e_r^{+}(u_j^{n+1},r_j^{n+1})&.\end{aligned} \tag{28}$$

Similarly, by multiplying the second equation in (16) with $e_s^{-}(u_j^{n+1},s_j^{n+1})$ we get

$$\begin{aligned} e_j^{-,n+1}-e_j^{-,n}-b\lambda\left(e_{j+1}^{-,n}-e_j^{-,n}\right)+A_j^{-,n}+\frac{m_1}{4}\left(s_j^{n+1}-s_j^n\right)^2&\\ +b\lambda\frac{m_1}{4}\left(s_{j+1}^n-s_j^n\right)^2-2b\lambda\frac{m_1}{m_2}e_{su}^{-}\left(\boldsymbol{\gamma}_j^{n+\frac{1}{2}}\right)\left(u_j^{n+1}-u_j^n\right)^2&\\ \le-\frac{\Delta t}{b\varepsilon}\left(g(u_j^{n+1},v_j^{n+1})-z_j^{n+1}\right)e_s^{-}(u_j^{n+1},s_j^{n+1})&,\end{aligned} \tag{29}$$

for some $\boldsymbol{\gamma}_j^{n+\frac{1}{2}}=(\gamma_j^{u,n+\frac{1}{2}},\gamma_j^{s,n+\frac{1}{2}})$ on the line between (u_j^{n+1},s_j^{n+1}) and (u_j^n,s_j^n), where

$$A_j^{-,n}=-b\lambda\left(e^{-}(u_j^n,s_{j+1}^n)-e^{-}(u_{j+1}^n,s_{j+1}^n)\right)-\left(e^{-}(u_j^{n+1},s_j^n)-e^{-}(u_j^n,s_j^n)\right).$$

Let

$$E_j^n=e_j^{+,n}+e_j^{-,n},\quad Q_{j+\frac{1}{2}}^n=be_j^{+,n}-be_{j+1}^{-,n}.$$

Then adding (28) and (29), and rearranging gives us

$$\begin{aligned}
&E_j^{n+1} - E_j^n + \lambda\left(Q_{j+\frac{1}{2}}^n - Q_{j-\frac{1}{2}}^n\right) \\
&\quad + \frac{m_1}{4}\left\{\left(r_j^{n+1} - r_j^n\right)^2 + \left(s_j^{n+1} - s_j^n\right)^2 + b\lambda\left(r_j^n - r_{j-1}^n\right)^2 + b\lambda\left(s_{j+1}^n - s_j^n\right)^2\right\} \\
&\qquad \le \frac{\Delta t}{b\varepsilon}\left(g(u_j^{n+1}, v_j^{n+1}) - z_j^{n+1}\right)\left(e_r^+(u_j^{n+1}, r_j^{n+1}) - e_s^-(u_j^{n+1}, r_j^{n+1})\right) \\
&\qquad + 2b\lambda\frac{m_1}{m_2}\left(e_{ru}^+\left(\zeta_j^{n+\frac{1}{2}}\right) + e_{su}^-\left(\gamma_j^{n+\frac{1}{2}}\right)\right)\left(u_j^{n+1} - u_j^n\right)^2 + \left|A_j^{+,n}\right| + \left|A_j^{-,n}\right|.
\end{aligned}$$

Now we use that $E_z(u,v,z) = \frac{1}{b}\left(e_r^+(u,r) - e_s^-(u,s)\right)$ and part (d) of Lemma 3.3 to get

$$\begin{aligned}
&\frac{\Delta t}{b\varepsilon}\left(g(u_j^{n+1}, v_j^{n+1}) - z_j^{n+1}\right)\left(e_r^+(u_j^{n+1}, r_j^{n+1}) - e_s^-(u_j^{n+1}, r_j^{n+1})\right) \\
&\qquad\qquad \le -\frac{\alpha\Delta t}{2b\varepsilon}\left(g(u_j^{n+1}, v_j^{n+1}) - z_j^{n+1}\right)^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
&E_j^{n+1} - E_j^n + \lambda\left(Q_{j+\frac{1}{2}}^n - Q_{j-\frac{1}{2}}^n\right) \\
&\quad + \frac{m_1}{4}\left\{\left(r_j^{n+1} - r_j^n\right)^2 + \left(s_j^{n+1} - s_j^n\right)^2 + b\lambda\left(r_j^n - r_{j-1}^n\right)^2 + b\lambda\left(s_{j+1}^n - s_j^n\right)^2\right\} \\
&\qquad\qquad + \frac{\alpha\Delta t}{2b\varepsilon}\left(g(u_j^{n+1}, v_j^{n+1}) - z_j^{n+1}\right)^2 \\
&\qquad \le 2b\lambda\frac{m_1}{m_2}\left(e_{ru}^+\left(\zeta_j^{n+\frac{1}{2}}\right) + e_{su}^-\left(\gamma_j^{n+\frac{1}{2}}\right)\right)\left(u_j^{n+1} - u_j^n\right)^2 + \left|A_j^{+,n}\right| + \left|A_j^{-,n}\right|.
\end{aligned} \tag{30}$$

Since the terms in (30) telescope, we can multiply by h and sum over n and j to obtain

$$\begin{aligned}
&h\frac{m_1}{4}\sum_{n=0}^{N-1}\sum_{j=-J}^{J}\Big[\left(r_j^{n+1} - r_j^n\right)^2 + \left(s_j^{n+1} - s_j^n\right)^2 \\
&\qquad\qquad + b\lambda\left(r_j^n - r_{j-1}^n\right)^2 + b\lambda\left(s_{j+1}^n - s_j^n\right)^2\Big] \\
&\quad + h\sum_{j=-J}^{J} E_j^N + \frac{\alpha h\Delta t}{2b\varepsilon}\sum_{n=0}^{N-1}\sum_{j=-J}^{J}\left(g(u_j^{n+1}, v_j^{n+1}) - z_j^{n+1}\right)^2 \\
&\quad \le 2b\Delta t\frac{m_1}{m_2}\sum_{n=0}^{N-1}\sum_{j=-J}^{J}\left(e_{ru}^+\left(\zeta_j^{n+\frac{1}{2}}\right) + e_{su}^-\left(\gamma_j^{n+\frac{1}{2}}\right)\right)\left(u_j^{n+1} - u_j^n\right)^2 \\
&\qquad + h\sum_{n=0}^{N-1}\sum_{j=-J}^{J}\left(\left|A_j^{+,n}\right| + \left|A_j^{-,n}\right|\right) \\
&\qquad + h\sum_{j=-J}^{J} E_j^0 + \Delta t\sum_{n=0}^{N-1}\left(Q_{J-\frac{1}{2}}^n - Q_{J+\frac{1}{2}}^n\right).
\end{aligned} \tag{31}$$

As in the proof of Lemma 3.4, we see that E_j^N is nonnegative when η is nonnegative. Since r_j^n and s_j^n are uniformly bounded, E_j^n and $Q_{j+1/2}^n$ are also uniformly bounded.

Hence there exists a constant $\widehat{C_1}$ such that

$$h \sum_{j=-J}^{J} E_j^0 + \Delta t \sum_{n=0}^{N-1} \left(Q_{J-\frac{1}{2}}^n - Q_{J+\frac{1}{2}}^n \right) \leq \widehat{C_1} \left(Jh + T \right) := C_1. \tag{32}$$

To conclude the proof, we need to bound the terms involving $A_j^{\pm,n}$ and $\left(u_j^{n+1} - u_j^n\right)^2$.

Remark 4.1. *Note that if we set $g(u,v) = f(v)$ and $b = a$, we get*

$$A_j^{+,n} = A_j^{-,n} = 0, \text{ and } e_{ru}^+ = e_{su}^- = 0.$$

Assume that the modified CFL condition

$$a\lambda \leq \frac{\widetilde{m_1}^2}{8\widetilde{m_2}^2} \tag{33}$$

holds, where $\widetilde{m_1}$ and $\widetilde{m_2}$ are constants such that

$$0 < \widetilde{m_1} \leq \frac{\frac{1}{2}\eta''(k^+(r))}{\left(1 + \frac{f'(k^+(r))}{a}\right)} \leq \widetilde{m_2}, \quad \widetilde{m_1} \leq \frac{\frac{1}{2}\eta''(k^-(s))}{\left(1 - \frac{f'(k^-(s))}{a}\right)} \leq \widetilde{m_2}, \quad r,s \in [0,1]. \tag{34}$$

Then inequality (31) takes the form

$$\begin{aligned}
h\frac{\widetilde{m_1}}{4} \sum_{n=0}^{N-1} \sum_{j=-J}^{J} \Big[\left(r_j^{n+1} - r_j^n\right)^2 + \left(s_j^{n+1} - s_j^n\right)^2 \\
+ a\lambda \left(r_j^n - r_{j-1}^n\right)^2 + a\lambda \left(s_{j+1}^n - s_j^n\right)^2 \Big] \\
+ h \sum_{j=-J}^{J} E_j^N + \frac{\alpha h \Delta t}{2a\varepsilon} \sum_{n=0}^{N-1} \sum_{j=-J}^{J} \left(f(v_j^{n+1}) - z_j^{n+1}\right)^2 \\
\leq h \sum_{j=-J}^{J} E_j^0 + \Delta t \sum_{n=0}^{N-1} \left(Q_{J-\frac{1}{2}}^n - Q_{J+\frac{1}{2}}^n \right) \leq C_1.
\end{aligned}$$

By relabeling $v_j^n \to u_j^n$, $z_j^n \to w_j^n$, $r_j^n \to \widetilde{r}_j^n$ and $s_j^n \to \widetilde{s}_j^n$ we achieve the following discrete L^2_{loc} estimate

$$\begin{aligned}
h \sum_{n=0}^{N-1} \sum_{j=-J}^{J} \left(u_j^{n+1} - u_j^n\right)^2 &= h \sum_{n=0}^{N-1} \sum_{j=-J}^{J} \left(\frac{\widetilde{r}_j^{n+1} + \widetilde{s}_j^{n+1}}{2} - \frac{\widetilde{r}_j^n + \widetilde{s}_j^n}{2} \right)^2 \\
&\leq h \sum_{n=0}^{N-1} \sum_{j=-J}^{J} \left[\left(\widetilde{r}_j^{n+1} - \widetilde{r}_j^n\right)^2 + \left(\widetilde{s}_j^{n+1} - \widetilde{s}_j^n\right)^2 \right] \leq \widehat{C_2},
\end{aligned}$$

and similarly

$$\Delta t \sum_{n=0}^{N-1} \sum_{j=-J}^{J} \left(u_j^n - u_{j-1}^n\right)^2 \leq \widehat{C_2},$$

for some constant $\widehat{C_2}$ that does not depend on h or ε.

Next we introduce a bound on λ that assures that both the modified CFL conditions (27) and (33) hold. We say that the strengthened CFL condition is satisfied if

$$\max\{a,b\}\,\lambda \leq \frac{1}{8} \min\left\{ \left(\frac{m_1}{m_2}\right)^2, \left(\frac{\widetilde{m_1}}{\widetilde{m_2}}\right)^2 \right\}, \tag{35}$$

where (m_1, m_2) and $(\widetilde{m_1}, \widetilde{m_2})$ are given by (25), (26) and (34). By Remark 4.1 and the boundedness of e^+_{ru} and e^-_{su} we can find a constant C_2 such that

$$2b\Delta t \frac{m_1}{m_2} \sum_{n=0}^{N-1} \sum_{j=-J}^{J} \left(e^+_{ru}\left(\zeta_j^{n+\frac12}\right) + e^-_{su}\left(\gamma_j^{n+\frac12}\right)\right)\left(u_j^{n+1} - u_j^n\right)^2 \le C_2, \tag{36}$$

uniformly in h and ε. Regarding $A_j^{+,n}$ we have that

$$\left|A_j^{+,n}\right| \le \left\|e^+\right\|_{\mathrm{Lip}} \left(b\lambda \left|u_j^n - u_{j-1}^n\right| + \left|u_j^{n+1} - u_j^n\right|\right).$$

Since u_j^n is given by the scheme (17) with $g(u,v) = f(v)$ we see that

$$u_j^{n+1} - u_j^n = \frac{\lambda a}{2}\left(u_{j-1}^n - 2u_j^n + u_{j+1}^n\right) - \frac{\lambda}{2}\left(w_{j+1}^n - w_{j-1}^n\right),$$

and so

$$\left|u_j^{n+1} - u_j^n\right| \le \frac{\lambda a}{2}\left|u_{j+1}^n - u_j^n\right| + \frac{\lambda a}{2}\left|u_j^n - u_{j-1}^n\right| + \frac{\lambda}{2}\left|w_{j+1}^n - w_{j-1}^n\right|.$$

Hence

$$\begin{aligned} h\sum_{n=0}^{N-1}\sum_{j=-J}^{J} |A_j^{+,n}| \le \frac{\Delta t}{2}\left\|e^+\right\|_{Lip([0,1]^2)} \sum_{n=0}^{N-1}\sum_{j=-J}^{J} \Big[& 2b\left|u_j^n - u_{j-1}^n\right| + a\left|u_{j+1}^n - u_j^n\right| \\ & + a\left|u_j^n - u_{j-1}^n\right| + \left|w_{j+1}^n - w_{j-1}^n\right|\Big] \\ \le T\left\|e^+\right\|_{Lip([0,1]\times\mathcal{K}_r)} \left((a+b) T.V._x(u^n) + \frac12 T.V._x(w^n)\right) & \le \widehat{C_3}, \end{aligned}$$

for some constant $\widehat{C_3}$, where we have used the uniform BV bounds on u^n and w^n from Lemma 4.1. By handling $A_j^{-,n}$ in a similar way, we get the bound

$$h\sum_{n=0}^{N-1}\sum_{j=-J}^{J} \left|A_j^{+,n}\right| + \left|A_j^{-,n}\right| \le C_3, \tag{37}$$

for some constant C_3 independent of h and ε. Going back to the inequality (31) we can now use the bounds (32), (36) and (37) to obtain

$$\begin{aligned} h\frac{m_1}{4}\sum_{n=0}^{N-1}\sum_{j=-J}^{J} & \left[\left(r_j^{n+1} - r_j^n\right)^2 + \left(s_j^{n+1} - s_j^n\right)^2 + b\lambda\left(r_j^n - r_{j-1}^n\right)^2 + b\lambda\left(s_{j+1}^n - s_j^n\right)^2\right] \\ & + h\sum_{j=-J}^{J} E_j^N + \frac{\alpha h\Delta t}{2b\varepsilon}\sum_{n=0}^{N-1}\sum_{j=-J}^{J}\left(g(u_j^{n+1}, v_j^{n+1}) - z_j^{n+1}\right)^2 \le C_1 + C_2 + C_3 \end{aligned}$$

which proves the lemma. □

Remark 4.2. *By choosing $\eta(v) = v^2/2$, we find that*

$$\begin{aligned} \frac{\frac12\left(b - \max|g_u|\right)}{b + \max|g_v|} \le & m_1 \le m_2 \le \frac{\frac12\left(b + \max|g_u|\right)}{b - \max|g_v|}, \\ \frac{\frac12 a}{a + \max|f'|} \le & \widetilde{m_1} \le \widetilde{m_2} \le \frac{\frac12 a}{a - \max|f'|}, \end{aligned}$$

and so the strengthened CFL condition (35) holds when

$$\max\{a,b\}\lambda \le \frac18 \min\left\{ \left(\frac{\left(b - \max|g_u|\right)\left(b - \max|g_v|\right)}{\left(b + \max|g_u|\right)\left(b + \max|g_v|\right)}\right)^2, \left(\frac{a - \max|f'|}{a + \max|f'|}\right)^2 \right\},$$

where the maximum and minimum of g_u, g_v and f' are taken over all $(u,v) \in [0,1]^2$.

4.1. Convergence of the relaxation scheme.

Lemma 4.4. *Assume that the strengthened CFL condition (35) holds, and let u be the entropy solution of (4). Then for all $c \in \mathbb{R}$, the set of distributions*

$$\left\{\partial_t \left|v^{\varepsilon,h} - c\right| + \partial_x \left(\operatorname{sign}(v^{\varepsilon,h} - c)(g(u, v^{\varepsilon,h}) - g(u,c))\right)\right\}_{\varepsilon>0,\, h>0}$$

lies in a compact subset of H^{-1}_{loc}.

Proof. We start the proof by approximating the Kružkov form

$$(|v-c|, \operatorname{sign}(v-c)(g(u,v) - g(u.c)))$$

with some smooth entropy/entropy-flux pair $(\eta_\varepsilon, q_\varepsilon)$ centered at $v=0$. Let (η_0, q_0) be as in (14) and $(\eta_\varepsilon, q_\varepsilon)$ as in (15). As in the proof of Theorem 3.1 we have that

$$\begin{aligned}\partial_t \left|v^{\varepsilon,h} - c\right| &+ \partial_x \left(\operatorname{sign}(v^{\varepsilon,h} - c)(g(u, v^{\varepsilon,h}) - g(u,c))\right) \\ &= \partial_t \eta_\varepsilon(v^{\varepsilon,h}) + \partial_x q_\varepsilon(u^{\varepsilon,h}, v^{\varepsilon,h}) + \mathcal{L}_1^{\varepsilon,h},\end{aligned}$$

in $\mathcal{D}'(\mathbb{R}\times(0,T))$, where $\mathcal{L}_1^{\varepsilon,h}$ is contained in a compact subset of $H^{-1}_{\mathrm{loc}}(\mathbb{R}\times(0,\infty))$. Continuing as in the proof of Theorem 3.1, we use the entropy/entropy-flux pair $(E_\varepsilon, Q_\varepsilon)$ corresponding to $(\eta_\varepsilon, q_\varepsilon)$ given by Lemma 3.3 to write

$$\partial_t \eta_\varepsilon(v^{\varepsilon,h}) + \partial_x q_\varepsilon(u^{\varepsilon,h}, v^{\varepsilon,h}) = \mathcal{L}_2^{\varepsilon,h} + \mathcal{L}_3^{\varepsilon,h} + \mathcal{L}_4^{\varepsilon,h},$$

where

$$\begin{aligned}\mathcal{L}_2^{\varepsilon,h} &= \partial_t \left(E_\varepsilon(u^{\varepsilon,h}, v^{\varepsilon,h}, g(u^{\varepsilon,h}, v^{\varepsilon,h})) - E_\varepsilon(u^{\varepsilon,h}, v^{\varepsilon,h}, z^{\varepsilon,h})\right),\\ \mathcal{L}_3^{\varepsilon,h} &= \partial_x \left(Q_\varepsilon(u^{\varepsilon,h}, v^{\varepsilon,h}, g(u^{\varepsilon,h}, v^{\varepsilon,h})) - Q_\varepsilon(u^{\varepsilon,h}, v^{\varepsilon,h}, z^{\varepsilon,h})\right),\\ \mathcal{L}_4^{\varepsilon,h} &= \partial_t E_\varepsilon(u^{\varepsilon,h}, v^{\varepsilon,h}, z^{\varepsilon,h}) + \partial_x Q_\varepsilon(u^{\varepsilon,h}, v^{\varepsilon,h}, z^{\varepsilon,h}).\end{aligned}$$

By Lemma 4.3 we then find that $\left|\langle \mathcal{L}_i^{\varepsilon,h}, \varphi\rangle\right| \leq \mathcal{O}(\sqrt{\varepsilon})\left\|\varphi\right\|_{H^1}$ for $i=2,3$. Hence

$$\mathcal{L}_2^{\varepsilon,h} \to 0 \text{ and } \mathcal{J}_3^{\varepsilon,h} \to 0 \text{ in } H^{-1}_{\mathrm{loc}}(\mathbb{R}\times(0,\infty)) \text{ when } \varepsilon \to 0.$$

To estimate $\mathcal{J}_4^{\varepsilon,h}$ we use that

$$\begin{aligned}\langle \mathcal{L}_4^{\varepsilon,h}, \varphi\rangle &= -\int_0^T \int E_\varepsilon(u^{\varepsilon,h}, v^{\varepsilon,h}, z^{\varepsilon,h})\varphi_t + Q_\varepsilon(u^{\varepsilon,h}, v^{\varepsilon,h}, z^{\varepsilon,h})\varphi_x \, dxdt \\ &= -\sum_{n,j} \iint\limits_{\chi_j^n} E_j^n \varphi_t + Q_j^n \varphi_x \, dxdt \\ &= \underbrace{-\sum_{n,j}\left[E_j^n \iint\limits_{\chi_j^n} \varphi_t(x_{j+\frac12}, t)\, dx\, dt + Q_j^n \iint\limits_{\chi_j^n} \varphi_x(x, t_{n+1})\, dx\, dt\right]}_{L_{4,1}(\varphi)} \\ &\quad \underbrace{-\sum_{n,j}\left[E_j^n \iint\limits_{\chi_j^n} \int_{x_{j+\frac12}}^x \varphi_{t\xi}(\xi,t)\, d\xi\, dx\, dt + Q_j^n \iint\limits_{\chi_j^n} \int_{t_{n+1}}^t \varphi_{x\tau}(x,\tau)\, d\tau\, dx\, dt\right]}_{L_{4,2}(\varphi)}\end{aligned}$$

where $\chi_j^n = I_j\times[t^n,t^{n+1})$, $E_j^n = E_\varepsilon(u_j^n,v_j^n,z_j^n) = e_j^{+,n}+e_j^{-,n}$, $Q_j^n = Q_\varepsilon(u_j^n,v_j^n,z_j^n) = be_j^{+,n} - be_j^{-,n}$. Let $\varphi_j^n = \varphi(x_j,t_n)$. Then using summation by parts

$$
\begin{aligned}
-\sum_{n,j} E_j^n \iint\limits_{\chi_j^n} \varphi_t(x_{j+\frac12},t)\,dx\,dt &= -h\sum_{n,j} E_j^n\left(\varphi_{j+\frac12}^{n+1} - \varphi_{j+\frac12}^n\right)\\
&= h\sum_{n,j}\varphi_{j+\frac12}^{n+1}\left(E_j^{n+1} - E_j^n\right),
\end{aligned}
$$

and

$$
\begin{aligned}
-\sum_{n,j} Q_j^n \iint\limits_{\chi_j^n} \varphi_x(x,t_{n+1})\,dx\,dt &= -hb\lambda\sum_{n,j}\left(e_j^{+,n} - e_j^{-,n}\right)\left(\varphi_{j+\frac12}^{n+1} - \varphi_{j-\frac12}^{n+1}\right)\\
&= hb\lambda\sum_{n,j}\varphi_{j-\frac12}^{n+1}\left(e_j^{+,n} - e_{j-1}^{+,n}\right) - hb\lambda\sum_{n,j}\varphi_{j+\frac12}^{n+1}\left(e_{j+1}^{-,n} - e_j^{-,n}\right)\\
&= hb\lambda\sum_{n,j}\varphi_{j+\frac12}^{n+1}\left(e_j^{+,n} - e_{j-1}^{+,n}\right) - hb\lambda\sum_{n,j}\varphi_{j+\frac12}^{n+1}\left(e_{j+1}^{-,n} - e_j^{-,n}\right)\\
&\quad - hb\lambda\sum_{n,j}\left(e_j^{+,n} - e_{j-1}^{+,n}\right)\left(\varphi_{j+\frac12}^{n+1} - \varphi_{j-\frac12}^{n+1}\right)\\
&= h\lambda\sum_{n,j}\varphi_{j+\frac12}^{n+1}\left(Q_{j+\frac12}^n - Q_{j-\frac12}^n\right) - hb\lambda\sum_{n,j}\left(e_j^{+,n} - e_{j-1}^{+,n}\right)\left(\varphi_{j+\frac12}^{n+1} - \varphi_{j-\frac12}^{n+1}\right),
\end{aligned}
$$

where $Q_{j+\frac12}^n = be_j^{+,n} - be_{j+1}^{-,n}$. Hence,

$$
\begin{aligned}
L_{4,1}(\varphi) = &\underbrace{h\sum_{n,j}\varphi_{j+\frac12}^{n+1}\left(\left(E_j^{n+1} - E_j^n\right) + \lambda\left(Q_{j+\frac12}^n - Q_{j-\frac12}^n\right)\right)}_{L_{4,1a}(\varphi)}\\
&\underbrace{-hb\lambda\sum_{n,j}\left(e_j^{+,n} - e_{j-1}^{+,n}\right)\left(\varphi_{j+\frac12}^{n+1} - \varphi_{j-\frac12}^{n+1}\right)}_{L_{4,1b}(\varphi)}.
\end{aligned}
$$

Let us define the distributions $\mathcal{L}_{4,1a}^{\varepsilon,h}$, $\mathcal{L}_{4,1b}^{\varepsilon,h}$ and $\mathcal{L}_{4,2}^{\varepsilon,h}$ by

$$
\left\langle \mathcal{L}_{4,1a}^{\varepsilon,h},\varphi\right\rangle = L_{41a}(\varphi),\quad \left\langle \mathcal{L}_{4,1b}^{\varepsilon,h},\varphi\right\rangle = L_{4,1b}(\varphi),\quad\text{and}\quad \left\langle \mathcal{L}_{4,2}^{\varepsilon,h},\varphi\right\rangle = L_{4,2}(\varphi),
$$

for $\varphi\in C_0^\infty$. We then have that $\mathcal{J}_4^{\varepsilon,h} = \mathcal{L}_{4,1a}^{\varepsilon,h} + \mathcal{L}_{4,1b}^{\varepsilon,h} + \mathcal{L}_{4,2}^{\varepsilon,h}$ in $\mathcal{D}'$.

The proof of Lemma 4.3, (30) shows that

$$
\begin{aligned}
&E_j^{n+1} - E_j^n + \lambda\left(Q_{j+\frac12}^n - Q_{j-\frac12}^n\right)\\
&\quad + \frac{m_1}{4}\left[\left(r_j^{n+1} - r_j^n\right)^2 + \left(s_j^{n+1} - s_j^n\right)^2 + b\lambda\left(r_j^n - r_{j-1}^n\right)^2 + b\lambda\left(s_{j+1}^n - s_j^n\right)^2\right]\\
&\qquad\qquad + \frac{\alpha\Delta t}{2b\varepsilon}\left(g(u_j^{n+1},v_j^{n+1}) - z_j^{n+1}\right)^2\\
&\quad\le 2b\lambda\frac{m_1}{m_2}\left(e_{ru}^+(\zeta_j^{n+\frac12}) + e_{su}^-(\gamma_j^{n+\frac12})\right)\left(u_j^{n+1} - u_j^n\right)^2 + \left|A_j^{+,n}\right| + \left|A_j^{-,n}\right|.
\end{aligned}
$$

Using that e^+_{ru} and e^+_{su} are bounded we get

$$\begin{aligned}
&\left|E_j^{n+1}-E_j^n+\lambda\left(Q^n_{j+\frac12}-Q^n_{j-\frac12}\right)\right| \\
&\quad\le \overline{C}\Big\{\left(r_j^{n+1}-r_j^n\right)^2+\left(s_j^{n+1}-s_j^n\right)^2+b\lambda\left(r_j^n-r_{j-1}^n\right)^2+b\lambda\left(s_{j+1}^n-s_j^n\right)^2 \\
&\qquad\quad+\frac{\Delta t}{b\varepsilon}\left(g(u_j^{n+1},v_j^{n+1})-z_j^{n+1}\right)^2+b\lambda\left(u_j^{n+1}-u_j^n\right)^2\Big\}+\left|A_j^{+,n}\right|+A_j^{-,n},
\end{aligned}$$

for some constant $\overline{C}$ independent of h and ε. Using Lemma 4.3, Remark 4.1 and (37), by summing over those j and n such that $\operatorname{supp}(\varphi)\cap\chi_j^n\neq\emptyset$ we obtain

$$|L_{4,1a}(\varphi)|\le h\,\|\varphi\|_{L^\infty}\sum_{n,j}\left|\left(E_j^{n+1}-E_j^n\right)+\lambda\left(Q^n_{j+\frac12}-Q^n_{j-\frac12}\right)\right|\le C\,\|\varphi\|_{L^\infty}\,.$$

Hence the set of distributions $\left\{\mathcal{L}^{\varepsilon,h}_{4,1a}\right\}_{\varepsilon>0,h>0}$ is bounded in $\mathcal{M}_{\mathrm{loc}}(\mathbb{R}\times(0,T))$.

By the Lipschitz continuity of e^+,

$$\left|e_j^{+,n}-e_{j-1}^{+,n}\right|\le C\left(\left|u_j^n-u_{j-1}^n\right|+\left|r_j^n-r_{j-1}^n\right|\right). \tag{38}$$

Furthermore $\varphi^{n+1}_{j+1/2}-\varphi^{n+1}_{j-1/2}=h\varphi_x(\overline{x}_j,t_{n+1})$ for some $\overline{x}_j\in[x_{j-1/2},x_{j+1/2}]$, and so

$$\begin{aligned}
|L_{4,1b}(\varphi)|&\le h^2b\lambda\sum_{n,j}\left|\varphi_x(\overline{x}_j,t_{n+1})\right|\left|e_j^{+,n}-e_{j-1}^{+,n}\right| \\
&\le Cb\sum_{n,j}h\Delta t\left|\varphi_x(\overline{x}_j,t_{n+1})\right|\left(\left|u_j^n-u_{j-1}^n\right|+\left|r_j^n-r_{j-1}^n\right|\right) \\
&\le Cb\Big(h\Delta t\sum_{n,j}\left(\varphi_x(\overline{x}_j,t_{n+1})\right)^2\Big)^{\frac12} \\
&\qquad\times\left[\Big(h\Delta t\sum_{n,j}\left(u_j^n-u_{j-1}^n\right)^2\Big)^{\frac12}+\Big(h\Delta t\sum_{n,j}\left(r_j^n-r_{j-1}^n\right)^2\Big)^{\frac12}\right].
\end{aligned}$$

Since $\varphi\in C_0^\infty$ we have $\left(h\Delta t\sum_{n,j}\left(\varphi_x(\overline{x}_j,t_{n+1})\right)^2\right)^{\frac12}\to\|\varphi_x\|_{L^2}$ as $h\downarrow 0$, so we can choose h small enough that

$$\Big(h\Delta t\sum_{n,j}\left(\varphi_x(\overline{x}_j,t_{n+1})\right)^2\Big)^{\frac12}\le 2\,\|\varphi\|_{H^1}\,.$$

Using this together with Lemma 4.3 and Remark 4.1 we can find some constant C that does not depend on h and ε such that

$$|L_{4,1b}(\varphi)|\le C\sqrt{h}\,\|\varphi\|_{H^1}\,. \tag{39}$$

By another use of summation by parts we find that

$$\begin{aligned}
L_{4,2}(\varphi)=&\underbrace{\sum_{n,j}\left(E_j^{n+1}-E_j^n\right)\int_{x_{j-\frac12}}^{x_{j+\frac12}}\left(\varphi(x,t_{n+1})-\varphi(x_{j+\frac12},t_{n+1})\right)dx}_{L_{4,2a}(\varphi)} \\
&\qquad+\underbrace{\sum_{n,j}\left(Q^n_{j+1}-Q^n_j\right)\int_{t_n}^{t_{n+1}}\left(\varphi(x_{j+\frac12},t)-\varphi(x_{j+\frac12},t_{n+1})\right)dt}_{L_{4,2b}(\varphi)}.
\end{aligned}$$

To estimate $L_{4,2a}$ and $L_{4,2b}$ we will need the following inequality. Let h be a H^1 function on an interval $[b_1, b_2]$. Then

$$\Big|\int_{b_1}^{b_2} (h(z) - h(b))\, dz\Big| \le \frac{1}{2}\,(b_2 - b_1)^2\, \|h'\|_{L^2((b_1,b_2))}, \qquad \text{for all } b \in [b_1, b_2]. \tag{40}$$

Furthermore E_ε is Lipschitz continuous, so that

$$\left|E_j^{n+1} - E_j^n\right| \le C\left(\left|u_j^{n+1} - u_j^n\right| + \left|r_j^{n+1} - r_j^n\right| + \left|s_j^{n+1} - s_j^n\right|\right).$$

Using the above, the Hölder inequality and (40), we get

$$\begin{aligned}
|L_{4,2a}(\varphi)| &\le \frac{C}{2\lambda}\Big(h\Delta t \sum_{n,j} \|\varphi_x(\cdot, t_{n+1})\|_{L^2}^2\Big)^{\frac{1}{2}} \\
&\qquad \times \Bigg[\Big(2h\Delta t \sum_{n,j} \left(u_j^{n+1} - u_j^n\right)^2\Big)^{\frac{1}{2}} \\
&\qquad\qquad + \Big(h\Delta t \sum_{n,j} \left(r_j^{n+1} - r_j^n\right)^2\Big)^{\frac{1}{2}} + \Big(h\Delta t \sum_{n,j} \left(s_j^{n+1} - s_j^n\right)^2\Big)^{\frac{1}{2}}\Bigg] \\
&\le C\sqrt{h}\, \|\varphi\|_{H^1}
\end{aligned}$$

for some constant C that is independent h and ε but dependent on $\operatorname{supp}(\varphi)$. Similarly, for $L_{2,2}(\varphi)$ we can find that

$$\begin{aligned}
|L_{4,2b}(\varphi)| &\le \frac{b\lambda C}{2}\Big(h\Delta t \sum_{n,j} \left\|\varphi_t(x_{j+\frac{1}{2}}, \cdot)\right\|_{L^2}^2\Big)^{\frac{1}{2}} \\
&\qquad \times \Bigg[\Big(2h\Delta t \sum_{n,j} \left(u_{j+1}^n - u_j^n\right)^2\Big)^{\frac{1}{2}} \\
&\qquad\qquad + \Big(h\Delta t \sum_{n,j} \left(r_{j+1}^n - r_j^n\right)^2\Big)^{\frac{1}{2}} + \Big(h\Delta t \sum_{n,j} \left(s_{j+1}^n - s_j^n\right)^2\Big)^{\frac{1}{2}}\Bigg] \\
&\le C\sqrt{h}\, \|\varphi\|_{H^1}.
\end{aligned}$$

Summing up, we have established that

$$\begin{aligned}
\partial_t \left|v^{\varepsilon,h} - c\right| &+ \partial_x\left(\operatorname{sign}(v^{\varepsilon,h} - c)(g(u, v^{\varepsilon,h}) - g(u,c))\right) \\
&= \mathcal{L}_1^{\varepsilon,h} + \mathcal{L}_2^{\varepsilon,h} + \mathcal{L}_3^{\varepsilon,h} + \mathcal{L}_{4,1a}^{\varepsilon,h} + \mathcal{L}_{4,1b}^{\varepsilon,h} + \mathcal{L}_{4,2a}^{\varepsilon,h} + \mathcal{L}_{4,2b}^{\varepsilon,h},
\end{aligned}$$

where $\mathcal{L}_1^{\varepsilon,h}$, $\mathcal{L}_2^{\varepsilon,h}$, $\mathcal{L}_3^{\varepsilon,h}$, $\mathcal{L}_{4,1b}^{\varepsilon,h}$, $\mathcal{L}_{4,2a}^{\varepsilon,h}$, and $\mathcal{L}_{4,2b}^{\varepsilon,h}$ are compact in H^{-1}_{loc}, and $\mathcal{L}_{4,1a}$ is bounded in $\mathcal{M}_{\text{loc}}$. Lemma 2.3 concludes the proof. □

Theorem 4.1. *Assume that the strengthened CFL condition (35) holds. Then there exists a sequence $\{(\varepsilon_n, h_n)\}_{n\in\mathbb{N}}$, $(\varepsilon_n, h_n) \downarrow (0,0)$, and a weak solution (u, v) of (1) such that*

$$u^{\varepsilon_n,h_n} \to u, \quad v^{\varepsilon_n,h_n} \to v \text{ in } L^p_{\text{loc}}(\mathbb{R} \times (0,\infty)),\ 1 \le p < \infty.$$

Proof. The family of functions $\left\{v^{\varepsilon,h}\right\}_{\varepsilon>0,\, h>0}$ defined on $\mathbb{R} \times (0,T)$ (and then on $\mathbb{R}\times(0,\infty)$ by letting $v^{\varepsilon,h}(x,t) = 0$ for $t \ge T$) lies in a bounded set of $L^1_{\text{loc}}(\mathbb{R}\times(0,T))$ by the L^∞ bounds in Lemma 4.2. By Lemma 4.4 and Lemma 2.2 there exists a converging subsequence $v^{\varepsilon_n,h_n} \to v$ in $L^p_{\text{loc}}(\mathbb{R} \times (0,\infty))$.

We know that $u^{\varepsilon,h} \to u$, where u is the entropy solution of (4). Multiplying the first equation in (17) by $h\Delta t \varphi_j^n$ and using summation by parts

$$\int_0^T \int \left(v^{\varepsilon,h}\varphi_t + z^{\varepsilon,h}\varphi_x\right) dxdt + \int v_0^{\varepsilon,h}(x)\varphi(x,0)\,dx$$
$$= \mathcal{O}(h) + \underbrace{\frac{b\Delta t}{4}\sum_{n=0}^{N-1}\sum_{j=-J}^{J}\left(\left(r_{j+1}^n - r_j^n\right) + \left(s_{j+1}^n - s_j^n\right)\right)\left(\varphi_{j+1}^n - \varphi_j^n\right)}_{E_1(h)}$$
$$+ \underbrace{\frac{h}{2}\sum_{n=0}^{N-1}\sum_{j=-J}^{J}\left(\left(r_j^{n+1} - r_j^n\right) + \left(s_j^{n+1} - s_j^n\right)\right)\left(\varphi_j^{n+1} - \varphi_j^n\right)}_{E_2(h)}$$

By Cauchy's inequality and Lemma 4.3 we have,

$$\begin{aligned}(E_1(h))^2 &\le \frac{b\Delta t^2}{4}\Big(\sum_{n=1}^{N}\sum_{j=-J}^{J}\left(\left(r_{j+1}^n - r_j^n\right)^2 + \left(s_{j+1}^n - s_j^n\right)^2\right)\Big)\\ &\qquad\times\Big(\sum_{n=1}^{N}\sum_{j=-J}^{J}\left(\frac{\varphi_{j+1}^n - \varphi_j^n}{h}\right)^2\Big)\\ &\le Ch,\end{aligned}$$

where C depends on X but not ε and h. Similarly $|E_2(h)| \le C\sqrt{h}$. For Ω bounded we have

$$\begin{aligned}\left\|g(u,v) - z^{\varepsilon,h}\right\|_{L^2(\Omega)} &\le \left\|g(u^{\varepsilon,h}, v^{\varepsilon,h}) - z^{\varepsilon,h}\right\|_{L^2(\Omega)}\\ &\qquad + C\left(\left\|u^{\varepsilon,h} - u\right\|_{L^2(\Omega)} + \left\|v^{\varepsilon,h} - v\right\|_{L^2(\Omega)}\right) \to 0\end{aligned}$$

as $\varepsilon, h \downarrow 0$. Then since $v^{\varepsilon,h} \to v$ and $z^{\varepsilon,h} \to g(u,v)$ strongly as $\varepsilon, h \downarrow 0$ we get

$$\begin{aligned}\int_0^T \int \left(v\varphi_t + g(u,v)\varphi_x\right) dx\,dt + \int v_0(x)\varphi(x,0)\,dx\\ = \lim_{\varepsilon,h\downarrow 0}\left(\mathcal{O}(h) + E_1(h) + E_2(h)\right) = 0.\end{aligned}$$

□

5. A numerical experiment

We test the scheme on the model problem studied in [9] where an exact solution is known. We choose the flux functions

$$f(u) = \frac{1}{2}u^2, \quad g(u,v) = 4uv(1-v),$$

and Riemann initial data

$$u_0(x) = \begin{cases} 0.75, & x < 0,\\ 0.25, & x \ge 0,\end{cases} \qquad v_0(x) = 0.5.$$

This problem has an exact solution given by

$$u(x,t) = \begin{cases} 3/4, & x < t/2,\\ 1/4, & x \ge t/2,\end{cases} \qquad v(x,t) = \begin{cases} 1/2, & x < -t,\\ 5/6, & x \le t/2,\\ 1/2, & x \ge t/2.\end{cases}$$

In Figure 1 we see the exact solution (u, v) together with the numerical approx-

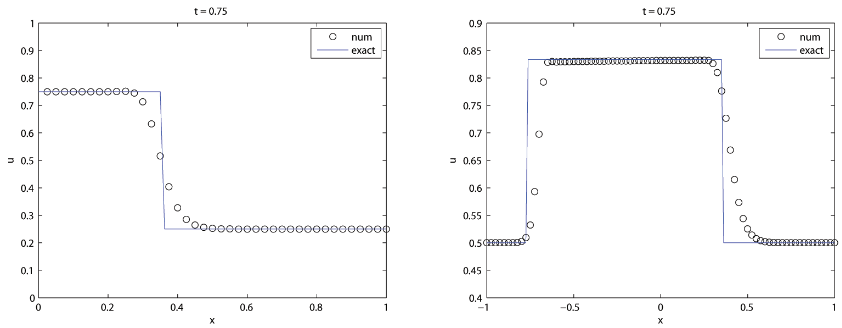

FIGURE 1. Exact and numerical solution for u and v.

imation at the time $t = 0.75$. In the numerical scheme we have used $h = 1/40$, $\Delta t = 1/100$, $a = 0.6$, $b = 1.7$ and $\varepsilon = 10^{-12}$. By other computations the scheme seems to be stable under the CFL condition (18) so the strengthened CFL condition (35) might be superfluous. As is expected, the scheme has some numerical diffusion depending on the parameters a, b and ε. Although ε can be chosen sufficiently small, this cannot be done with a and b without losing stability. In the numerical experiments different values for a and b have been tested, and the smallest values that still seem to give a stable solution are used in the approximations.

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