

## CORRIGENDUM: A POSTERIORI ERROR ESTIMATION FOR NON-CONFORMING QUADRILATERAL FINITE ELEMENTS

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**Abstract.** An invalid assumption on the equivalence of alternative sets of degrees of freedom for the rotated  $\mathbb{Q}_1$  element led to an incompatibility between the analysis and the application in our earlier article [1]. We outline minor modifications needed to restore the validity of all results and conclusions in the previous article.

**Key Words.**

### 1. Introduction

In an earlier article [1], we presented an a posteriori error bound for finite element approximation using the non-conforming rotated  $\mathbb{Q}_1$  element. This element may be viewed as a triple  $(S, P, \Sigma)$ , where  $S$  is the reference square,  $P$  is the approximation space  $\{1, \hat{x}, \hat{y}, \hat{x}^2 - \hat{y}^2\}$  and  $\Sigma$  is a unisolvent set of degrees of freedom that may be chosen in two distinct ways: either **(C)** function evaluation at the midpoints of the sides of  $S$ , or **(C')** line integrals over the sides of  $S$ . However, as pointed out in [2], the actual finite element approximation is *not* the same for both sets of degrees of freedom<sup>1</sup>! This means that the analysis in [1] is not applicable in case **(C)** but, after some minor modifications which we outline below, is applicable in case **(C')**.

### 2. Modifications

The spaces  $X_{\mathcal{P}}$  and  $X_{\mathcal{P},E}$  appearing in [1, p. 4] should be changed to read

$$X_{\mathcal{P}} = \left\{ v : \Omega \rightarrow \mathbb{R} : v|_K \circ \mathbf{F}_K \in P \quad \forall K \in \mathcal{P}, \quad \int_{\gamma} [v] ds = 0 \quad \forall \gamma \in \partial\mathcal{P} \setminus \partial\Omega \right\}$$

where  $[v]$  denotes the jump across an interface, with the subspace  $X_{\mathcal{P},E}$  defined by

$$X_{\mathcal{P},E} = \left\{ v \in X_{\mathcal{P}} : \int_{\gamma} v ds = 0 \text{ for } \gamma \subset \Gamma_D \right\}.$$

With this modification, the range of the operator  $\Pi_{\mathcal{P}}$  defined in [1, eq.(6)] is the space  $X_{\mathcal{P},E}$ . This would not be the case with the degrees of freedom chosen as in **(C)** and is the reason why the previous analysis is not applicable to **(C)**. The remaining analysis in [1] is then correct as written as far as [1, eq.(45)] which should be replaced by

$$u^*(\mathbf{m}_{\gamma}) = \begin{cases} 0, & \text{if } \mathbf{m}_{\gamma} \in \Gamma_D \\ h_{\gamma}^{-1} \int_{\gamma} u_{\mathcal{P}} ds, & \text{otherwise.} \end{cases}$$

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The choice of degrees of freedom according to  $(\mathbf{C}')$  means that the values of the function  $u^*$  at the midpoints are well-defined by this formula. This modified definition means the estimate [1, eq.(51)] is now proved as follows. Firstly, thanks to the modified definition of  $X_{\mathcal{P},E}$ ,

$$|u_{\mathcal{P}}(\mathbf{x}_n)|_{K'} - u_{\mathcal{P}}(\mathbf{x}_n)|_K| = |[u_{\mathcal{P}}(\mathbf{x}_n)]| = \left| [u_{\mathcal{P}}(\mathbf{x}_n)] - h_{\gamma}^{-1} \int_{\gamma} [u_{\mathcal{P}}] ds \right|$$

and then, using the following identity

$$w(\mathbf{x}_n) - h_{\gamma}^{-1} \int_{\gamma} w ds = h_{\gamma}^{-1} \int_{\gamma} ds \int_s^{\mathbf{x}_n} d\tau \frac{\partial w}{\partial \tau},$$

along with a Cauchy-Schwarz inequality, we derive the estimate

$$\left| [u_{\mathcal{P}}(\mathbf{x}_n)] - h_{\gamma}^{-1} \int_{\gamma} [u_{\mathcal{P}}] ds \right| \leq Ch_{\gamma}^{1/2} \|J_{\gamma}^{\tau}\|_{\gamma}$$

which gives the same estimate as [1, eq.(51)]. A similar modification is needed to obtain estimate [1, eq.(52)].

Finally, the definition of the function  $g_K$  on an interior edge  $\gamma$ , given on Page 6, should be should be modified to read

$$g_K = \begin{cases} \frac{1}{2|\gamma|} \int_{\gamma} \mathbf{n}_K \cdot (a_K \mathbf{grad}_{\mathcal{P}} u_{\mathcal{P}}|_K + a_{K'} \mathbf{grad}_{\mathcal{P}} u_{\mathcal{P}}|_{K'}) ds & \text{on } \gamma = \partial K \cap \partial K' \\ \frac{1}{|\gamma|} \int_{\gamma} \mathbf{n}_K \cdot a_K \mathbf{grad}_{\mathcal{P}} u_{\mathcal{P}}|_K ds & \text{on } \gamma \subset \Gamma_D \end{cases}$$

which then leads to equation (19) holding with the jump residual  $J^{\nu}$  defined on the boundary of element  $K$  by

$$-\frac{1}{2} J_{|\gamma}^{\nu} = g_K - \mathbf{n}_K \cdot a_K \mathbf{grad}_{\mathcal{P}} u_{\mathcal{P}}|_K.$$

The remainder of the analysis in [1] then applies without further modification.

In summary, with these modifications, all analysis and conclusions given in the previous work [1] are correct.

## References

- [1] Mark Ainsworth. A posteriori error estimation for non-conforming quadrilateral finite elements. *Int. J. Numer. Anal. Model.*, 2(1):1–18, 2005.
- [2] Carsten Carstensen. Personal communication. 2006.

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