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THE h-p VERSION OF THE CONTINUOUS PETROV-GALERKIN METHOD FOR NONLINEAR VOLTERRA FUNCTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH VANISHING DELAYS

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In memory of Professor Ben-yu Guo

Abstract. We investigate an *h-p* version of the continuous Petrov-Galerkin method for the nonlinear Volterra functional integro-differential equations with vanishing delays. We derive *h-p* version a priori error estimates in the L^2-, H^1- and L^∞-norms, which are completely explicit in the local discretization and regularity parameters. Numerical computations supporting the theoretical results are also presented.

Key words. *h-p* version, continuous Petrov-Galerkin method, nonlinear Volterra functional integro-differential equations, vanishing delays.

1. Introduction

We study the numerical solutions for the nonlinear Volterra functional integro-differential equation (VFIDE) with vanishing delays:

$$\begin{cases} u'(t) = f(t, u(t), u(\theta(t))) + (\mathcal{V}u)(t) + (\mathcal{V}_\theta u)(t), & t \in I := [0, T], \\ u(0) = u_0, \end{cases} \tag{1}$$

corresponding to the Volterra integral operators

$$(\mathcal{V}u)(t) := \int_0^t K_1(t,s)G_1(s,u(s))ds, \qquad (\mathcal{V}_\theta u)(t) := \int_0^{\theta(t)} K_2(t,s)G_2(s,u(s))ds,$$

where the delay function θ is subject to the following conditions:
(C1) $\theta(0) = 0$ and $\theta(t) < t$ for $t > 0$,
(C2) $\theta'(t) \geq q_0 > 0$ for all $t \in I$.
We assume that f and G_i with $i = 1, 2$ are given functions. Moreover, the kernels $K_1(t,s)$ and $K_2(t,s)$ are continuous on $D := \{(t,s) : 0 \leq s \leq t,\ t \in I\}$ and $D_\theta := \{(t,s) : 0 \leq s \leq \theta(t),\ t \in I\}$, respectively.

During the past few decades, many numerical methods have been proposed and analyzed for the VFIDEs. Among those a large number of methods are based on the h-version approach, which means that the convergence is achieved by decreasing the size of time steps at a fixed and typically low approximation order. For an overview of the lower-order methods developed for the VFIDEs, the reader can refer to monographs [3, 5] and the references therein. In contrast, the higher-order methods, for example, the p- and h-p version methods employ (varying) high order approximation polynomials. Particulary, the h-p version method allows for locally varying time steps and approximation orders, which can significantly enhance the numerical accuracy. The h-p version continuous and discontinuous

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Galerkin methods were introduced for initial-value problems in [9, 17, 19], for delay differential equations in [6], for parabolic problems in [10], and for Volterra integro-differential equations in [4, 8, 18, 20]. Moreover, some other high-order methods, such as the spectral methods were developed for various Volterra integro-differential equations with delays; see, e.g., [1, 13, 14, 15, 16, 21]. However, to the best of our knowledge, there is no work that considers the h-p version Galerkin method for nonlinear VFIDEs.

The purpose of the current work is to present and analyze an h-p version of the continuous Petrov-Galerkin (CPG) discretization scheme for the numerical approximation of the VFIDE (1) with vanishing delays. The Petrov-Galerkin method allows the trial and test spaces to be different, and it has become powerful tools for solving many kinds of differential equations (see e.g., [7, 12]). The CPG method presented in this paper is a hybrid of the continuous and discontinuous Galerkin methods with respect to time. More precisely, one uses continuous and piecewise polynomials for the trial spaces, but uses discontinuous and piecewise polynomials for the test spaces. With such choice of the trial and test spaces, we show that the CPG scheme defines a unique approximate solution, provided that a certain condition on the time steps is satisfied (which is completely independent of the approximation orders). We also describe in detail our implementation for the CPG scheme according to certain relationship between the delay function $\theta(t)$ and nodal points of the time partition. Moreover, we derive h-p version a priori error estimates that are completely explicit with respect to the local time steps, the local approximation orders, and the local regularity properties of the exact solution.

The remainder of this paper is organized as follows. In Section 2, we introduce the h-p version of the CPG method for the VFIDE (1) and prove existence and uniqueness of approximate solutions. We also give a detailed description of the computational form of the CPG scheme. In Section 3, we carry out a complete h-p version error analysis of the CPG method. In Section 4, we present some numerical experiments to verify the theoretical results. We end the paper with a summary and discussion in Section 5.

2. The h-p version of continuous Petrov-Galerkin method

In this section, we first introduce the h-p version of the CPG method for the VFIDE (1). We then show the existence and uniqueness of the approximate solutions. Finally, we discuss the numerical implementation of the CPG scheme.

2.1. Continuous Petrov-Galerkin discretization.

Let $\mathcal{T}_h$ be a partition of the time interval I given by the points

$$0 = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = T.$$

We set $I_n = (t_{n-1}, t_n)$ and $k_n = t_n - t_{n-1}$ for $1 \le n \le N$. Let $k = \max_{1\le n\le N}\{k_n\}$. Moreover, we assign to each time interval I_n an approximation order $r_n \ge 1$ and introduce the degree vector $\mathbf{r} = \{r_n\}_{n=1}^{N}$. Then, the tuple $(\mathcal{T}_h, \mathbf{r})$ is called an h-p discretization of I. Next, we introduce the h-p version trial and test spaces

$$S^{\mathbf{r},1}(\mathcal{T}_h) = \{u \in H^1(I) : u|_{I_n} \in P_{r_n}(I_n), 1 \le n \le N\}$$

and

$$S^{\mathbf{r}-1,0}(\mathcal{T}_h) = \{u \in L^2(I) : u|_{I_n} \in P_{r_n-1}(I_n), 1 \le n \le N\},$$

respectively, where $P_{r_n}(I_n)$ denotes the space of polynomials of degree at most r_n on I_n.

The h-p version CPG approximation of the VFIDE (1) is now defined as follows: find $U \in S^{\mathbf{r},1}(\mathcal{T}_h)$ such that $U(0) = u_0$ and

$$\begin{aligned} & \sum_{n=1}^{N} \int_{I_n} U'(t)\varphi(t)dt \\ = & \sum_{n=1}^{N} \int_{I_n} \Big(f(t, U(t), U(\theta(t))) + (\mathcal{V}U)(t) + (\mathcal{V}_\theta U)(t) \Big) \varphi(t)dt \end{aligned} \tag{2}$$

for all $\varphi \in S^{\mathbf{r}-1,0}(\mathcal{T}_h)$.

Remark 2.1. Due to the discontinuous character of the test space $S^{\mathbf{r}-1,0}(\mathcal{T}_h)$, the CPG method in (2) can be regarded as a time stepping scheme: if U is given on the time intervals $I_m, 1 \le m \le n-1$, we find $U|_{I_n} \in P_{r_n}(I_n)$ on I_n by solving

$$\begin{aligned} \int_{I_n} U'(t)\varphi(t)dt &= \int_{I_n} \Big(f(t, U(t), U(\theta(t))) + (\mathcal{V}U)(t) + (\mathcal{V}_\theta U)(t) \Big) \varphi(t)dt, \\ U|_{I_n}(t_{n-1}) &= U|_{I_{n-1}}(t_{n-1}) \end{aligned} \tag{3}$$

for all $\varphi \in P_{r_n-1}(I_n)$. Here, $U|_{I_1}(0) = u_0$.

2.2. Existence and uniqueness of discrete solutions. We start by showing the following well-known Poincaré-Friedrichs inequality (see, e.g., [2]).

Lemma 2.1. *Let $u \in H^1(J), J = (a, b) \subset R$. Assume that $u(a) = 0$. Then, there holds*

$$\|u\|_{L^2(J)} \le h\|u'\|_{L^2(J)},$$

where $h = b - a$.

We next address the well-posedness of the discrete solutions. For our purpose, let

$$\bar{K}_1 := \max_{(t,s)\in D} |K_1(t, s)|, \qquad \bar{K}_2 := \max_{(t,s)\in D_q} |K_2(t, s)|. \tag{4}$$

Further, we assume that $f(t, u, v), G_1(t, u)$ and $G_2(t, u)$ fulfill the following Lipschitz conditions:

$$|f(t, u_1, v) - f(t, u_2, v)| \le L_1|u_1 - u_2|, \tag{5}$$

$$|f(t, u, v_1) - f(t, u, v_2)| \le L_2|v_1 - v_2|, \tag{6}$$

$$|G_1(t, u_1) - G_1(t, u_2)| \le L_3|u_1 - u_2|, \tag{7}$$

and

$$|G_2(t, u_1) - G_2(t, u_2)| \le L_4|u_1 - u_2| \tag{8}$$

for all $t \in I$, $|u| < \infty$, $|v_i| < \infty$ and $|u_i| < \infty$ $(i = 1, 2)$, where L_1, L_2, L_3 and L_4 are positive constants independent of t, u and v.

Theorem 2.1. *Assume that the partition $\mathcal{T}_h$ satisfies*

$$\lambda_n := \left(L_1 + \frac{L_2}{\sqrt{q_0}} + \frac{\bar{K}_1 L_3}{\sqrt{2}} k_n + \frac{\bar{K}_2 L_4}{\sqrt{2}} k_n \right) k_n < 1, \quad 1 \le n \le N. \tag{9}$$

Then the discrete problem (2) has a unique solution $U \in S^{\mathbf{r},1}(\mathcal{T}_h)$.

Proof. Owing to Remark 2.1, it suffices to prove that problem (3) admits a unique solution $U|_{I_n} \in P_{r_n}(I_n), 1 \le n \le N$. Since the CPG solution is constructed step by step, it is enough to show the existence and uniqueness on the first time interval I_1, namely, we only have to consider $n = 1$ in (3) (for $n \ge 2$ the proof is completely analogous).

To this end, we shall show that on I_1 there is a unique solution $U \in P_{r_1}(I_1)$ satisfying

$$\begin{aligned} \int_{I_1} U'(t)\varphi(t)dt &= \int_{I_1} \Big(f(t, U(t), U(\theta(t)) + (\mathcal{V}U)(t) + (\mathcal{V}_\theta U)(t)\Big)\varphi(t)dt, \\ U(0) &= u_0 \end{aligned} \tag{10}$$

for all $\varphi \in P_{r_1-1}(I_1)$.

Select $U_0 \in P_{r_1}(I_1)$ with $U_0(0) = u_0$. For $m \ge 1$, let $U_m \in P_{r_1}(I_1)$ be the solution of the linear problem

$$\begin{aligned} \int_{I_1} U_m'(t)\varphi(t)dt &= \int_{I_1} \Big(f(t, U_{m-1}(t), U_{m-1}(\theta(t)) + (\mathcal{V}U_{m-1})(t) \\ &\quad +(\mathcal{V}_\theta U_{m-1})(t)\Big)\varphi(t)dt, \\ U_m(0) &= u_0 \end{aligned} \tag{11}$$

for all $\varphi \in P_{r_1-1}(I_1)$. Then, we have

$$\begin{aligned} &\int_{I_1} (U_m - U_{m-1})'(t)\varphi(t)dt \\ =\ &\int_{I_1} \Big(f(t, U_{m-1}(t), U_{m-1}(\theta(t)) + (\mathcal{V}U_{m-1})(t) + (\mathcal{V}_\theta U_{m-1})(t)\Big)\varphi(t)dt \\ &-\int_{I_1} \Big(f(t, U_{m-2}(t), U_{m-2}(\theta(t)) + (\mathcal{V}U_{m-2})(t) + (\mathcal{V}_\theta U_{m-2})(t)\Big)\varphi(t)dt \end{aligned}$$

for all $\varphi \in P_{r_1-1}(I_1)$. Choosing $\varphi = (U_m - U_{m-1})'$ in the above equation, using (4)-(8) and the Cauchy-Schwarz inequality yields

$$\begin{aligned} &\|(U_m - U_{m-1})'\|^2_{L^2(I_1)} \\ \le\ & L_1\|U_{m-1} - U_{m-2}\|_{L^2(I_1)}\|(U_m - U_{m-1})'\|_{L^2(I_1)} \\ &+L_2\|(U_{m-1} - U_{m-2})(\theta(t))\|_{L^2(I_1)}\|(U_m - U_{m-1})'\|_{L^2(I_1)} \\ &+\bar{K_1}L_3\left\|\int_0^t |(U_{m-1} - U_{m-2})(s)|ds\right\|_{L^2(I_1)}\|(U_m - U_{m-1})'\|_{L^2(I_1)} \\ &+\bar{K_2}L_4\left\|\int_0^{\theta(t)} |(U_{m-1} - U_{m-2})(s)|ds\right\|_{L^2(I_1)}\|(U_m - U_{m-1})'\|_{L^2(I_1)}, \end{aligned}$$

which implies

$$\begin{aligned} &\|(U_m - U_{m-1})'\|_{L^2(I_1)} \\ \le\ & L_1\|U_{m-1} - U_{m-2}\|_{L^2(I_1)} + L_2\|(U_{m-1} - U_{m-2})(\theta(t))\|_{L^2(I_1)} \\ &+\bar{K_1}L_3\left\|\int_0^t |(U_{m-1} - U_{m-2})(s)|ds\right\|_{L^2(I_1)} \\ &+\bar{K_2}L_4\left\|\int_0^{\theta(t)} |(U_{m-1} - U_{m-2})(s)|ds\right\|_{L^2(I_1)}. \end{aligned} \tag{12}$$

In view of the conditions (C1) and (C2), we find that

$$\begin{aligned}\|(U_{m-1}-U_{m-2})(\theta(t))\|^2_{L^2(I_1)} &\le \frac{1}{q_0}\int_0^{\theta(t_1)} |(U_{m-1}-U_{m-2})(s)|^2 ds\\ &\le \frac{1}{q_0}\|U_{m-1}-U_{m-2}\|^2_{L^2(I_1)},\end{aligned}$$

and by the Cauchy-Schwarz inequality we have

$$\begin{aligned}\left\|\int_0^t |(U_{m-1}-U_{m-2})(s)|ds\right\|^2_{L^2(I_1)} &\le \int_{I_1} t\left(\int_0^t |(U_{m-1}-U_{m-2})(s)|^2 ds\right)dt\\ &\le \frac{k_1^2}{2}\|U_{m-1}-U_{m-2}\|^2_{L^2(I_1)},\end{aligned}$$

$$\begin{aligned}\left\|\int_0^{\theta(t)} |(U_{m-1}-U_{m-2})(s)|ds\right\|^2_{L^2(I_1)} &\le \left\|\int_0^t |(U_{m-1}-U_{m-2})(s)|ds\right\|^2_{L^2(I_1)}\\ &\le \frac{k_1^2}{2}\|U_{m-1}-U_{m-2}\|^2_{L^2(I_1)},\end{aligned}$$

which together with (12) gives

$$\begin{aligned}&\|(U_m-U_{m-1})'\|_{L^2(I_1)}\\ \le\ &\left(L_1+\frac{L_2}{\sqrt{q_0}}+\frac{\bar{K}_1L_3k_1}{\sqrt{2}}+\frac{\bar{K}_2L_4k_1}{\sqrt{2}}\right)\|U_{m-1}-U_{m-2}\|_{L^2(I_1)}.\end{aligned}$$

Then, by Lemma 2.1 we get

$$\begin{aligned}&\|(U_m-U_{m-1})'\|_{L^2(I_1)}\\ \le\ &\left(L_1+\frac{L_2}{\sqrt{q_0}}+\frac{\bar{K}_1L_3k_1}{\sqrt{2}}+\frac{\bar{K}_2L_4k_1}{\sqrt{2}}\right)k_1\|(U_{m-1}-U_{m-2})'\|_{L^2(I_1)}\\ =\ &\lambda_1\|(U_{m-1}-U_{m-2})'\|_{L^2(I_1)}\le \lambda_1^{m-1}\|(U_1-U_0)'\|_{L^2(I_1)}\end{aligned}\tag{13}$$

and

$$\begin{aligned}&\|U_m-U_{m-1}\|_{L^2(I_1)}\\ \le\ &k_1\|(U_m-U_{m-1})'\|_{L^2(I_1)}\le\lambda_1\|U_{m-1}-U_{m-2}\|_{L^2(I_1)}\\ \le\ &\lambda_1^{m-1}\|U_1-U_0\|_{L^2(I_1)},\end{aligned}\tag{14}$$

which implies

$$\|U_m-U_{m-1}\|_{H^\ell(I_1)}\le\lambda_1^{m-1}\|U_1-U_0\|_{H^\ell(I_1)},\qquad \ell=0,1.$$

For our purpose, we denote by $[\alpha]$ the smallest integer larger or equal to α. For any $\varepsilon>0$, there is an integer $N=\left[\ln\frac{\varepsilon(1-\lambda_1)}{\|U_1-U_0\|_{L^2(I_1)}}\Big/\ln\lambda_1\right]$ such that for $m>n>N$ there holds

$$\begin{aligned}\|U_m-U_n\|_{H^1(I_1)} &\le \|U_m-U_{m-1}\|_{H^1(I_1)}+\cdots+\|U_{n+1}-U_n\|_{H^1(I_1)}\\ &\le (\lambda_1^{m-1}+\cdots+\lambda_1^n)\|U_1-U_0\|_{H^1(I_1)}\\ &= \frac{\lambda_1^n(1-\lambda_1^{m-n})}{1-\lambda_1}\|U_1-U_0\|_{H^1(I_1)}\\ &\le \frac{\lambda_1^N}{1-\lambda_1}\|U_1-U_0\|_{H^1(I_1)}<\varepsilon,\end{aligned}\tag{15}$$

which implies that $\{U_m\}$ is a Cauchy sequence in $H^1(I_1)$. Hence, $\{U_m\}$ has a limit $U \in P_{r_1}(I_1)$ such that $\lim_{m\to\infty} U_m(t) = U(t)$ in $H^1(I_1)$. Taking the limit on both sides of (11), then $U(t)$ satisfies the equation (11). Thus the existence is proved. Similarly, using the above arguments can easily lead to the uniqueness. In fact, suppose there are two solutions U and $\widetilde{U}$ of the problem (11), then we have

$$\|U - \widetilde{U}\|_{L^2(I_1)} \leq \lambda_1 \|U - \widetilde{U}\|_{L^2(I_1)}$$

for $0 < \lambda_1 < 1$, which implies that $U = \widetilde{U}$. This proves the uniqueness.

□

2.3. Computational form of the continuous Petrov-Galerkin method. Let $L_l(x), x \in [-1,1]$ be the standard Legendre polynomial of degree l. The shifted Legendre polynomials $L_{n,l}(t)$ on the interval I_n are defined by

$$L_{n,l}(t) = L_l\Big(\frac{2t - t_{n-1} - t_n}{k_n}\Big), \quad t \in I_n, \quad l \geq 0.$$

Let $U_n(t) = U|_{I_n}$ be the solution of the discrete problem (3) on the interval $I_n, 1 \leq n \leq N$. We expand $U_n(t)$ as

$$U_n(t) = \sum_{l=0}^{r_n} \hat{u}_{n,l} L_{n,l}(t).$$

Inserting the above expression into (3) and choosing $\varphi = L_{n,j}(t),\ 0 \leq j \leq r_n - 1$, we can rewrite (3) as a system of nonlinear algebraic equations for the unknown vector

$$\widehat{\mathbf{U}}_n := (\hat{u}_{n,0}, \hat{u}_{n,1}, \ldots, \hat{u}_{n,r_n})^T \in \mathbb{R}^{r_n+1}.$$

We emphasize that, the structure of the resulted nonlinear system depends strongly on the delay terms in (3) and changes for each value of n as we pass from Phase I to Phase III (described below).

For our purpose, we introduce the matrices

$$A_n = (a_{jl})_{0\leq j\leq r_n, 0\leq l\leq r_n} \in \mathbb{R}^{(r_n+1)\times(r_n+1)}, \quad 1 \leq n \leq N,$$

with the entries given by

$$a_{jl} = \int_{I_n} L'_{n,l}(t) L_{n,j}(t)dt = \int_{-1}^{1} L'_l(x) L_j(x)dx, \quad 0 \leq j \leq r_n - 1,\ 0 \leq l \leq r_n,$$

and $a_{r_n l} = L_{n,l}(t_{n-1}) = L_l(-1),\ 0 \leq l \leq r_n$. Further, for $0 \leq j \leq r_n - 1$, we set

$$\begin{aligned} b^1_{n,j} &:= \int_{I_n} (\mathcal{V}U)(t) L_{n,j}(t)dt \\ &= \int_{I_n} \Big(\int_0^t K_1(t,s) G_1(s,U(s))ds\Big) L_{n,j}(t)dt \\ &= \sum_{m=1}^{n-1} \int_{I_n} \Big(\int_{I_m} K_1(t,s) G_1(s,U_m(s))ds\Big) L_{n,j}(t)dt \\ &\quad + \int_{I_n} \bigg(\int_{t_{n-1}}^t K_1(t,s) G_1(s,U_n(s))ds\bigg) L_{n,j}(t)dt \end{aligned} \tag{16}$$

and

$$
\begin{aligned}
b_{n,j}^2 &:= \int_{I_n} (\mathcal{V}_\theta U)(t) L_{n,j}(t)dt \\
&= \int_{I_n} \left(\int_0^{\theta(t)} K_2(t,s) G_2(s,U(s))ds \right) L_{n,j}(t)dt \\
&= \int_{I_n} \left(\int_0^{\theta(t_{n-1})} K_2(t,s) G_2(s,U(s))ds \right) L_{n,j}(t)dt \\
&\quad + \int_{I_n} \left(\int_{\theta(t_{n-1})}^{\theta(t)} K_2(t,s) G_2(s,U(s))ds \right) L_{n,j}(t)dt \\
&:= J_{1,j} + J_{2,j}.
\end{aligned}
\tag{17}
$$

Obviously, if $n = 1$, the summation term in (16) and the term $J_{1,j}$ in (17) will vanish.

We now introduce the following three distinct phases inspired by [6].

• *Phase* I: $n = 1$. In this initial phase we have complete overlap, i.e., for any $t \in I_1$ the images $\theta(t) \in I_1$. For $0 \le j \le r_1 - 1$, we define

$$
f_{1,j}^I := \int_{I_1} f(t, U(t), U(\theta(t))) L_{1,j}(t)dt = \int_{I_1} f(t, U_1(t), U_1(\theta(t))) L_{1,j}(t)dt.
$$

Moreover, we have

$$
b_{1,j}^2 = \int_{I_1} \left(\int_0^{\theta(t)} K_2(t,s) G_2(s,U_1(s))ds \right) L_{1,j}(t)dt.
$$

Let $c_{1,j}^I = f_{1,j}^I + b_{1,j}^1 + b_{1,j}^2$ and

$$
C^I(\widehat{\mathbf{U}}_1) := \left(c_{1,0}^I, c_{1,1}^I, \cdots, c_{1,r_1-1}^I, u_0 \right)^T.
$$

Then we can rewrite (3) as the nonlinear system

$$
A_1 \widehat{\mathbf{U}}_1 = C^I(\widehat{\mathbf{U}}_1). \tag{18}
$$

• *Phase* II: If $n > 1$ and $\theta(t_n) > t_{n-1}$, then we will encounter partial overlap, i.e., for some $t \in I_n$ the images $\theta(t)$ are still in I_n, while for some other (smaller) $t \in I_n$ we have $\theta(t) \notin I_n$. Clearly, there is an integer $z \ge 1$ such that $\theta(t_{n-1}) \in I_z$.

Let $t_0^* = t_{n-1}, t_{n-z+1}^* = t_n$ and $t_m^* = \theta^{-1}(t_{z+m-1}) \in I_n$ for $1 \le m \le n - z$. For $0 \le j \le r_n - 1$, we define

$$
\begin{aligned}
f_{n,j}^{II} &:= \int_{I_n} f(t, U(t), U(\theta(t))) L_{n,j}(t)dt \\
&= \sum_{m=1}^{n-z+1} \int_{t_{m-1}^*}^{t_m^*} f(t, U_n(t), U_{z+m-1}(\theta(t))) L_{n,j}(t)dt.
\end{aligned}
$$

In this phase, we have

$$\begin{aligned}
J_{1,j} &= \int_{I_n}\left(\int_0^{t_{z-1}} K_2(t,s)G_2(s,U(s))ds\right)L_{n,j}(t)dt \\
&\quad + \int_{I_n}\left(\int_{t_{z-1}}^{\theta(t_{n-1})} K_2(t,s)G_2(s,U(s))ds\right)L_{n,j}(t)dt \\
&= \sum_{\alpha=1}^{z-1}\int_{I_n}\left(\int_{I_\alpha} K_2(t,s)G_2(s,U_\alpha(s))ds\right)L_{n,j}(t)dt \\
&\quad + \int_{I_n}\left(\int_{t_{z-1}}^{\theta(t_{n-1})} K_2(t,s)G_2(s,U_z(s))ds\right)L_{n,j}(t)dt
\end{aligned}$$

and

$$J_{2,j} = \sum_{m=1}^{n-z+1}\int_{t^*_{m-1}}^{t^*_m}\left(\int_{\theta(t_{n-1})}^{\theta(t)} K_2(t,s)G_2(s,U(s))ds\right)L_{n,j}(t)dt := \sum_{m=1}^{n-z+1} J_{2,j}^{II,m},$$

with

$$\begin{aligned}
J_{2,j}^{II,m} &= \sum_{\beta=1}^{m-1}\int_{t^*_{m-1}}^{t^*_m}\left(\int_{\theta(t^*_{\beta-1})}^{\theta(t^*_\beta)} K_2(t,s)G_2(s,U_{z+\beta-1}(s))ds\right)L_{n,j}(t)dt \\
&\quad + \int_{t^*_{m-1}}^{t^*_m}\left(\int_{\theta(t^*_{m-1})}^{\theta(t)} K_2(t,s)G_2(s,U_{z+m-1}(s))ds\right)L_{n,j}(t)dt.
\end{aligned}$$

Let $c_{n,j}^{II} = f_{n,j}^{II} + b_{n,j}^1 + b_{n,j}^2$ and

$$C^{II}(\widehat{\mathbf{U}}_n) := \left(c_{n,0}^{II}, c_{n,1}^{II}, \cdots, c_{n,r_n-1}^{II}, U_{n-1}(t_{n-1})\right)^T.$$

Then we can rewrite (3) as the nonlinear system

$$A_n\widehat{\mathbf{U}}_n = C^{II}(\widehat{\mathbf{U}}_n). \tag{19}$$

• *Phase* III: If $n > 1$ and $\theta(t_n) \le t_{n-1}$, then we will encounter the pure delay phase, i.e., there is no overlap between I_n and the images $\theta(t)$ for any $t \in I_n$. In this phase, there are two integers z_1 and z_2 ($z_1 \le z_2$) such that $\theta(t_{n-1}) \in I_{z_1}$ and $\theta(t_n) \in I_{z_2}$.

Let $t_0^* = t_{n-1}$, $t^*_{z_2-z_1+1} = t_n$ and $t^*_m = \theta^{-1}(t_{z_1+m-1}) \in I_n$ for $1 \le m \le z_2 - z_1$ (if $z_1 < z_2$). For $0 \le j \le r_n - 1$, we define

$$\begin{aligned}
f_{n,j}^{III} &:= \int_{I_n} f(t,U(t),U(\theta(t)))L_{n,j}(t)dt \\
&= \sum_{m=1}^{z_2-z_1+1}\int_{t^*_{m-1}}^{t^*_m} f(t,U_n(t),U_{z_1+m-1}(\theta(t)))L_{n,j}(t)dt.
\end{aligned}$$

In this phase, we have

$$\begin{aligned}
J_{1,j} &= \int_{I_n}\left(\int_0^{t_{z_1-1}} K_2(t,s)G_2(s,U(s))ds\right)L_{n,j}(t)dt \\
&\quad + \int_{I_n}\left(\int_{t_{z_1-1}}^{\theta(t_{n-1})} K_2(t,s)G_2(s,U(s))ds\right)L_{n,j}(t)dt \\
&= \sum_{\alpha=1}^{z_1-1}\int_{I_n}\left(\int_{I_\alpha} K_2(t,s)G_2(s,U_\alpha(s))ds\right)L_{n,j}(t)dt \\
&\quad + \int_{I_n}\left(\int_{t_{z_1-1}}^{\theta(t_{n-1})} K_2(t,s)G_2(s,U_{z_1}(s))ds\right)L_{n,j}(t)dt
\end{aligned}$$

and

$$J_{2,j} = \sum_{m=1}^{z_2-z_1+1} \int_{t_{m-1}^*}^{t_m^*} \left(\int_{\theta(t_{n-1})}^{\theta(t)} K_2(t,s)U(s)ds \right) L_{n,j}(t)dt := \sum_{m=1}^{z_2-z_1+1} J_{2,j}^{III,m},$$

with

$$\begin{aligned} J_{2,j}^{III,m} &= \sum_{\beta=1}^{m-1} \int_{t_{m-1}^*}^{t_m^*} \left(\int_{\theta(t_{\beta-1}^*)}^{\theta(t_\beta^*)} K_2(t,s)G_2(s, U_{z_1+\beta-1}(s))ds \right) L_{n,j}(t)dt \\ &\quad + \int_{t_{m-1}^*}^{t_m^*} \left(\int_{\theta(t_{m-1}^*)}^{\theta(t)} K_2(t,s)G_2(s, U_{z_1+m-1}(s))ds \right) L_{n,j}(t)dt. \end{aligned} \tag{20}$$

Evidently, for $m = 1$ the summation term in the first line of (20) will vanish.

Let $c_{n,j}^{III} = f_{n,j}^{III} + b_{n,j}^1 + b_{n,j}^2$ and

$$C^{III}(\widehat{\mathbf{U}}_n) := \left(c_{n,0}^{III}, c_{n,1}^{III}, \cdots, c_{n,r_n-1}^{III}, U_{n-1}(t_{n-1})\right)^T.$$

Then we can rewrite (3) as the nonlinear system

$$A_n \widehat{\mathbf{U}}_n = C^{III}(\widehat{\mathbf{U}}_n). \tag{21}$$

Remark 2.2. In actual computation, the nonlinear systems (18)-(21) can be solved by an iterative process, for example, the Newton-Raphson iteration method or the successive substitution method.

3. Error analysis

In this section, we carry our a priori error analysis of the h-p version of the CPG method.

3.1. Preliminaries. Let $\Lambda = (-1, 1)$. For a function $u \in H^1(\Lambda)$, we introduce a projection operator $\Pi_\Lambda^r : H^1(\Lambda) \to P_r(\Lambda)$ with $r \geq 1$ by

$$\begin{cases} \displaystyle\int_\Lambda (u - \Pi_\Lambda^r u)' \varphi dt &= 0, \quad \forall\, \varphi \in P_{r-1}(\Lambda), \\ \Pi_\Lambda^r u(-1) &= u(-1). \end{cases} \tag{22}$$

Setting $\varphi = 1$ in (22) and using integration by parts gives $u(1) - \Pi_\Lambda^r u(1) = u(-1) - \Pi_\Lambda^r u(-1) = 0$, which implies $\Pi_\Lambda^r u(1) = u(1)$. It is well-known that the projection operator Π_Λ^r is well defined (see, e.g., [19]) and there holds

$$\Pi_\Lambda^r u(x) = \int_{-1}^x \Big(\sum_{i=0}^{r-1} a_i L_i(\xi) \Big) d\xi + u(-1),$$

where $a_i = \frac{2i+1}{2} \int_\Lambda u' L_i dx$ is the Legendre expansion coefficients of u'.

For any interval $J = (a, b)$ of length $h = b - a$, we define $\Pi_J^r u = [\Pi_\Lambda^r (u \circ \mathcal{M})] \circ \mathcal{M}^{-1}$, where $\mathcal{M} : \Lambda \to J$ is the linear transformation $x \mapsto t = \frac{a+b+hx}{2}$. Then for the exact solution u of (1) we can define an approximation polynomial $\mathcal{I}u \in S^{\mathbf{r},1}(\mathcal{T}_h)$ as

$$\mathcal{I}u|_{I_n} = \Pi_{I_n}^{r_n} u, \quad 1 \leq n \leq N.$$

Thanks to the definition of Π_Λ^r, it is straightforward to show that $\mathcal{I}u(t_n) = u(t_n)$ for $0 \leq n \leq N$, and there holds

$$\int_{I_n} (u - \mathcal{I}u)' \varphi dt = 0, \quad \forall\, \varphi \in P_{r_n-1}(I_n). \tag{23}$$

The polynomial $\mathcal{I}u$ constructed above has the following approximation properties (cf. [11, 20]).

Lemma 3.2. *Let $\mathcal{T}_h$ be any mesh in I and assume that $u \in H^1(I)$ satisfies $u|_{I_n} \in H^{s_{0,n}+1}(I_n)$ for $s_{0,n} \geq 0$. Then*

$$\|u - \mathcal{I}u\|^2_{L^2(I)} \leq \sum_{n=1}^{N} \big(\frac{k_n}{2}\big)^{2s_n+2} \frac{\Gamma(r_n+1-s_n)}{r_n(r_n+1)\Gamma(r_n+1+s_n)} \|u\|^2_{H^{s_n+1}(I_n)}, \tag{24}$$

$$|u - \mathcal{I}u|^2_{H^1(I)} \leq \sum_{n=1}^{N} \big(\frac{k_n}{2}\big)^{2s_n} \frac{\Gamma(r_n+1-s_n)}{\Gamma(r_n+1+s_n)} \|u\|^2_{H^{s_n+1}(I_n)} \tag{25}$$

for any real s_n, $0 \leq s_n \leq \min\{r_n, s_{0,n}\}$.

Moreover, if $u \in H^1(I)$ satisfies $u|_{I_n} \in W^{s_{0,n}+1,\infty}(I_n)$ for $s_{0,n} \geq 0$. Then

$$\|u - \mathcal{I}u\|^2_{L^\infty(I_n)} \leq C\big(\frac{k_n}{2}\big)^{2s_n+2} \frac{\Gamma(r_n+1-s_n)}{\Gamma(r_n+1+s_n)} \|u\|^2_{W^{s_n+1,\infty}(I_n)} \tag{26}$$

for any real s_n, $0 \leq s_n \leq \min\{r_n, s_{0,n}\}$.

We note that, the following discrete Gronwall inequality have been proved, for instance, in [3].

Lemma 3.3. *Let $\{a_n\}_{n=1}^N$ and $\{b_n\}_{n=1}^N$ be two sequences of nonnegative real numbers with $b_1 \leq b_2 \leq \cdots \leq b_N$. Assume that for $C \geq 0$ and weights $w_i > 0, 1 \leq i \leq N-1$, there holds*

$$a_1 \leq b_1, \quad a_n \leq b_n + C\sum_{i=1}^{n-1} w_i a_i, \quad 2 \leq n \leq N.$$

Then

$$a_n \leq b_n \exp(C\sum_{i=1}^{n-1} w_i), \quad 1 \leq n \leq N.$$

3.2. Abstract error bounds. Let u be the exact solution of (1) and U be the h-p version of the CPG approximation defined by (2). We proceed in a standard way and decompose the error $e = u - U$ into two parts:

$$e = (u - \mathcal{I}u) + (\mathcal{I}u - U) := \eta + \xi. \tag{27}$$

Lemma 3.2 can be used to bound η, and we are left with the task of estimating the term ξ.

In view of (1) and (3), there holds

$$\begin{aligned} \int_{I_n} e'\varphi dt &= \int_{I_n} (f(t, u(t), u(\theta(t))) - f(t, U(t), U(\theta(t))))\varphi dt \\ &\quad + \int_{I_n} (\mathcal{V}u - \mathcal{V}U)\varphi dt + \int_{I_n} (\mathcal{V}_\theta u - \mathcal{V}_\theta U)\varphi dt \end{aligned}$$

for all $\varphi \in P_{r_n-1}(I_n)$. Then, by (23) we have

$$\begin{aligned} \int_{I_n} \xi'\varphi dt &= \int_{I_n} (f(t, u(t), u(\theta(t))) - f(t, U(t), U(\theta(t))))\varphi dt \\ &\quad + \int_{I_n} (\mathcal{V}u - \mathcal{V}U)\varphi dt + \int_{I_n} (\mathcal{V}_\theta u - \mathcal{V}_\theta U)\varphi dt \end{aligned} \tag{28}$$

for all $\varphi \in P_{r_n-1}(I_n)$.

For any $v \in L^2(I_n)$, we define the L^2 projection of v onto $P_{r_n-1}(I_n)$ by $\Pi^{r_n-1}v$, namely,

$$\int_{I_n} (v - \Pi^{r_n-1}v)\varphi dt = 0, \quad \forall\, \varphi \in P_{r_n-1}(I_n).$$

First, we show the following bounds.

Lemma 3.4. *Assume that k is sufficiently small, there holds*

$$\|\xi\|_{L^2(0,t_n)} \le C\|\eta\|_{L^2(0,t_n)}, \tag{29}$$

$$|\xi|_{H^1(0,t_n)} \le C\|\eta\|_{L^2(0,t_n)}, \tag{30}$$

and

$$|\xi(t_n)| \le C\|\eta\|_{L^2(0,t_n)} \tag{31}$$

for $1 \le n \le N$*, where the constant* $C > 0$ *solely depends on* $q_0, L_1, L_2, L_3, L_4, \bar{K_1}, \bar{K_2}$*, and* t_n*.*

Proof. By choosing $\varphi = \Pi^{r_n-1}\xi$ in (28) and using (4)-(8) we get

$$\begin{aligned}
\int_{I_n} \xi'\xi dt &= \int_{I_n} (f(t,u(t),u(\theta(t))) - f(t,U(t),U(\theta(t))))\Pi^{r_n-1}\xi dt \\
&\quad + \int_{I_n} (\mathcal{V}u - \mathcal{V}U)\Pi^{r_n-1}\xi dt + \int_{I_n} (\mathcal{V}_\theta u - \mathcal{V}_\theta U)\Pi^{r_n-1}\xi dt \\
&\le L_1 \int_{I_n} |e| \cdot |\Pi^{r_n-1}\xi| dt + L_2 \int_{I_n} |e(\theta(t))| \cdot |\Pi^{r_n-1}\xi| dt \\
&\quad + \bar{K_1}L_3 \int_{I_n} \Big(\int_0^t |e(s)|ds\Big)|\Pi^{r_n-1}\xi| dt \\
&\quad + \bar{K_2}L_4 \int_{I_n} \Big(\int_0^{\theta(t)} |e(s)|ds\Big)|\Pi^{r_n-1}\xi| dt,
\end{aligned}$$

which together with the Cauchy-Schwarz inequality and the L^2-stability of Π^{r_n-1} yields

$$\begin{aligned}
&\frac{1}{2}(|\xi(t_n)|^2 - |\xi(t_{n-1})|^2) \\
\le\ & L_1\|e\|_{L^2(I_n)}\|\xi\|_{L^2(I_n)} + L_2\|e(\theta(t))\|_{L^2(I_n)}\|\xi\|_{L^2(I_n)} \\
&+ \bar{K_1}L_3 \int_{I_n} \Big(\int_{t_{n-1}}^t |e(s)|ds\Big)|\Pi^{r_n-1}\xi| dt \\
&+ \bar{K_1}L_3 \int_{I_n} |\Pi^{r_n-1}\xi| dt \int_0^{t_{n-1}} |e(s)|ds \\
&+ \bar{K_2}L_4 \left\|\int_0^{\theta(t)} |e(s)|ds\right\|_{L^2(I_n)} \|\xi\|_{L^2(I_n)} \\
\le\ & L_1\|e\|_{L^2(I_n)}\|\xi\|_{L^2(I_n)} + L_2\|e(\theta(t))\|_{L^2(I_n)}\|\xi\|_{L^2(I_n)} \\
&+ \frac{\bar{K_1}L_3}{\sqrt{2}} k_n \|e\|_{L^2(I_n)}\|\xi\|_{L^2(I_n)} \\
&+ \bar{K_1}L_3 k_n^{\frac{1}{2}} \|\xi\|_{L^2(I_n)} \sum_{i=1}^{n-1} k_i^{\frac{1}{2}} \|e\|_{L^2(I_i)} \\
&+ \bar{K_2}L_4 \left\|\int_0^{\theta(t)} |e(s)|ds\right\|_{L^2(I_n)} \|\xi\|_{L^2(I_n)}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{L_1}{2}\Big(\|e\|^2_{L^2(I_n)} + \|\xi\|^2_{L^2(I_n)}\Big) + \frac{L_2}{2}\Big(\|e(\theta(t))\|^2_{L^2(I_n)} + \|\xi\|^2_{L^2(I_n)}\Big) \\
&\quad + \frac{\bar{K}_1 L_3 k_n}{2\sqrt{2}}\Big(\|e\|^2_{L^2(I_n)} + \|\xi\|^2_{L^2(I_n)}\Big) \\
&\quad + \frac{\bar{K}_1 L_3}{2}\Big(\|\xi\|^2_{L^2(I_n)} + k_n t_{n-1}\|e\|^2_{L^2(0,t_{n-1})}\Big) \\
&\quad + \frac{\bar{K}_2 L_4}{2}\Big(\Big\|\int_0^{\theta(t)} |e(s)|ds\Big\|^2_{L^2(I_n)} + \|\xi\|^2_{L^2(I_n)}\Big).
\end{aligned}
$$

Here, we have used the fact that

$$
\begin{aligned}
\Big\|\int_{t_{n-1}}^{t} |e(s)|ds\Big\|_{L^2(I_n)} &\leq \Big\{\int_{t_{n-1}}^{t_n} (t-t_{n-1})\Big(\int_{t_{n-1}}^{t} |e(s)|^2 ds\Big)dt\Big\}^{\frac{1}{2}} \\
&\leq \frac{k_n}{\sqrt{2}}\|e\|_{L^2(I_n)}.
\end{aligned}
\tag{32}
$$

Consequently,

$$
\begin{aligned}
|\xi(t_n)|^2 &\leq |\xi(t_{n-1})|^2 + \Big(L_1 + \frac{\bar{K}_1 L_3 k_n}{\sqrt{2}}\Big)\|e\|^2_{L^2(I_n)} \\
&\quad + \Big(L_1 + L_2 + \frac{\bar{K}_1 L_3 k_n}{\sqrt{2}} + \bar{K}_1 L_3 + \bar{K}_2 L_4\Big)\|\xi\|^2_{L^2(I_n)} \\
&\quad + L_2\|e(\theta(t))\|^2_{L^2(I_n)} + \bar{K}_1 L_3 k_n t_{n-1}\|e\|^2_{L^2(0,t_{n-1})} \\
&\quad + \bar{K}_2 L_4\Big\|\int_0^{\theta(t)} |e(s)|ds\Big\|^2_{L^2(I_n)} \\
&\leq |\xi(t_{n-1})|^2 + 2\Big(L_1 + \frac{\bar{K}_1 L_3 k_n}{\sqrt{2}}\Big)\|\eta\|^2_{L^2(I_n)} \\
&\quad + \Big(3L_1 + L_2 + \frac{3\bar{K}_1 L_3 k_n}{\sqrt{2}} + \bar{K}_1 L_3 + \bar{K}_2 L_4\Big)\|\xi\|^2_{L^2(I_n)} \\
&\quad + L_2\|e(\theta(t))\|^2_{L^2(I_n)} + \bar{K}_1 L_3 k_n t_{n-1}\|e\|^2_{L^2(0,t_{n-1})} \\
&\quad + \bar{K}_2 L_4\Big\|\int_0^{\theta(t)} |e(s)|ds\Big\|^2_{L^2(I_n)}.
\end{aligned}
\tag{33}
$$

Additionally, taking $\varphi = \Pi^{r_n-1}((t_{n-1}-t)\xi)$ in (28), we find that

$$
\begin{aligned}
&\int_{I_n} (t_{n-1}-t)\xi'\xi dt \\
= &\int_{I_n} (f(t,u(t),u(\theta(t))) - f(t,U(t),U(\theta(t))))\Pi^{r_n-1}((t_{n-1}-t)\xi)dt \\
&+ \int_{I_n} (\mathcal{V}u - \mathcal{V}U)\Pi^{r_n-1}((t_{n-1}-t)\xi)dt \\
&+ \int_{I_n} (\mathcal{V}_\theta u - \mathcal{V}_\theta U)\Pi^{r_n-1}((t_{n-1}-t)\xi)dt,
\end{aligned}
$$

which together with (4)-(8) gives

$$
\begin{aligned}
& \frac{1}{2}\big(-k_n|\xi(t_n)|^2+\|\xi\|^2_{L^2(I_n)}\big)\\
\le\ & L_1\|e\|_{L^2(I_n)}\|\Pi^{r_n-1}((t_{n-1}-t)\xi)\|_{L^2(I_n)}\\
& +L_2\|e(\theta(t))\|_{L^2(I_n)}\|\Pi^{r_n-1}((t_{n-1}-t)\xi)\|_{L^2(I_n)}\\
& +\bar{K}_1L_3\int_{I_n}\Big(\int_{t_{n-1}}^{t}|e(s)|ds\Big)|\Pi^{r_n-1}((t_{n-1}-t)\xi)|dt\\
& +\bar{K}_1L_3\int_{I_n}\Big(\int_0^{t_{n-1}}|e(s)|ds\Big)|\Pi^{r_n-1}((t_{n-1}-t)\xi)|dt\\
& +\bar{K}_2L_4\left\|\int_0^{\theta(t)}|e(s)|ds\right\|_{L^2(I_n)}\|\Pi^{r_n-1}((t_{n-1}-t)\xi)\|_{L^2(I_n)}.
\end{aligned}
\tag{34}
$$

We notice that

$$
\|\Pi^{r_n-1}((t_{n-1}-t)\xi)\|_{L^2(I_n)}\le\|(t_{n-1}-t)\xi\|_{L^2(I_n)}\le k_n\|\xi\|_{L^2(I_n)}.
$$

Then, by (34) and (32) we readily find that

$$
\begin{aligned}
& \|\xi\|^2_{L^2(I_n)}\\
\le\ & k_n|\xi(t_n)|^2+2L_1k_n\|e\|_{L^2(I_n)}\|\xi\|_{L^2(I_n)}+2L_2k_n\|e(\theta(t))\|_{L^2(I_n)}\|\xi\|_{L^2(I_n)}\\
& +\frac{2\bar{K}_1L_3k_n^2}{\sqrt{2}}\|e\|_{L^2(I_n)}\|\xi\|_{L^2(I_n)}+2\bar{K}_1L_3k_n^{\frac{3}{2}}\|\xi\|_{L^2(I_n)}\Big(\sum_{i=1}^{n-1}k_i^{\frac{1}{2}}\|e\|_{L^2(I_i)}\Big)\\
& +2\bar{K}_2L_4k_n\left\|\int_0^{\theta(t)}|e(s)|ds\right\|_{L^2(I_n)}\|\xi\|_{L^2(I_n)}\\
\le\ & k_n|\xi(t_n)|^2+L_1k_n\Big(\|e\|^2_{L^2(I_n)}+\|\xi\|^2_{L^2(I_n)}\Big)\\
& +L_2k_n\Big(\|e(\theta(t))\|^2_{L^2(I_n)}+\|\xi\|^2_{L^2(I_n)}\Big)\\
& +\frac{\bar{K}_1L_3k_n^2}{\sqrt{2}}\Big(\|e\|^2_{L^2(I_n)}+\|\xi\|^2_{L^2(I_n)}\Big)\\
& +\bar{K}_1L_3k_n\Big(k_nt_{n-1}\|e\|^2_{L^2(0,t_{n-1})}+\|\xi\|^2_{L^2(I_n)}\Big)\\
& +\bar{K}_2L_4k_n\left(\left\|\int_0^{\theta(t)}|e(s)|ds\right\|^2_{L^2(I_n)}+\|\xi\|^2_{L^2(I_n)}\right)\\
\le\ & k_n|\xi(t_n)|^2+2\Big(L_1+\frac{\bar{K}_1L_3k_n}{\sqrt{2}}\Big)k_n\|\eta\|^2_{L^2(I_n)}\\
& +\Big(3L_1+L_2+\frac{3\bar{K}_1L_3k_n}{\sqrt{2}}+\bar{K}_1L_3+\bar{K}_2L_4\Big)k_n\|\xi\|^2_{L^2(I_n)}\\
& +L_2k_n\|e(\theta(t))\|^2_{L^2(I_n)}+\bar{K}_1L_3k_n^2t_{n-1}\|e\|^2_{L^2(0,t_{n-1})}\\
& +\bar{K}_2L_4k_n\left\|\int_0^{\theta(t)}|e(s)|ds\right\|^2_{L^2(I_n)}.
\end{aligned}
$$

For convenience, we define

$$A_n = \Big(3L_1 + L_2 + \frac{3\bar{K}_1 L_3 k_n}{\sqrt{2}} + \bar{K}_1 L_3 + \bar{K}_2 L_4\Big) k_n.$$

We observe after elementary manipulation that

$$\begin{aligned}
\|\xi\|^2_{L^2(I_n)} \quad \le \quad & \frac{k_n}{1-A_n}|\xi(t_n)|^2 + \frac{2\Big(L_1 + \frac{\bar{K}_1 L_3 k_n}{\sqrt{2}}\Big)k_n}{1-A_n}\|\eta\|^2_{L^2(I_n)} \\
& + \frac{L_2 k_n}{1-A_n}\|e(\theta(t))\|^2_{L^2(I_n)} + \frac{\bar{K}_1 L_3 k_n^2 t_{n-1}}{1-A_n}\|e\|^2_{L^2(0,t_{n-1})} \\
& + \frac{\bar{K}_2 L_4 k_n}{1-A_n}\left\| \int_0^{\theta(t)} |e(s)| ds \right\|^2_{L^2(I_n)}.
\end{aligned} \tag{35}$$

By inserting (35) into (33), we obtain

$$\begin{aligned}
|\xi(t_n)|^2 \quad \le \quad & |\xi(t_{n-1})|^2 + 2\Big(L_1 + \frac{\bar{K}_1 L_3 k_n}{\sqrt{2}}\Big)\|\eta\|^2_{L^2(I_n)} \\
& + \frac{A_n}{1-A_n}|\xi(t_n)|^2 + \frac{2\Big(L_1 + \frac{\bar{K}_1 L_3 k_n}{\sqrt{2}}\Big)A_n}{1-A_n}\|\eta\|^2_{L^2(I_n)} \\
& + \frac{L_2 A_n}{1-A_n}\|e(\theta(t))\|^2_{L^2(I_n)} + \frac{\bar{K}_1 L_3 k_n t_{n-1} A_n}{1-A_n}\|e\|^2_{L^2(0,t_{n-1})} \\
& + \frac{\bar{K}_2 L_4 A_n}{1-A_n}\left\| \int_0^{\theta(t)} |e(s)| ds \right\|^2_{L^2(I_n)} + L_2\|e(\theta(t))\|^2_{L^2(I_n)} \\
& + \bar{K}_1 L_3 k_n t_{n-1}\|e\|^2_{L^2(0,t_{n-1})} + \bar{K}_2 L_4 \left\| \int_0^{\theta(t)} |e(s)| ds \right\|^2_{L^2(I_n)},
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
|\xi(t_n)|^2 \quad \le \quad & \Big(1 + \frac{A_n}{1-2A_n}\Big)|\xi(t_{n-1})|^2 + \frac{2\Big(L_1 + \frac{\bar{K}_1 L_3 k_n}{\sqrt{2}}\Big)}{1-2A_n}\|\eta\|^2_{L^2(I_n)} \\
& + \frac{L_2}{1-2A_n}\|e(\theta(t))\|^2_{L^2(I_n)} + \frac{\bar{K}_1 L_3 k_n t_{n-1}}{1-2A_n}\|e\|^2_{L^2(0,t_{n-1})} \\
& + \frac{\bar{K}_2 L_4}{1-2A_n}\left\| \int_0^{\theta(t)} |e(s)| ds \right\|^2_{L^2(I_n)}.
\end{aligned} \tag{36}$$

Assume that k_n is sufficiently small, then there exists a positive constant γ such that

$$2A_n \le \gamma < 1, \quad 1 \le n \le N.$$

Summing up (36) over all element $I_i, 1 \le i \le n$, and using the facts that $\xi(t_0) = 0$ and $\xi|_{I_i}(t_i) = \xi|_{I_{i+1}}(t_i), 1 \le i \le n-1$, we readily conclude that

(37)

$$\begin{aligned}
|\xi(t_n)|^2 \quad \le \quad & \sum_{i=1}^{n-1} \frac{A_{i+1}}{1-2A_{i+1}} |\xi(t_i)|^2 + \sum_{i=1}^{n} \frac{2\Big(L_1 + \dfrac{\bar{K}_1 L_3 k_i}{\sqrt{2}}\Big)}{1-2A_i} \|\eta\|^2_{L^2(I_i)} \\
& + \sum_{i=1}^{n} \frac{L_2}{1-2A_i} \|e(\theta(t))\|^2_{L^2(I_i)} + \sum_{i=1}^{n-1} \frac{\bar{K}_1 L_3 k_{i+1} t_i}{1-2A_{i+1}} \|e\|^2_{L^2(0,t_i)} \\
& + \sum_{i=1}^{n} \frac{\bar{K}_2 L_4}{1-2A_i} \left\| \int_0^{\theta(t)} |e(s)| ds \right\|^2_{L^2(I_i)} \\
\le \quad & \frac{3L_1 + L_2 + \dfrac{3\bar{K}_1 L_3 k}{\sqrt{2}} + \bar{K}_1 L_3 + \bar{K}_2 L_4}{1-\gamma} \sum_{i=1}^{n-1} k_{i+1} |\xi(t_i)|^2 \\
& + \frac{2\Big(L_1 + \dfrac{\bar{K}_1 L_3 k}{\sqrt{2}}\Big)}{1-\gamma} \|\eta\|^2_{L^2(0,t_n)} + \frac{L_2}{1-\gamma} \|e(\theta(t))\|^2_{L^2(0,t_n)} \\
& + \frac{\bar{K}_1 L_3}{1-\gamma} \sum_{i=1}^{n-1} k_{i+1} t_i \|e\|^2_{L^2(0,t_i)} + \frac{\bar{K}_2 L_4}{1-\gamma} \left\| \int_0^{\theta(t)} |e(s)| ds \right\|^2_{L^2(0,t_n)}.
\end{aligned}$$

Combining the facts that

$$\sum_{i=1}^{n-1} k_{i+1} t_i \|e\|^2_{L^2(0,t_i)} \le t_{n-1} \|e\|^2_{L^2(0,t_{n-1})} \sum_{i=1}^{n-1} k_{i+1} \le t_n^2 \|e\|^2_{L^2(0,t_{n-1})},$$

$$\|e(\theta(t))\|^2_{L^2(0,t_n)} \le \frac{1}{q_0} \int_0^{\theta(t_n)} |e(s)|^2 ds \le \frac{1}{q_0} \|e\|^2_{L^2(0,t_n)}, \tag{38}$$

$$\begin{aligned}
\left\| \int_0^{\theta(t)} |e(s)| ds \right\|^2_{L^2(0,t_n)} \quad & \le \quad \int_0^{t_n} \theta(t) \Big(\int_0^{\theta(t)} |e(s)|^2 ds \Big) dt \\
& \le \quad \int_0^{t_n} t \Big(\int_0^{\theta(t)} |e(s)|^2 ds \Big) dt \\
& \le \quad \frac{t_n^2}{2} \int_0^{\theta(t_n)} |e(s)|^2 ds \le \frac{t_n^2}{2} \|e\|^2_{L^2(0,t_n)},
\end{aligned} \tag{39}$$

and applying Lemma 3.3 to (37) yields

$$\begin{aligned}
|\xi(t_n)|^2 \quad \le \quad & \exp\left(\frac{3L_1 + L_2 + \dfrac{3\bar{K}_1 L_3 k}{\sqrt{2}} + \bar{K}_1 L_3 + \bar{K}_2 L_4}{1-\gamma} \sum_{i=1}^{n-1} k_{i+1} \right) \\
& \cdot \left(\frac{2\Big(L_1 + \dfrac{\bar{K}_1 L_3 k}{\sqrt{2}}\Big)}{1-\gamma} \|\eta\|^2_{L^2(0,t_n)} + \frac{L_3}{q_0(1-\gamma)} \|e\|^2_{L^2(0,t_n)} \right. \\
& \left. + \frac{\bar{K}_1 L_3 t_n^2}{1-\gamma} \|e\|^2_{L^2(0,t_{n-1})} + \frac{\bar{K}_2 L_4 t_n^2}{2(1-\gamma)} \|e\|^2_{L^2(0,t_n)} \right) \\
\le \quad & C e^{C t_n} \Big(\|\eta\|^2_{L^2(0,t_n)} + \|e\|^2_{L^2(0,t_n)} \Big),
\end{aligned} \tag{40}$$

where the constant $C > 0$ depends on $q_0, \bar{K_1}, \bar{K_2}, L_1, L_2, L_3, L_4, \gamma$ and t_n. Inserting (40) into (35), and then using the estimates (38) and (39), we obtain

$$
\begin{aligned}
\|\xi\|^2_{L^2(I_n)} \quad \leq \quad & Ce^{Ct_n}k_n\Big(\|\eta\|^2_{L^2(0,t_n)} + \|e\|^2_{L^2(0,t_n)}\Big) \\
& +Ck_n\|\eta\|^2_{L^2(I_n)} + Ck_n\|e(\theta(t))\|^2_{L^2(I_n)} \\
& +Ck_n^2 t_{n-1}\|e\|^2_{L^2(0,t_{n-1})} + Ck_n\Big\|\int_0^{\theta(t)}|e(s)|ds\Big\|^2_{L^2(I_n)} \\
\leq \quad & Ck_n\|\eta\|^2_{L^2(0,t_n)} + Ck_n\|\xi\|^2_{L^2(0,t_n)}.
\end{aligned} \tag{41}
$$

Assume that k_n is sufficiently small, then (41) can be rewritten as

$$
\|\xi\|^2_{L^2(I_n)} \quad \leq \quad Ck_n\|\eta\|^2_{L^2(0,t_n)} + Ck_n\|\xi\|^2_{L^2(0,t_{n-1})},
$$

or equivalently,

$$
\frac{\|\xi\|^2_{L^2(I_n)}}{k_n} \leq C\|\eta\|^2_{L^2(0,t_n)} + C\sum_{i=1}^{n-1} k_i \frac{\|\xi\|^2_{L^2(I_i)}}{k_i}. \tag{42}
$$

Then, we apply Lemma 3.3 to (42) get

$$
\frac{\|\xi\|^2_{L^2(I_n)}}{k_n} \leq C\|\eta\|^2_{L^2(0,t_n)} \exp\Big(C\sum_{i=1}^{n-1} k_i\Big) \leq Ce^{Ct_{n-1}}\|\eta\|^2_{L^2(0,t_n)},
$$

which leads to

$$
\|\xi\|^2_{L^2(I_n)} \leq Ck_n\|\eta\|^2_{L^2(0,t_n)}. \tag{43}
$$

Summing up (43) over all element $I_i, 1 \leq i \leq n$, gives

$$
\|\xi\|^2_{L^2(0,t_n)} \leq C\sum_{i=1}^{n} k_i\|\eta\|^2_{L^2(0,t_i)} \leq C\|\eta\|^2_{L^2(0,t_n)}\sum_{i=1}^{n} k_i \leq Ct_n\|\eta\|^2_{L^2(0,t_n)}.
$$

This completes the proof of (29).

By choosing $\varphi = \xi'$ in (28) we find that

$$
\begin{aligned}
\|\xi'\|^2_{L^2(I_n)} \quad \leq \quad & L_1\|e\|_{L^2(I_n)}\|\xi'\|_{L^2(I_n)} + L_2\|e(\theta(t))\|_{L^2(I_n)}\|\xi'\|_{L^2(I_n)} \\
& +\bar{K_1}L_3\Big\|\int_0^t|e(s)|ds\Big\|_{L^2(I_n)}\|\xi'\|_{L^2(I_n)} \\
& +\bar{K_2}L_4\Big\|\int_0^{\theta(t)}|e(s)|ds\Big\|_{L^2(I_n)}\|\xi'\|_{L^2(I_n)},
\end{aligned}
$$

which implies

$$
\begin{aligned}
|\xi|_{H^1(I_n)} \quad \leq \quad & L_1\|e\|_{L^2(I_n)} + L_2\|e(\theta(t))\|_{L^2(I_n)} + \bar{K_1}L_3\Big\|\int_0^t|e(s)|ds\Big\|_{L^2(I_n)} \\
& +\bar{K_2}L_4\Big\|\int_0^{\theta(t)}|e(s)|ds\Big\|_{L^2(I_n)}.
\end{aligned}
$$

Iterating this estimate, then using (38), (39), and (29) we conclude that

$$
\begin{aligned}
|\xi|^2_{H^1(0,t_n)} &\le C\bigg(\|e\|^2_{L^2(0,t_n)} + \|e(\theta(t))\|^2_{L^2(0,t_n)} + \Big\|\int_0^t |e(s)|ds\Big\|^2_{L^2(0,t_n)} \\
&\quad + \Big\|\int_0^{\theta(t)} |e(s)|ds\Big\|^2_{L^2(0,t_n)}\bigg) \\
&\le C\|e\|^2_{L^2(0,t_n)} \le C\|\xi\|^2_{L^2(0,t_n)} + C\|\eta\|^2_{L^2(0,t_n)} \\
&\le C\|\eta\|^2_{L^2(0,t_n)},
\end{aligned}
$$

which implies (30). Here, we have used the fact that

$$
\Big\|\int_0^t |e(s)|ds\Big\|^2_{L^2(0,t_n)} \le \int_0^{t_n} t\Big(\int_0^t |e(s)|^2 ds\Big)dt \le \frac{t_n^2}{2}\|e\|^2_{L^2(0,t_n)}. \tag{44}
$$

Finally, combining (40) and (29) we obtain

$$
|\xi(t_n)|^2 \le Ce^{Ct_n}\Big(\|\eta\|^2_{L^2(0,t_n)} + \|\xi\|^2_{L^2(0,t_n)}\Big) \le C\|\eta\|^2_{L^2(0,t_n)}.
$$

This ends the proof of (31). □

We next bound the derivative of ξ as follows.

Lemma 3.5. *For $1 \le n \le N$, there holds*

$$
\int_{I_n} |\xi'|^2(t-t_{n-1})dt \le Ck_n\|\eta\|^2_{L^2(0,t_n)}, \tag{45}
$$

where the constant $C > 0$ solely depends on $q_0, L_1, L_2, L_3, L_4, \bar{K_1}, \bar{K_2}$, and t_n.

Proof. By selecting $\varphi = \Pi^{r_n-1}((t-t_{n-1})\xi')$ in (28), we deduce that

$$
\begin{aligned}
&\int_{I_n} (t-t_{n-1})|\xi'|^2 dt \\
&= \int_{I_n} (f(t,u(t),u(\theta(t))) - f(t,U(t),U(\theta(t))))\Pi^{r_n-1}((t-t_{n-1})\xi')dt \\
&\quad + \int_{I_n} (\mathcal{V}u - \mathcal{V}U)\Pi^{r_n-1}((t-t_{n-1})\xi')dt \\
&\quad + \int_{I_n} (\mathcal{V}_\theta u - \mathcal{V}_\theta U)\Pi^{r_n-1}((t-t_{n-1})\xi')dt \\
&\le L_1\int_{I_n} |e|\cdot|\Pi^{r_n-1}((t-t_{n-1})\xi')|dt \\
&\quad + L_2\int_{I_n} |e(\theta(t))|\cdot|\Pi^{r_n-1}((t-t_{n-1})\xi')|dt \\
&\quad + \bar{K_1}L_3\int_{I_n}\Big(\int_0^t |e(s)|ds\Big)|\Pi^{r_n-1}((t-t_{n-1})\xi')|dt \\
&\quad + \bar{K_2}L_4\int_{I_n}\Big(\int_0^{\theta(t)} |e(s)|ds\Big)|\Pi^{r_n-1}((t-t_{n-1})\xi')|dt \\
&:= A_1 + A_2 + A_3 + A_4.
\end{aligned} \tag{46}
$$

Thanks to the Cauchy-Schwarz inequality and the L^2-stability of Π^{r_n-1}, we have

$$
\begin{aligned}
A_1 &\le L_1\|e\|_{L^2(I_n)}\|(t-t_{n-1})\xi'\|_{L^2(I_n)} \\
&\le L_1 k_n^{\frac{1}{2}}\|e\|_{L^2(I_n)}\Big\{\int_{I_n}(t-t_{n-1})|\xi'|^2dt\Big\}^{\frac{1}{2}}.
\end{aligned} \tag{47}
$$

Similarly, by (38), (44) and (39) , we readily find that

$$
\begin{aligned}
A_2 &\leq L_2\|e(\theta(t))\|_{L^2(I_n)}\|(t-t_{n-1})\xi'\|_{L^2(I_n)} \\
&\leq \frac{L_2}{\sqrt{q_0}}k_n^{\frac{1}{2}}\|e\|_{L^2(0,t_n)}\Big\{\int_{I_n}(t-t_{n-1})|\xi'|^2dt\Big\}^{\frac{1}{2}},
\end{aligned}
\tag{48}
$$

$$
\begin{aligned}
A_3 &\leq \bar{K_1}L_3\left\|\int_0^t|e(s)|ds\right\|_{L^2(I_n)}\|(t-t_{n-1})\xi'\|_{L^2(I_n)} \\
&\leq \frac{\bar{K_1}L_3t_n}{\sqrt{2}}k_n^{\frac{1}{2}}\|e\|_{L^2(0,t_n)}\Big\{\int_{I_n}(t-t_{n-1})|\xi'|^2dt\Big\}^{\frac{1}{2}},
\end{aligned}
\tag{49}
$$

and

$$
\begin{aligned}
A_4 &\leq \bar{K_2}L_4\left\|\int_0^{\theta(t)}|e(s)|ds\right\|_{L^2(I_n)}\|(t-t_{n-1})\xi'\|_{L^2(I_n)} \\
&\leq \frac{\bar{K_2}L_4t_n}{\sqrt{2}}k_n^{\frac{1}{2}}\|e\|_{L^2(0,t_n)}\Big\{\int_{I_n}(t-t_{n-1})|\xi'|^2dt\Big\}^{\frac{1}{2}}.
\end{aligned}
\tag{50}
$$

Hence, combing (46)-(50) and (29) yields

$$
\begin{aligned}
\Big\{\int_{I_n}|\xi'|^2(t-t_{n-1})dt\Big\}^{\frac{1}{2}} &\leq Ck_n^{\frac{1}{2}}\|e\|_{L^2(0,t_n)} \leq Ck_n^{\frac{1}{2}}(\|\eta\|_{L^2(0,t_n)}+\|\xi\|_{L^2(0,t_n)}) \\
&\leq Ck_n^{\frac{1}{2}}\|\eta\|_{L^2(0,t_n)}.
\end{aligned}
$$

This implies the assertion. □

We also need the following inverse inequality (cf. [9]).

Lemma 3.6. *On each interval I_n there holds*

$$
\|\varphi\|^2_{L^\infty(I_n)} \leq C\left(\log(r_n+1)\int_{I_n}|\varphi'(t)|^2(t-t_{n-1})dt+|\varphi(t_n)|^2\right)
$$

for any $\varphi \in P_{r_n}(I_n)$, where the constant $C>0$ is independent of k_n and r_n. Moreover, the estimate cannot be improved asymptotically as $r_n \to \infty$.

The following results state abstract error bounds of the CPG method.

Theorem 3.2. *Let u be the exact solution of* (1) *and U be the h-p CPG approximation defined by* (2)*. For k sufficiently small, there holds*

$$
\|u-U\|_{L^2(I)} \leq C\|u-\mathcal{I}u\|_{L^2(I)}, \tag{51}
$$

$$
|u-U|_{H^1(I)} \leq C\|u-\mathcal{I}u\|_{H^1(I)}, \tag{52}
$$

and

$$
\|u-U\|_{L^\infty(I)} \leq C\left(1+k\log(r+1)\right)^{\frac{1}{2}}\|u-\mathcal{I}u\|_{L^\infty(I)}, \tag{53}
$$

where $r=\max\{r_n\}_{n=1}^N$ and the constants $C>0$ solely depend on $q_0, L_1, L_2, L_3, L_4, \bar{K_1}, \bar{K_2}$, and T.

Proof. With the aid of (29) we get

$$
\|u-U\|_{L^2(I)} \leq \|\eta\|_{L^2(I)}+\|\xi\|_{L^2(I)} \leq C\|\eta\|_{L^2(I)},
$$

which completes the proof of (51).

Similarly, by (30) we obtain

$$
|u-U|_{H^1(I)} \leq |\eta|_{H^1(I)}+|\xi|_{H^1(I)} \leq |\eta|_{H^1(I)}+C\|\eta\|_{L^2(I)} \leq C\|\eta\|_{H^1(I)},
$$

which implies (52).

By using (31) and employing Lemmas 3.5 and 3.6, we deduce that

$$\begin{aligned}
&\|u-U\|^2_{L^\infty(I)}\\
\le\;& 2\|\eta\|^2_{L^\infty(I)}+2\max_{1\le n\le N}\|\xi\|^2_{L^\infty(I_n)}\\
\le\;& 2\|\eta\|^2_{L^\infty(I)}+C\max_{1\le n\le N}\Big\{\log(r_n+1)\int_I|\xi'(t)|^2(t-t_{n-1})dt+|\xi(t_n)|^2\Big\}\\
\le\;& 2\|\eta\|^2_{L^\infty(I)}+C\max_{1\le n\le N}\Big\{k_n\log(r_n+1)\|\eta\|^2_{L^2(0,t_n)}+\|\eta\|^2_{L^2(0,t_n)}\Big\}\\
\le\;& 2\|\eta\|^2_{L^\infty(I)}+Ck\log(r+1)\|\eta\|^2_{L^2(I)}+C\|\eta\|^2_{L^2(I)}\\
\le\;& C\Big(1+k\log(r+1)\Big)\|\eta\|^2_{L^\infty(I)}.
\end{aligned}$$

Here, we have used the fact that $\|\eta\|^2_{L^2(I)}\le T\|\eta\|^2_{L^\infty(I)}$. This completes the proof of (53). □

3.3. Global L^2, H^1, and L^∞-error estimates. We are now in a position to present our main result.

Theorem 3.3. *Let $\mathcal{T}_h$ be any mesh in I, u be the exact solution of* (1) *and U be the h-p CPG approximation defined by* (2). *We assume that $u\in H^1(I)$ satisfies $u|_{I_n}\in H^{s_{0,n}+1}(I_n)$ for $s_{0,n}\ge 0$. Then, for k sufficiently small, there holds*

$$\|u-U\|^2_{L^2(I)}\le C\sum_{n=1}^N\big(\frac{k_n}{2}\big)^{2s_n+2}\frac{\Gamma(r_n+1-s_n)}{r_n(r_n+1)\Gamma(r_n+1+s_n)}\|u\|^2_{H^{s_n+1}(I_n)},$$

$$|u-U|^2_{H^1(I)}\le C\sum_{n=1}^N\big(\frac{k_n}{2}\big)^{2s_n}\frac{\Gamma(r_n+1-s_n)}{\Gamma(r_n+1+s_n)}\|u\|^2_{H^{s_n+1}(I_n)}$$

for any real s_n, $0\le s_n\le\min\{r_n,s_{0,n}\}$.

Moreover, if $u\in H^1(I)$ satisfies $u|_{I_n}\in W^{s_{0,n}+1,\infty}(I_n)$ for $s_{0,n}\ge 0$, there holds

$$\begin{aligned}
&\|u-U\|^2_{L^\infty(I)}\\
\le\;& C\left(1+k\log(r+1)\right)\max_{1\le n\le N}\Big\{\big(\tfrac{k_n}{2}\big)^{2s_n+2}\frac{\Gamma(r_n+1-s_n)}{\Gamma(r_n+1+s_n)}\|u\|^2_{W^{s_n+1,\infty}(I_n)}\Big\}
\end{aligned}$$

for any real s_n, $0\le s_n\le\min\{r_n,s_{0,n}\}$.

Proof. The assertions follow readily from Theorem 3.2 and Lemma 3.2. □

Remark 3.1. These estimates show that the error bounds are explicit with respect to the time steps k_n, the approximation order r_n, and the regularity of the exact solution s_n.

From the error bounds in Theorem 3.3, the following convergence rates can be obtained for the h- and p-version of the CPG method.

Corollary 3.1. *Let $r_n=r, 1\le n\le N$ and $\mathcal{T}_h$ be a quasi-uniform mesh in I. If $u\in H^{s+1}(I)$ for $s\ge 0$, then*

$$\|u-U\|_{L^2(I)}\le C\frac{k^{\min\{s,r\}+1}}{r^{s+1}}\|u\|_{H^{s+1}(I)},$$

$$|u-U|_{H^1(I)}\le C\frac{k^{\min\{s,r\}}}{r^{s}}\|u\|_{H^{s+1}(I)}.$$

Moreover, if $u \in W^{s+1,\infty}(I)$, *there holds*

$$\|u - U\|_{L^\infty(I)} \le C\,(1 + k\log(r+1))^{\frac{1}{2}}\,\frac{k^{\min\{s,r\}+1}}{r^s}\|u\|_{W^{s+1,\infty}(I)}.$$

Proof. The assertions follows from Theorem 3.3 and Stirling's formula. □

Remark 3.2. These estimates show that the h-p version CPG method converges either as the time step k is decreased or as the polynomial degrees r is increased. Moreover, the p-version (with fixed time partition) can yields arbitrarily high-order algebraic convergence rates (i.e., spectral convergence) if the solution u is smooth enough. Moreover, it can be proved that the p-version converges exponentially if u is analytic on $[0, T]$ (see, for instance, [11]).

4. Numerical experiments

In this section, we illustrate the performance of the h-p version of the CPG method for the following VFIDE:

$$(54)\quad \begin{cases} u'(t) &= g(t) + e^{-u(t)} + e^{-t}e^{-u(\theta(t))} + \int_0^t e^{s-t}(u(s) + e^{-u(s)})ds \\ & \quad + \int_0^{\theta(t)} e^{s-t}(u(s) + e^{-u(s)})ds, \qquad t \in [0,1], \\ u(0) &= 1, \end{cases}$$

with $g(t) = -\ln(t+e) + 2e^{-t} - e^{\theta(t)-t}\ln(\theta(t)+e) - \dfrac{e^{-t}}{\theta(t)+e}$ and $\theta(t) = \frac{4}{5}\sin(t)$ such that the exact solution $u(t) = \ln(t+e)$.

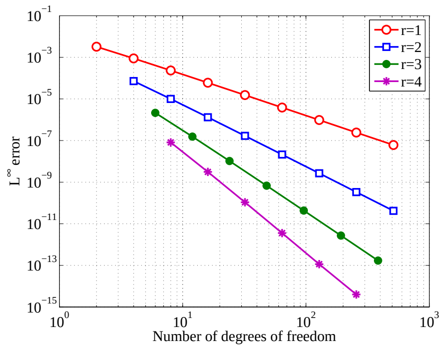

FIGURE 1. L^∞-errors of the h-version.

We begin by considering the behaviour of the h-version of the CPG method on uniform time partitions for problem (54). The L^∞-errors are shown in Fig. 1. Obviously, the straight error curves correspond to algebraic convergence in the step-size k, for each polynomial degree r. Moreover, we list the L^2-, H^1-(seminorm), and L^∞-errors of the h-version CPG method in Table 1, the convergence rates confirm the sharpness prediction in Corollary 3.1.

In Fig. 2, we present the L^∞-errors of the p-version of the CPG method. The results show that exponential rates of convergence are achieved for each fixed uniform time partitions. In addition, we note that the global L^∞-error of 10^{-15} can be obtained with less than 15 degrees of freedom for the p-version. However, this is not the case for the h-version as shown in Fig. 1. This implies that, for smooth

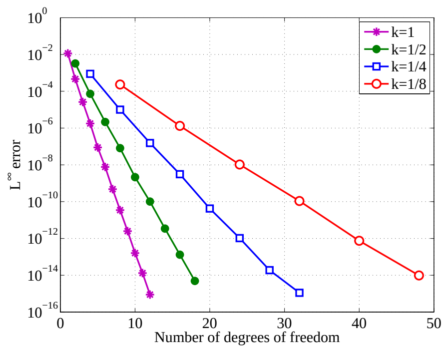

FIGURE 2. L^∞-errors of the p-version.

TABLE 1. Numerical errors and convergence rates of the h-version.

degree r	step-size k	L^2-errors	order	H^1-errors	order	L^∞-errors	order
	1/128	5.00 E-07	2.00	2.27 E-04	1.00	9.70 E-07	1.99
1	1/256	1.25 E-07	2.00	1.13 E-04	1.00	2.43 E-07	2.00
	1/512	3.12 E-08	2.00	5.67 E-05	1.00	6.08 E-08	2.00
	1/64	1.43 E-09	3.00	5.94 E-07	2.00	2.67 E-9	2.99
2	1/128	1.79 E-10	3.00	1.49 E-07	2.00	3.35 E-10	3.00
	1/256	2.24 E-11	3.00	3.71 E-08	2.00	4.19 E-11	3.00
	1/32	2.04 E-11	4.00	6.20 E-09	3.00	4.31 E-11	3.97
3	1/64	1.28 E-12	4.00	7.76 E-10	3.00	2.72 E-12	3.99
	1/128	7.98 E-14	4.00	9.69 E-11	3.00	1.70 E-13	4.00

solution it is advantageous to increase r and keep k fixed (p-version of the CPG method) rather than to reduce k for r fixed (h-version of the CPG method).

5. Concluding Remarks

In this paper, we have presented an h-p version of the CPG method for the nonlinear VFIDEs with vanishing delays. We have proved that the CPG scheme is well-defined as long as the time steps are sufficiently small. Moreover, we have obtained a priori error bounds in the L^2-, H^1- and L^∞-norms that are explicit with respect to the local time steps, the local approximation orders, and the local regularity of the exact solutions. Extensions of the analysis presented herein to the h-p version of the CPG method might be possible for VFIDEs with weakly singular kernels by following along the lines of this paper, in conjunction with our recent work [20] for Volterra integro-differential equations with weakly singular kernels. This will be a topic for our future research.

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