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ON THE SINGULARLY PERTURBED SEMILINEAR REACTION-DIFFUSION PROBLEM AND ITS NUMERICAL SOLUTION

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Abstract. We obtain improved derivative estimates for the solution of the semilinear singularly perturbed reaction-diffusion problem in one dimension. This enables us to modify the transition points between the fine and coarse parts of the Shishkin discretization mesh. We prove that the numerical solution, obtained by using the central finite-difference scheme on the modified mesh, retains the same order of convergence uniform in the perturbation parameter as on the standard Shishkin mesh. However, the modified mesh may be denser in the layers than the standard one, and, when this is the case, numerical results show an improvement in the accuracy of the computed solution.

Key words. singularly perturbed boundary-value problem, reaction-diffusion, Shishkin mesh, finite differences, and uniform convergence.

1. Introduction

We consider the semilinear singularly perturbed boundary-value problem

$$\begin{gathered} Tu := -\varepsilon^2 u'' + b(x,u) = 0, \quad x \in I := [0,1], \quad u(0) = u(1) = 0, \\ b_u(x,u) \geq \beta^2 > 0, \quad x \in I, \quad u \in D, \quad \beta > 0, \end{gathered} \tag{1}$$

where $0 < \varepsilon \leq \varepsilon^* \ll 1$, b is a sufficiently smooth function, $b \in C^k(I \times D)$, $k \geq 0$, and D is some closed bounded domain which we specify in Section 2. The problem has a unique solution $u \in C^{k+2}(I)$ for which the following derivative estimates hold true (cf. [17]):

$$|u^{(i)}(x)| \leq M \left(1 + \varepsilon^{-i} e^{-\beta x/\varepsilon} + \varepsilon^{-i} e^{\beta(x-1)/\varepsilon}\right), \quad i = 0, 1, \ldots, k, \quad x \in I, \tag{2}$$

with M denoting a generic positive constant independent of ε. The estimates show that, in general, the solution u has boundary layers near $x = 0$ and $x = 1$.

Numerical methods for problems of type (1), sometimes in the linear version, are studied in [1, 2, 3, 8, 9, 11, 13, 14, 16, 17, 19]. The semilinear problem $Tu = 0$ is considered in [6, 15, 7] under relaxed conditions on b, which allow for multiple solutions with boundary or interior layers. We do not consider these relaxed conditions here. Instead, we focus on the condition on b stated in (1), which is assumed in most of the above-cited works. Our aim is to show that even with this condition it is possible to improve numerical results obtained when the problem (1) is discretized on a mesh of Shishkin type.

Shishkin meshes [3, 11, 9, 13] are arguably the most popular meshes for discretizing singular perturbation problems. The presence of layers is characteristic of solutions to singularly perturbed boundary-value problems and Shishkin meshes are layer-adapted. This is why they enable ε-uniform convergence of numerical

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solutions, which is the main goal of numerical methods for singularly perturbed problems. For the problem (1), the Shishkin mesh is divided into two fine parts in the layers and the coarse one outside the layers. The points at which the mesh step size changes are called transition points. The standard definition of the left transition point for (1) is $a\varepsilon \ln N/\beta$ and the right transition point is $1 - a\varepsilon \ln N/\beta$. The quantities N and a in this definition are respectively the total number of mesh steps and a sufficiently large positive parameter, which is related to the order of convergence of the numerical method. In general, the influence of the choice of the transition points and the complete mechanism of the Shishkin mesh are explained in details in [5]. A discussion of generalizations of the Shishkin mesh can be found in [9].

The crucial result of this paper is the modification of the estimates (2) to

$$|u^{(i)}(x)| \le M \left(1 + \varepsilon^{-i} e^{-\beta_0 x/\varepsilon} + \varepsilon^{-i} e^{\beta_1 (x-1)/\varepsilon}\right), \quad i = 0, 1, \ldots, k, \quad x \in I, \tag{3}$$

where $\beta_i > 0$ and $b_u(i, u) > \beta_i^2$, $i = 0, 1$, for $u \in D$. This is obtained without any additional conditions on (1). The estimates in (3) may be sharper than those in (2). They also enable a redefinition of the transition points as $a\varepsilon \ln N/\beta_0$ and $1 - a\varepsilon \ln N/\beta_1$. It immediately follows that the standard central discretization of (1) on this modified Shishkin mesh (with $a \ge 2$) yields ε-uniform pointwise convergence of order almost 2. This is the same result as on the standard Shishkin mesh. However, it is possible that $\beta_i > \beta$, $i = 0, 1$, and we may get a better layer-resolving mesh since the transition points are moved closer to the end points where the layers occur. If this happens, the density of mesh points in the layers increases, because of which we can expect more accurate numerical results. This expectation has already been confirmed in [19] for the linear case of the problem (1).

The motivation for [19] and the present paper comes from [10], where a similarly modified transition point is used in numerical experiments with the quasilinear singularly perturbed convection-diffusion problem. However, the theoretical analysis in [10] is carried out for the standard Shishkin mesh, since no improved derivative estimates of the solution were available for use.

The outline of the paper is as follows. In Section 2 we analyze the continuous solution of the problem (1). We prove the estimates in (3), as well as some other estimates. Then, in Section 3 we consider the linear case of the problem and improve the results from Section 2. Section 3 also contains a discussion of the proof technique used for the semilinear problem (1) and the one in [19] for the linear problem. This shows that our present analysis is not a straightforward generalization of the analysis in [19]. In Section 4, the modified Shishkin mesh is defined and the ε-uniform convergence result for the central discretization scheme is proved. This is followed by Section 5, where we show that the piecewise-linear interpolation of the numerical solution retains the accuracy of the numerical solution. The results of sections 4 and 5 for the linear case are sharper than the results in [19]. Finally, Section 6 provides results of numerical experiments, which demonstrate improvements in the computed solution when compared to the results on the standard Shishkin mesh.

2. The general continuous problem

We assume that there exist constants u_* and u^* such that

$$u_* \le 0 \le u^*, \quad b(x, u_*) \le 0 \le b(x, u^*), \quad x \in I,$$

From $Tu_* \leq 0 \leq Tu^*$ we get that u_* and u^* are respectively lower and upper solutions of the problem (1). Therefore, this problem has a solution u satisfying

$$u(x) \in D := [u_*, u^*], \quad x \in I. \tag{4}$$

The domain D defined above is the domain referred to in the condition on b in (1). This condition also implies that the operator T (together with the boundary-condition operator) is inverse monotone, which means that u is unique. Moreover, (4) gives that $|u(x)| \leq M$, $x \in I$. Recall that M is used to denote any (in the sense of $O(1)$) positive constant independent of ε. Some particular constants of this kind are denoted by M_1, M_2, M_*, $\widetilde{M}$, etc. We also use some sufficiently small positive constants m, m_*, m_0, etc., which are independent of ε.

The proof of (2) can be found in [17] in the case when it is assumed that $b_u > \beta^2$. If $b_u \geq \beta^2$, the proof requires the approach from [8] after a suitable linearization of the operator T. As discussed in the introduction, the constant β is used to construct the transition points in the standard Shishkin mesh. From now on, we set

$$\beta = \min_{x \in I, y \in D} \sqrt{b_u(x,y)}$$

in order to emphasize that we are interested in using the greatest possible value of β. We are going to modify the transition points of the Shishkin mesh by using some other constants which may be greater than β. We define these constants in terms of b_u and an arbitrarily small fixed value $\delta \in (0,1)$, independent of ε:

$$\beta_i := (1-\delta) \min_{y \in D} \sqrt{b_u(i,y)}, \quad i = 0, 1.$$

The constant δ is used because we need β_i to be strictly less than the minimum occurring in the definition of β_i. Let

$$y_0(x) = e^{-\beta_0 x/\varepsilon} \quad \text{and} \quad y_1(x) = e^{\beta_1 (x-1)/\varepsilon}.$$

We can now prove estimates (3).

Theorem 1. *The solution u of the problem (1) satisfies*

$$|u^{(i)}(x)| \leq M[1 + \varepsilon^{-i} y_0(x) + \varepsilon^{-i} y_1(x)], \quad i = 0, 1, \ldots, k, \quad x \in I,$$

and

$$|u^{(i)}(x)| \leq M[\varepsilon^{k-i} + \varepsilon^{-i} y_0(x) + \varepsilon^{-i} y_1(x)], \quad i = k+1, k+2, \quad x \in I.$$

Proof. The case $k = 0$ is trivial, so let us consider $k \geq 1$. To prove the first estimates, it suffices to show that for all $i = 0, 1, \ldots, k$ we have

$$|u^{(i)}(x)| \leq M[1 + \varepsilon^{-i} y_0(x)], \quad x \in [0, m], \tag{5}$$

where m is a sufficiently small positive constant independent of ε, $m < \frac{1}{2}$. This is because (2) implies that $|u^{(i)}(x)| \leq M$ for $x \in [m, 1-m]$ and because the proof of

$$|u^{(i)}(x)| \leq M[1 + \varepsilon^{-i} y_1(x)], \quad x \in [1-m, 1],$$

is analogous.

As we have seen, $|u(x)| \leq M$ for $x \in I$. Thus, (5) hold true for $i = 0$. Next, assume that (5) is satisfied for $i = 0, 1, ..., j-1$, where $1 \leq j \leq k$. We need to prove (5) for $i = j$ and we do this by using the linear operator L and the barrier function φ_j, which are defined as follows:

$$Lv := -\varepsilon^2 v'' + b_u(x, u(x))v \quad \text{(for any } C^2(I)\text{-function } v),$$

$$\varphi_j(x) = M_j^* + M_j \varepsilon^{-j} y_0(x).$$

Since $b_u(x,u(x)) > 0$, $x \in I$, the operator L is inverse monotone.

After differentiating the differential equation in (1) j times, we get that

$$Lu^{(j)}(x) = -\frac{\partial^j}{\partial x^j} b(x,u(x)) + f_j(x),$$

where $f_1(x) \equiv 0$ and for $j \ge 2$, $f_j(x)$ contains terms the absolute value of which can be estimated by

$$M \prod_{n=1}^{\ell} |u^{(i_n)}(x)|$$

with $1 \le \ell \le j$, $1 \le i_n \le j-1$, and $1 \le i_1 + i_2 + \cdots + i_\ell \le j$. Using the inductive assumption, we get

$$\prod_{n=1}^{\ell} |u^{(i_n)}(x)| \le M\left[1 + \varepsilon^{-j} y_0(x)\right], \quad x \in [0,m].$$

Therefore,

$$|Lu^{(j)}(x)| \le \overline{M}_j + \widetilde{M}_j \varepsilon^{-j} y_0(x), \quad x \in [0,m], \tag{6}$$

with some constants $\overline{M}_j$ and $\widetilde{M}_j$. We now use the function φ_j to get

$$L\varphi_j(x) \ge M_j^* \beta^2 + [b_u(x,u(x)) - \beta_0^2] M_j \varepsilon^{-j} y_0(x), \quad x \in [0,m].$$

There exist positive constants m and m_* such that

$$\begin{aligned} b_u(x,u(x)) - \beta_0^2 &= b_u(x,u(x)) - b_u(0,u(x)) + b_u(0,u(x)) - \beta_0^2 \\ &\ge -x \cdot \max_{x\in I, y \in D} |b_{xu}(x,y)| + \delta\beta_0^2 \ge m_*, \quad x \in [0,m]. \end{aligned} \tag{7}$$

Therefore,

$$L\varphi_j(x) \ge M_j^* \beta^2 + m_* M_j \varepsilon^{-j} y_0(x), \quad x \in [0,m]. \tag{8}$$

We now choose sufficiently large constants M_j^* and M_j,

$$M_j^* \beta^2 \ge \overline{M}_j, \quad m_* M_j \ge \widetilde{M}_j,$$

so that (6) and (8) imply

$$L\varphi_j(x) \ge |Lu^{(j)}(x)|, \quad x \in [0,m].$$

At the same time, M_j and M_j^* should be chosen large enough to give, respectively,

$$|u^{(j)}(0)| \le M\varepsilon^{-j} \le \varphi_j(0)$$

and

$$|u^{(j)}(m)| \le M \le \varphi_j(m).$$

Then inverse monotonicity implies that

$$|u^{(j)}(x)| \le \varphi_j(x), \quad x \in [0,m],$$

which gives (5) for $i = j$.

Finally, the estimates of $|u^{(k+1)}(x)|$ and $|u^{(k+2)}(x)|$ can be proved using the technique from [1] (see [9, p. 51] as well). □

Remark 1. *It is easy to see that the results of Theorem 1 also hold true for a somewhat more general problem*

$$Tu = g_\varepsilon(x), \quad x \in I, \quad u(0) = u(1) = 0,$$

where

$$|g_\varepsilon^{(i)}(x)| \le M[1 + \varepsilon^{-i} y_0(x) + \varepsilon^{-i} y_1(x)], \quad i = 0, 1, \dots, k, \quad x \in I. \tag{9}$$

Theorem 1 suffices in the consistency-error analysis of the discrete operator approximating the continuous operator T, see Section 4. However, in order to analyze the error of the piecewise-linear interpolant of the numerical solution (Section 5), we need some results which involve the solution u_0 of the reduced problem corresponding to (1), $b(x, u_0) \equiv 0$ on I. There exists a unique reduced solution u_0 such that $u_0(x) \in D$, $x \in I$, and $u_0 \in C^k(I)$. Below we need $k = 2$.

Let

$$v_0(x) = e^{-\beta x/\varepsilon}, \quad v_1(x) = e^{\beta(x-1)/\varepsilon}.$$

Also, define the linear operator

$$\tilde{L}v := -\varepsilon^2 v'' + r(x)v,$$

where

$$\begin{gathered} r(x) = \textstyle\int_0^1 b_u(x, \theta(x; s))\, ds \ge \beta^2 > 0, \quad x \in I, \\ \theta(x; s) = u_0(x) + s(u - u_0)(x), \end{gathered} \tag{10}$$

so that

$$\tilde{L}(u - u_0) = Tu - Tu_0 = \varepsilon^2 u_0''. \tag{11}$$

Using the fact that $\tilde{L}$ is inverse monotone, we can prove the following lemma.

Lemma 1. *The solution u of the problem (1) with $k = 2$ and the corresponding reduced solution u_0 satisfy*

$$|u(x) - u_0(x)| \le M[\varepsilon^2 + v_0(x) + v_1(x)], \quad x \in I.$$

Proof. Use the barrier function

$$\psi(x) = M_* \varepsilon^2 + M_{*0} v_0(x) + M_{*1} v_1(x)$$

and choose the constants M_*, M_{*0}, and M_{*1} so that

$$\psi(x) \ge |(u - u_0)(x)|, \quad x = 0, 1, \quad \tilde{L}\psi(x) \ge |\tilde{L}(u - u_0)(x)|, \quad x \in I,$$

keeping (10) and (11) in mind. □

We next use the technique from the proof of Theorem 1 to improve the result of Lemma 1.

Theorem 2. *The solution u of the problem (1) with $k = 2$ and the corresponding reduced solution u_0 satisfy*

$$|u(x) - u_0(x)| \le M[\varepsilon^2 + y_0(x) + y_1(x)], \quad x \in I.$$

Proof. It follows from Lemma 1 that

$$|u(x) - u_0(x)| \le M\varepsilon^2, \quad x \in [m_0, 1 - m_0],$$

for any $m_0 < \frac{1}{2}$. Because of this, we just need to prove that

$$|u(x) - u_0(x)| \le M[\varepsilon^2 + y_0(x)], \quad x \in [0, m_0], \tag{12}$$

the proof of

$$|u(x) - u_0(x)| \le M[\varepsilon^2 + y_1(x)], \quad x \in [1 - m_0, 1],$$

being analogous.

Let

$$\tilde{\psi}(x) = M^* \varepsilon^2 + M_0 y_0(x),$$

where M^* and M_0 are chosen so that

$$\tilde{\psi}(0) \ge M_0 \ge |u_0(0)| = |(u - u_0)(0)| \quad \text{and} \quad \tilde{\psi}(m_0) \ge M^* \varepsilon^2 \ge |(u - u_0)(m_0)|.$$

It also holds true that

$$\tilde{L}\tilde{\psi}(x) \ge M^* \varepsilon^2 \beta^2 + M_0[r(x) - \beta_0^2] y_0(x).$$

Analogously to (7), we now get that

$$r(x) - \beta_0^2 \ge 0, \quad x \in [0, m_0],$$

for some m_0. Therefore, M^* can be chosen so that

$$\tilde{L}\tilde{\psi}(x) \ge M^* \varepsilon^2 \beta^2 \ge |\tilde{L}(u - u_0)(x)|, \quad x \in [0, m_0],$$

which gives (12). □

3. The linear case

In this section we consider the linear case of the problem (1) with the aim of improving the results from the preceding section.

The linear problem is

$$\begin{gathered} \Lambda u := -\varepsilon^2 u'' + \tilde{b}^2(x) u = f(x), \quad x \in I, \quad u(0) = u(1) = 0, \\ \tilde{b}(x) > 0, \quad x \in I, \end{gathered} \tag{13}$$

where $\tilde{b}, f \in C^k(I)$, $k \ge 0$. We have

$$\beta = \min_{x \in I} \tilde{b}(x) \quad \text{and} \quad \beta_i = (1 - \delta)\tilde{b}(i), \quad i = 0, 1. \tag{14}$$

In citeVT, the solution u to the problem (13) is decomposed as follows:

$$\begin{gathered} u(x) = p w_0(x) + q w_1(x) + z(x), \\ w_0(x) = e^{-\tilde{b}(0)x/\varepsilon}, \quad w_1(x) = e^{\tilde{b}(1)(x-1)/\varepsilon}, \end{gathered} \tag{15}$$

with constants p and q satisfying $|p|, |q| \le M$ and

$$|z^{(i)}(x)| \le M[1 + \varepsilon^{1-i} v_0(x) + \varepsilon^{1-i} v_1(x)], \quad i = 0, 1, \dots, k, \quad x \in I, \tag{16}$$

with β as in (14).

Because of the functions v_0 and v_1 in (16), the estimates of the derivatives of u which follow from (15) and (16) are not generally sharper than those in Theorem 1. However, using the technique of Theorem 1 the estimates in (16) can be improved to give a stronger result than the derivative estimates in Theorem 1. The following holds true.

Theorem 3. *The solution u of the linear problem (13) can be decomposed like in (15), where the function z satisfies*

$$|z^{(i)}(x)| \le M[1 + \varepsilon^{1-i} y_0(x) + \varepsilon^{1-i} y_1(x)], \quad i = 0, 1, \dots, k, \quad x \in I,$$

and

$$|z^{(i)}(x)| \le M[\varepsilon^{k-i} + \varepsilon^{1-i} y_0(x) + \varepsilon^{1-i} y_1(x)], \quad i = k+1, k+2, \quad x \in I,$$

with β_0 and β_1 defined like in (14).

Proof. By construction (see [16, 19]), $z'(0) = z'(1) = 0$, which gives that $|p|, |q| \leq M$, and thus $|z(x)| \leq M$, $x \in I$. Moreover, z' satisfies $\Lambda z' = g_\varepsilon(x)$, $x \in I$, with some function g_ε for which (9) holds true. Because of Remark 1, the derivatives of z' can be estimated in the same way as the derivatives of u in Theorem 1. □

We now prove a stronger, linear version of Theorem 2. This result is needed to analyze the error of the piecewise-linear interpolant of the numerical solution approximating the solution u of (13). We consider the following asymptotic expansion:

$$u_A(x) := u_0(x) - u_0(0)w_0(x) - u_0(1)w_1(x), \tag{17}$$

where $u_0(x) = f(x)/\tilde{b}^2(x)$ is the solution of the reduced problem corresponding to (13). Like in the semilinear case, it is needed that $u_0 \in C^2(I)$.

Theorem 4. *For the solution u of the linear problem (13) with $k = 2$ we have*

$$|u(x) - u_A(x)| \leq M[\varepsilon^2 + \varepsilon y_0(x) + \varepsilon y_1(x)], \quad x \in I,$$

with β_0 and β_1 defined like in (14).

Proof. Let $\bar{u} = u - u_A$. It is proved in [19] that

$$|\bar{u}(x)| \leq M\left[\varepsilon^2 + \varepsilon e^{-\beta' x/\varepsilon} + \varepsilon e^{\beta'(1-x)/\varepsilon}\right], \quad x \in I,$$

where $\beta' \in (0, \beta)$, with β as in (14). This implies that $|\bar{u}(x)| \leq M\varepsilon^2$, for $x \in [0, \bar{m}]$ and any $\bar{m} < \frac{1}{2}$. Therefore, like in the proof of Theorem 2, we only need to prove that

$$|\bar{u}(x)| \leq M[\varepsilon^2 + \varepsilon y_0(x)], \quad x \in [0, \bar{m}]. \tag{18}$$

We have

$$\begin{aligned} |\Lambda \bar{u}(x)| &\leq M\left[\varepsilon^2 + [\tilde{b}^2(x) - \tilde{b}^2(0)]w_0(x)\right] \\ &\leq M\left[\varepsilon^2 + x e^{-\delta \tilde{b}(0)x/\varepsilon} y_0(x)\right] \\ &\leq M[\varepsilon^2 + \varepsilon y_0(x)], \quad x \in [0, \bar{m}]. \end{aligned} \tag{19}$$

On the other hand, the function

$$\varphi(x) = \widetilde{M}\varepsilon^2 + \widetilde{M_0}\varepsilon y_0(x)$$

satisfies

$$\begin{aligned} \Lambda\varphi(x) &\geq \widetilde{M}\varepsilon^2\beta^2 + \widetilde{M_0}\varepsilon[\tilde{b}^2(x) - \beta_0^2]y_0(x) \\ &\geq \widetilde{M}\varepsilon^2\beta^2 + \widetilde{M_0}\bar{m}_*\varepsilon y_0(x), \quad x \in [0, \bar{m}], \end{aligned}$$

where $\bar{m}$ and $\bar{m}_*$ are selected analogously to the procedure in (7). In view of (19), this means that $\widetilde{M}$ and $\widetilde{M_0}$ can be chosen so that

$$\Lambda\varphi(x) \geq |\Lambda\bar{u}(x)|, \quad x \in [0, \bar{m}],$$

and, moreover, so that

$$\varphi(x) \geq |\bar{u}(x)|, \quad x = 0, \bar{m}.$$

Then (18) follows from inverse monotonicity. □

As mentioned in the introduction, the modified Shishkin mesh for the linear problem (13) is introduced in [19]. However, the z component of the decomposition (15) is estimated there only using (16) and not the improved estimates of Theorem 3. Consequently, the modified Shishkin mesh we present in Section 4 can be denser in the layers than the mesh proposed in [19]. The decomposition (15), cf. [16], is the reaction-diffusion counterpart of the Kellogg-Tsan decomposition of the linear convection-diffusion problem, [4]. The Kellogg-Tsan technique cannot be used for nonlinear problems and this is why we have to take a different approach here, that of Theorem 1. Therefore, our proof of (3) for (1) is not a straightforward generalization of the of the proof of the derivative estimates in [19]. Moreover, when the result for the semilinear problem is applied to the linear case in combination with the Kellogg-Tsan approach, we get Theorem 3, which improves the result of [19].

4. The discretization

We now consider a finite-difference discretization of the problem (1). Let $\bar{\omega}$ be a general mesh with points x_i, $i = 0, 1, \ldots, N$, such that $0 = x_0 < x_1 < \cdots < x_N = 1$. Let also $\omega = \bar{\omega} \backslash \{0, 1\}$ and $h_i = x_i - x_{i-1}$, $i = 1, 2, \ldots, N$. Mesh functions on ω are denoted by U^N, W^N, etc. If g is a function defined on $(0, 1)$, we write g_i instead of $g(x_i)$ and g^N for the corresponding mesh function. The maximum norm used for the mesh function $W^N = (W_1^N, \ldots, W_{N-1}^N)$ is given by

$$\|W^N\|_\infty = \max_{1 \le i \le N-1} |W_i^N|.$$

The constants M are also assumed to be independent of N.

The discretization of the problem (1) on ω is

$$T^N U_i^N := -\varepsilon^2 D'' U_i^N + b(x_i, U_i^N) = 0 \quad \text{for} \quad i = 1, 2, \ldots, N-1, \tag{20}$$

where $U_0^N := 0$, $U_N^N := 0$, and D'' is the standard central approximation of u_i'' :

$$D'' U_i^N = \frac{1}{\hbar_i} \left(\frac{U_{i+1}^N - U_i^N}{h_{i+1}} - \frac{U_i^N - U_{i-1}^N}{h_i} \right), \quad \hbar_i = \frac{h_i + h_{i+1}}{2}.$$

The discrete problem (20) has a unique solution U^N such that $U_i^N \in D$, $i = 1, 2, \ldots, N-1$, cf. e.g. [17].

We consider next two arbitrary mesh functions V^N and W^N with components in D. It holds true that

$$T^N V_i^N - T^N W_i^N = \mathcal{L}^N (V_i^N - W_i^N),$$

where $\mathcal{L}^N$ is a linear operator,

$$\mathcal{L}^N V_i^N := -\varepsilon^2 D'' V_i^N + \rho_i V_i^N$$

and

$$\rho_i = \int_0^1 b_u(x_i, W_i^N - t(V_i^N - W_i^N))\, dt.$$

The discrete Green's function $G(x_i, \xi_j)$, associated with the operator $\mathcal{L}^N$ as a function of $x_i \in \omega$ for a fixed $\xi_j \in \omega$, is defined by

$$\mathcal{L}^N G(x_i, \xi_j) = \delta(x_i, \xi_j)/\hbar_j, \quad i = 1, 2, \ldots, N-1,$$
$$G(0, \xi_j) = G(1, \xi_j) = 0,$$

where $\delta(x_i, \xi_j)$ is the Kronecker delta. Then we have

$$V_i^N - W_i^N = \sum_{j=1}^{N-1} \hbar_j G(x_i, \xi_j)(T^N V_i^N - T^N W_i^N), \quad i = 1, 2, \ldots, N-1. \tag{21}$$

Lemma 2. [14, Lemma 1] *For the discrete Green's function G on the mesh ω, we have*

$$\sum_{j=1}^{N-1} \hbar_j |G(x_i, \xi_j)| \leq \frac{1}{\beta^2}.$$

Lemma 3. [14, Lemma 2] *For the discrete Green's function G on the mesh ω, we have*

$$|G(x_i, \xi_j)| \leq \frac{M}{\varepsilon + \hbar_j}.$$

A minor modification of the Shishkin mesh is described next, cf. [18] for instance. Let $J = QN$ be a positive integer such that $Q < \frac{1}{2}$ and $Q^{-1} \leq M$. Let also $L = L(N)$ satisfy $\ln(\ln N) \leq L \leq \ln N$ and

$$e^{-L} \leq \frac{L}{N}.$$

The smallest L satisfying the above inequality is L^*, which solves the equation $e^{-L} = L/N$. This equation can be solved using Newton's method. It holds true that $L^* < \ln N$, but when $N \to \infty$, L^* behaves like $\ln N$, [18]. Let also

$$\sigma = \min\left\{Q, \frac{a\varepsilon L}{\beta}\right\}, \quad a \geq 2. \tag{22}$$

The mesh, denoted by $S(L)$, is constructed by forming a fine equidistant mesh with J mesh steps in each of the intervals $[0, \sigma]$ and $[1-\sigma, 1]$, and a coarse equidistant mesh with $N-2J$ mesh steps in $[\sigma, 1-\sigma]$. Therefore, the transition points between the fine parts of the mesh and the coarse one are σ and $1-\sigma$. The standard Shishkin mesh for the reaction-diffusion problem (1) typically has $L = \ln N$ and $Q = \frac{1}{4}$. The use of the general L and Q is a minor modification and we still refer to the mesh $S(L)$ as the standard Shishkin mesh. The modified Shishkin mesh, which we are about to introduce, has different transition points.

As usual, we assume that $\sigma = a\varepsilon L/\beta$ because $\sigma = Q$ is unrealistic in practice, N growing exponentially when $\varepsilon \to 0$. This choice of σ is based on the derivative estimates (2). Since the improved estimates (3) are now available, we can use modified transition points σ_0 and $1-\sigma_1$ instead of the standard ones. The quantities σ_0 and σ_1 are defined as follows:

$$\sigma_i = \min\left\{Q, \frac{a\varepsilon L}{\beta_i}\right\}, \quad i = 0, 1, \quad a \geq 2. \tag{23}$$

Like in the case of the standard Shishkin mesh, we only consider $\sigma_i = a\varepsilon L/\beta_i$, $i = 0, 1$.

Therefore, the mesh points of the modified Shishkin mesh, denoted by $S'(L)$, are

$$x_i = \begin{cases} ih^0, & 0 \leq i \leq J, \\ \sigma_0 + (i-J)H, & J \leq i \leq N-J, \\ 1 - \sigma_1 + (i-N+J)h^1, & N-J \leq i \leq N, \end{cases}$$

with mesh-step sizes

$$h^i = a\varepsilon L/(\beta_i J), \quad i = 0, 1, \quad \text{and} \quad H = (1-\sigma_0-\sigma_1)/(N-2J).$$

We have the following ε-uniform pointwise convergence result. Since no more than $u^{(4)}$ is needed in the consistency-error analysis of the central scheme (20), it is assumed that $b \in C^2(I \times D)$, i.e., $k = 2$.

Theorem 5. *Let u be the solution of (1) with $k = 2$ and let U^N be the solution of (20) on the mesh $S'(L)$. Then*

$$\|u^N - U^N\|_\infty \le MN^{-2}L^2.$$

Proof. We use (21) with $V^N = u^N$ and $W^N = U^N$, together with Lemmas 2 and 3, to prove this result like in [19, Theorem 1]. $\square$

Remark 2. *Theorem 5 gives the same ε-uniform convergence result as the one that can be obtained on the standard Shishkin mesh $S(L)$. However, it is possible that both β_0 and β_1 are significantly greater than β and when this is the case, the modified mesh is denser in the layers and we can expect better numerical results. It should be kept in mind that this is just an expectation and, although our numerical experiments confirm this expectation, we do not claim that the use of the modified Shishkin mesh guarantees better numerical results in all instances when $\beta_i > \beta$, $i = 0, 1$. By the way, it is also possible for β to be greater than both β_0 and β_1, but only negligibly so.*

For the linear problem (13), as a special case of (1), we have the solution decomposition of Theorem 3. This can be used to modify the mesh even further and to improve the result from [19]. The discretization of (13) is analogous to (20),

$$-\varepsilon^2 D'' U_i^N + \tilde{b}_i U_i^N = f_i, \quad i = 1, 2, \dots, N-1. \tag{24}$$

Let us consider this discretization on a different modification of the Shishkin mesh which uses $\tilde{b}(i)$ instead of β_i, $i = 0, 1$, in the formulas for the two transition points. This newly modified mesh is denoted by $\tilde{S}(L)$.

Theorem 6. *Let u be the solution of the linear problem (13) with $k = 2$ and let U^N be the solution of (24) on the mesh $\tilde{S}(L)$. Then*

$$\|u^N - U^N\|_\infty \le MN^{-2}L^2.$$

Proof. A result of this kind is proved in [19] on the modified Shishkin mesh with $\beta_i = \min\{\tilde{b}(i), 2\beta\}$, $i = 0, 1$. This definition of β_i is adopted there because the consistency error is analyzed using the solution decomposition (15) with the original derivative estimates of z, (16). In view of the improved estimates in Theorem 3, β_i can be redefined as

$$\beta_i = \min\{\tilde{b}(i), 2\beta_i\} = \min\{\tilde{b}(i), 2(1-\delta)\tilde{b}(i)\} = \tilde{b}(i),$$

where in the last step we use the fact that δ is positive and arbitrarily close to 0. $\square$

Remark 3. *Since $\tilde{b}(i) \ge \beta$, $i = 0, 1$, the mesh $\tilde{S}(L)$ can be either more or equally dense in the layers than the standard Shishkin mesh $S(L)$. When $\tilde{b}(i) > \beta$, $i = 0, 1$, $\tilde{S}(L)$ is expected to produce more accurate numerical results than $S(L)$, but, as already stated in Remark 2, this cannot be guaranteed. In any case, because of the solution representations in (15), it is certainly more natural to define the mesh transition points in terms of $\tilde{b}(0)$ and $\tilde{b}(1)$, rather than in terms of β. Moreover, $\tilde{b}(0)$ and $\tilde{b}(1)$ are easier to calculate than β. Even in the semilinear case, it may be easier to find β_0 and β_1 than β since the minimization with respect to x is not required.*

5. Piecewise-linear interpolation

In this section we analyze the piecewise-linear interpolant of the numerical solution. For a function g, defined on $\bar{\omega}$, the piecewise-linear interpolant g^I is given by

$$g^I(x) = g_i + \frac{g_{i+1} - g_i}{h_{i+1}}(x - x_i), \quad x \in [x_i, x_{i+1}], \quad i = 0, 1, \dots, N-1.$$

If $g \in C^2(I)$, we have

$$|(g - g^I)(x)| \le M h_{i+1}^2 \|g''\|_{[x_i, x_{i+1}]}, \quad x \in [x_i, x_{i+1}], \tag{25}$$

where

$$\|g\|_X = \max_{x \in X} |g(x)|$$

for any closed interval X, $X \subseteq I$.

Theorem 7. *Let u be the solution of (1) with $k = 2$ and let U^N be the solution of the discrete problem (20) on the mesh $S'(L)$. Then,*

$$\|u - U^{N,I}\|_I \le M N^{-2} L^2$$

where $U^{N,I}$ is the piecewise-linear interpolant of U^N on I.

Proof. The proof is analogous to the one given in [19] for the linear problem. We nevertheless provide some details here in order to show that Theorem 1 is not sufficient for the proof and that we also need Theorem 2.

Let $x \in [x_i, x_{i+1}] \subseteq [0, \frac{1}{2}]$. Using the triangle inequality we get

$$|(u - U^{N,I})(x)| \le |(u - u^I)(x)| + |(u^I - U^{N,I})(x)|.$$

Then, Theorem 5 immediately gives that

$$|(u^I - U^{N,I})(x)| \le \max\{|u_i - U_i^N|, |u_{i+1} - U_{i+1}^N|\} \le M N^{-2} L^2.$$

As for $|(u - u^I)(x)|$, (25) and Theorem 1 imply

$$\|u - u^I\|_{[0,\sigma_0]} \le M (h^0)^2 [1 + \varepsilon^{-2} e^{-\beta_0 x/\varepsilon}] \le M N^{-2} L^2$$

and

$$|u - u^I|_{[\sigma_0, \frac{1}{2}]} \le M H^2 \left[1 + \varepsilon^{-2} e^{-\beta_0 \sigma_0/\varepsilon}\right] \le M N^{-2} \left[1 + L^2 (N\varepsilon)^{-2}\right].$$

Thus, if $\varepsilon N \ge 1$, we have

$$\|u - u^I\|_{[\sigma_0, \frac{1}{2}]} \le M N^{-2} L^2.$$

On the other hand, if $\varepsilon N \le 1$ and $x \in [\sigma_0, \frac{1}{2}]$, we use

$$|(u - u^I)(x)| \le |(u - u_0)(x)| + |(u_0 - u_0^I)(x)| + |(u_0^I - u^I)(x)|.$$

The interpolation-error estimate (25) and Theorem 2 give respectively

$$|(u_0 - u_0^I)(x)| \le M N^{-2}$$

and

$$|(u - u_0)(x)| + |(u_0^I - u^I)(x)| \le M \left(\varepsilon^2 + e^{-\beta_0 \sigma_0/\varepsilon}\right) \le M \left(N^{-2} + N^{-2} L^2\right),$$

which finally proves

$$\|u - u^I\|_{[\sigma_0, \frac{1}{2}]} \le M N^{-2} L^2.$$

The proof is analogous for $x \in [x_i, x_{i+1}] \subseteq [\frac{1}{2}, 1]$. If $x \in [x_i, x_{i+1}]$ and $x_i < \frac{1}{2} \leq x_{i+1}$, the boundary-layer functions w_1 and y_1 are still exponentially small when $\varepsilon \to 0$ and the proof follows the same lines as in the above case $x \in [\sigma_0, \frac{1}{2}]$. □

In the linear case, the result of Theorem 7 is valid on the mesh $\tilde{S}(L)$. The proof is analogous to the one above, but, due to the mesh $\tilde{S}(L)$, u_A has to replace u_0 in the relevant part of the proof. This is why Theorem 4 is needed here. Because of the use of the mesh $\tilde{S}(L)$, the result of the theorem below is an improvement over the corresponding result in [19].

Theorem 8. *Let u be the solution of the linear problem (13) with $k = 2$ and let U^N be the solution of the discrete problem (24) on the mesh $\tilde{S}(L)$. Then,*

$$\|u - U^{N,I}\|_I \leq MN^{-2}L^2$$

where $U^{N,I}$ is the piecewise-linear interpolant of U^N on I.

6. Numerical results

In this section, we present numerical results for two test problems in order to illustrate our theoretical findings. The first test problem is nonlinear and of the type discussed in [12, p. 194]:

$$(26) \qquad -\varepsilon^2 u'' + 1 + (1 - x + x^2)u - u^2 = 0, \quad x \in I, \quad u(0) = u(1) = 0.$$

The exact solution is not known and for upper and lower solutions we can take $u^* = 0$ and $u_* = -1$, respectively. The function b in (26) satisfies $b_u(x, u) = 1 - x + x^2 - 2u$ and it holds true that

$$\min_{x \in I,\ y \in [-1,0]} b_u(x, y) = b_u(\tfrac{1}{2}, 0) = \tfrac{3}{4}$$

and

$$\min_{y \in [-1,0]} b_u(t, y) = b_u(t, 0) = 1, \quad t = 0, 1.$$

Based on this, we set $\beta = 0.866$ and $\beta_0 = \beta_1 = 0.999$.

In Table 1 we present the results for the modified mesh with $Q = \frac{3}{8}$ and $L = L^*$. The errors $E^I(N)$ are shown, where

$$E^I(N) = \max_{1 \leq i \leq N-1} |U_i^N - U_i^{65536,I}|$$

and $U^{65536,I}$ is the piecewise-linear interpolant of the numerical solution obtained on the mesh with $N = 2^{16}$ steps. Numerical approximations of the ε-uniform order of convergence, which are determined using the double mesh method like in [3], are also presented. We calculate the double mesh error

$$e(N) = \max_{1 \leq i \leq N-1} |U_i^N - U_i^{2N,I}|,$$

where $U_i^N - U_i^{2N,I}$ is the difference between the value of the solution at x_i on a mesh with N steps and the interpolated value of the solution, at the same point x_i, on a mesh with $2N$ intervals. For each value of N the quantities

$$(27) \qquad ord(N) = \frac{\ln e(N) - \ln e(2N)}{\ln 2} \quad \text{and} \quad \widetilde{ord}(N) = \frac{\ln e(N) - \ln e(2N)}{\ln 2L(N) - \ln L(2N)}$$

are computed. The values of $ord(N)$ and $\widetilde{ord}(N)$ are taken as numerical approximations of the rates of convergence as powers of N^{-1} and as powers of $N^{-1}L(N)$, respectively.

TABLE 1. $E^I(N)$, $ord(N)$, $\widetilde{ord}(N)$ on $S'(L^*)$ with $Q = 0.375$, $a = 2$.

	$\log_2 N$						
$-2\log_2\varepsilon$	6	7	8	9	10	11	12
16	7.906(−4)	2.719(−4)	9.072(−5)	2.930(−5)	9.218(−6)	2.843(−6)	8.572(−6)
	1.455	1.329	1.532	1.648	1.664	1.687	1.742
	1.895	1.675	1.880	1.979	1.964	1.962	2.000
20	7.895(−4)	2.715(−4)	9.058(−5)	2.925(−5)	9.204(−6)	2.839(−6)	8.558(−6)
	1.454	1.329	1.532	1.648	1.664	1.688	1.742
	1.895	1.675	1.880	1.980	1.964	1.962	2.000
24	7.892(−4)	2.714(−4)	9.055(−5)	2.925(−5)	9.201(−6)	2.838(−6)	8.555(−6)
	1.454	1.330	1.532	1.648	1.664	1.688	1.742
	1.895	1.675	1.880	1.980	1.964	1.962	2.000
$28-36$	7.891(−4)	2.714(−4)	9.054(−5)	2.924(−5)	9.200(−6)	2.837(−6)	8.554(−6)
	1.454	1.330	1.532	1.648	1.664	1.688	1.742
	1.895	1.675	1.880	1.980	1.964	1.962	2.000

TABLE 2. $r(N)$ for $\varepsilon^2 = 2^{-k}$, $k = 16, 20, \ldots, 36$, $a = 2$.

		$\log_2 N$						
Q	L	6	7	8	9	10	11	12
0.375	L^*	1.329	1.331	1.330	1.331	1.331	1.326	1.332
0.375	$\ln N$	1.315	1.334	1.332	1.331	1.331	1.331	1.331
0.250	$\ln N$	1.277	1.324	1.336	1.328	1.330	1.331	1.331

Table 2 presents the ratio of the errors $E^I(N)$ on the standard and modified Shishkin meshes,

$$r(N) = \frac{E^I_S(N)}{E^I_{S'}(N)},$$

for different values of the mesh parameters. It turns out that, for any choice of N and the mesh parameters, $r(N)$ remains the same for all values of ε considered in Table 1. The results clearly show that the errors $E^I(N)$ are greater on the standard mesh. Therefore, the modified Shishkin mesh performs better than the standard one regardless of how the parameters Q and L are chosen (the density of the mesh in the layer is greater if Q is greater and L is less). However, as expected, there is no essential difference in the orders of convergence on the standard and modified meshes, which is why we do not provide the rates (27) for the standard mesh.

In order to illustrate the linear interpolation results, we use the test problem

$$-\varepsilon^2 y'' + (1 - x + x^2)y = f(x, \varepsilon), \quad x \in (0, 1), \quad y(0) = 0,\ y(1) = 0, \tag{28}$$

where the right-hand side $f(x, \varepsilon)$ is determined by the following exact solution y:

$$y(x) = \cos(2\pi x) - \frac{e^{-x/\varepsilon} + e^{(x-1)/\varepsilon} + e^{-1/\varepsilon}}{1 + 2e^{-1/\varepsilon}}.$$

This problem is also considered in [19]. Here $\beta = 0.865$ and $\beta_0 = \beta_1 = 1$. In Table 3, we present

$$E^r(N) = \max_{x_i \in I_r} |Y_i^{N,I} - y_i|,$$

where $Y^{N,I}$ is the piecewise-linear interpolant of the numerical solution and I_r is a set of 80 points chosen as follows: 20 random points from each interval $[0, \mu]$ and $[1 - \mu, 1]$, and 40 random points from $[\mu, 1 - \mu]$, where $\mu := \varepsilon \ln |2|$ represents the width of the layer as determined in [2, p. 78]. Table 4 contains the ratio of the errors $E^r(N)$ on the standard and modified Shishkin meshes,

$$R^r(N) = \frac{E^r_S(N)}{E^r_{\tilde{S}}(N)},$$

for each pair of ε and N. The results show that the interpolation errors $E^r(N)$ are also greater on the standard mesh.

In this paper, we theoretically analyze the simple central finite-difference scheme. It is also of interest to compare how higher-order schemes perform on the standard and modified Shishkin meshes. To this end, we present numerical results for the test problem (28) solved by the quadratic finite-element method [20]. The abstract variational problem for the general linear problem (13) is: Find $y \in H^1_0(0,1)$ such that

$$B_\varepsilon(y,w) = (f,w), \qquad \forall w \in H^1_0(0,1),$$

where

$$B_\varepsilon(y,w) = \varepsilon^2(y,w) + (\tilde{b}^2 y, w), \qquad \left((y,w) = \int_0^1 y(x)w(x)dx\right).$$

The finite-element problem is to find $y^N \in V_N^{\varepsilon,p} \subset H^1_0(0,1)$ such that

$$B_\varepsilon(y^N,w) = (f,w), \qquad \forall w \in V_N^{\varepsilon,p},$$

where $V_N^{\varepsilon,p}$ is the space of the standard C^0 piecewise-continuous polynomials of degree p. For the quadratic finite-element method, $p = 2$ and the transition-point parameter $a = p + 1$.

Table 5 contains the following ratio of the maximum pointwise errors:

$$R(N) = \frac{E_S(N)}{E_{\tilde{S}}(N)}, \quad \text{where} \quad E_{\cdot}(N) = \max_{1\le i\le N-1} |Y_i^N - y_i|.$$

The theoretical analysis of the quadratic finite-element method in [20] is based on a generalized Shishkin mesh introduced in the same paper. We also present

TABLE 3. $E^r(N)$ on $\tilde{S}(L^*)$ with $Q = 0.375$, $a = 2$.

	$\log_2 N$						
$-2\log_2\varepsilon$	6	7	8	9	10	11	12
16	7.388(−3)	2.394(−3)	8.926(−4)	2.571(−4)	8.566(−5)	2.580(−5)	7.187(−6)
20	6.185(−3)	2.455(−3)	8.370(−4)	2.432(−4)	7.718(−5)	2.762(−5)	8.503(−6)
24	7.342(−3)	2.370(−3)	8.910(−4)	2.920(−4)	8.965(−5)	2.788(−5)	8.564(−6)
28	7.353(−3)	2.500(−3)	8.450(−4)	2.476(−4)	7.894(−5)	2.568(−5)	7.734(−6)
32	7.293(−3)	2.628(−3)	8.847(−4)	2.890(−4)	8.940(−5)	2.743(−5)	8.658(−6)
36	7.361(−3)	2.622(−3)	8.927(−4)	2.725(−4)	8.612(−5)	2.497(−5)	8.556(−6)

TABLE 4. $R^r(N)$ for $Q = 0.375$, $L = L^*$, $a = 2$.

	$\log_2 N$						
$-2\log_2\varepsilon$	6	7	8	9	10	11	12
16	1.305	1.305	1.347	1.421	1.433	1.344	1.472
20	1.219	1.268	1.417	1.382	1.313	1.179	1.288
24	1.320	1.312	1.341	1.336	1.348	1.370	1.228
28	1.317	1.245	1.381	1.465	1.426	1.364	1.396
32	1.321	1.325	1.380	1.274	1.334	1.342	1.269
36	1.317	1.328	1.331	1.350	1.404	1.458	1.174

TABLE 5. $R(N)$ for $Q = 0.25$, $L = \ln N$, $a = 3$.

	$\log_2 N$						
$-2\log_2 \varepsilon$	6	7	8	9	10	11	12
16	1.847	1.662	1.709	1.741	1.760	1.772	1.778
20	1.848	1.662	1.709	1.741	1.760	1.772	1.778
24	1.848	1.662	1.709	1.741	1.760	1.772	1.777
28	1.848	1.662	1.709	1.741	1.760	1.771	1.783
32	1.848	1.662	1.709	1.741	1.760	1.774	1.778
36	1.848	1.662	1.709	1.741	1.760	1.772	1.739

TABLE 6. $\bar{R}(N)$ for $p = 2$.

	$\log_2 N$						
$-2\log_2 \varepsilon$	6	7	8	9	10	11	12
16	1.665	1.678	1.717	1.745	1.762	1.773	1.779
20	1.631	1.678	1.717	1.745	1.762	1.773	1.778
24	1.204	1.678	1.717	1.745	1.762	1.773	1.779
28	1.115	1.678	1.717	1.745	1.762	1.772	1.769
32	1.093	1.678	1.717	1.745	1.762	1.773	1.779
36	1.088	1.678	1.717	1.745	1.762	1.766	1.779

numerical results for this kind of mesh. Table 6 shows the values of the ratio

$$\bar{R}(N) = \frac{E_{\bar{S}}(N)}{E_{\bar{S}'}(N)},$$

where $\bar{S}$ denotes the generalized Shishkin mesh from [20], with

$$\sigma = \min\left\{\frac{1}{4}, \frac{\varepsilon}{\beta}(p + 1.5)\ln(\frac{N}{4} + 1)\right\},$$

and where $\bar{S}'$ is a modification of $\bar{S}$ in the sense of (23).

We can see in Tables 2, 4, 5, and 6 that the maximum pointwise errors are less on the Shishkin mesh with modified transition points than on those with the standard transition points, all other mesh parameters being equal.

Finally, in Table 7 we present the error ratio in which the numerical errors are measured in the energy norm that naturally accompanies finite-element methods for the problem (13). The energy norm is defined by

$$\|w\|_\varepsilon^2 = \varepsilon^2(w', w') + \beta^2(w, w).$$

We calculate the integrals in this norm and all integrals in the finite element method using the three-point Gauss-Legendre quadrature formula. The ratio in Table 7 is

$$\bar{r}_\varepsilon(N) = \frac{e^\varepsilon_{\bar{S}}(N)}{e^\varepsilon_{\bar{S}'}(N)}, \quad \text{where} \quad e^\varepsilon_\cdot(N) = \|y^N - y\|_\varepsilon.$$

As opposed to the maximum norm, the energy norm shows cases when the modified and standard Shishkin meshes behave equally. In fact, the values of the ratio $\bar{r}_\varepsilon(N)$ decrease when $\varepsilon \to 0$ and N is kept fixed. In any case, the modified mesh performs at least as equally well as the standard one.

TABLE 7. $\bar{r}_\varepsilon(N)$ for $p = 2$.

	$\log_2 N$						
$-2\log_2 \varepsilon$	6	7	8	9	10	11	12
16	1.285	1.327	1.334	1.336	1.336	1.336	1.336
20	1.100	1.268	1.325	1.335	1.336	1.336	1.336
24	1.009	1.069	1.229	1.316	1.333	1.336	1.336
28	1.001	1.005	1.040	1.167	1.292	1.330	1.335
32	1.000	1.000	1.003	1.020	1.099	1.099	1.317
36	1.000	1.000	1.000	1.001	1.009	1.049	1.173

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