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ON THE INFLUENCE OF THE WAVENUMBER ON COMPRESSION IN A WAVELET BOUNDARY ELEMENT METHOD FOR THE HELMHOLTZ EQUATION

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Abstract. We examine how the wavenumber influences the compression in a wavelet boundary element method for the Helmholtz equation. We show that for wavelets with high vanishing moments the number of nonzeros in the resulting compressed matrix is approximately proportional to the square of the wavenumber. When the wavenumber is fixed, the wavelet boundary element method has optimal complexity with respect to the number of unknowns. When the mesh spacing is proportional to the wavelength, the complexity of the wavelet boundary element method is approximately proportional to the square of the number of unknowns.

Key Words. wavelets, boundary element method, and Helmholtz equation.

1. Introduction

Techniques based on boundary integral equations (BIEs) are attractive for three dimensional partial differential equations (PDEs) because they are defined on a two dimensional surface rather than a three dimensional domain. Two dimensional problems can typically be solved with many fewer unknowns than three dimensional problems. BIE formulations are particularly attractive when the three dimensional domain is infinite, for example in an exterior problem.

Traditional boundary element methods (BEMs) for solving integral equations are very effective but have $O(N^2)$ complexity and require $O(N^2)$ storage, where N is the number of unknowns. There is much interest in fast solution techniques for BIEs. Prominent fast techniques include the fast multipole method (FMM) [16, 9], equivalent sources [4, 3], spectral basis functions [1, 8], and wavelets [2, 7, 15, 17, 14, 6].

The use of wavelets for the compression of operators was pioneered by Beylkin, Coifman and Rokhlin [2], who showed that operators of Calderon-Zygmund type can be represented, to a specified accuracy, by a sparse matrix. Dahmen, Prössdorf and Schneider [7] linked the accuracy to the discretisation error. Later developments by Rathsfeld [15], von Petersdorff and Schwab [17], and Lage and Schwab [14] yielded sophisticated discretisation schemes with (a) complexity $O(N \log^a N)$ for some small integer a, and (b) asymptotic order of convergence that is not changed by the compression. The wavelet Galerkin scheme of Dahmen, Harbrecht and Schneider [6] has the optimal complexity $O(N)$. For the Laplace problem these wavelet schemes have been successfully demonstrated. In contrast, for the Helmholtz problem wavelet schemes have not been widely used. The oscillatory nature of the kernel

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in BIEs for the Helmholtz equation is considered an obstacle to compression using wavelets.

Wagner and Chew [18] showed that their wavelet discretisation scheme for the related electromagnetic scattering problem yields a compressed matrix with $O(N^2)$ nonzeros. Thus wavelets are not useful in that context. However, in that study the number of unknowns was tied to the wavenumber k so that the number of nodes per wavelength remained constant. Huybrechs, Simoens and Vandewalle [13] showed that, for the two dimensional Helmholtz problem, the wavelet Galerkin scheme has approximate complexity $O(kN)$. When the number of unknowns is tied to the wavenumber this becomes $O(N^2)$ and is in agreement with [18].

Here we consider application of the wavelet Galerkin scheme in [6, 10] for the three dimensional Helmholtz problem. We analyse the wavenumber dependence of the compression obtained with the wavelets so that we can assess how useful the wavelet discretisation scheme is in this context. We explicitly consider several integral operators, including the single and double layer potentials and the hypersingular operator. We show that there are approximately $O(k^2N)$ nonzeros in the compressed matrix. The exact exponent of k depends on the order of the integral operator considered and on the vanishing moments of the wavelets.

Practitioners might increase the number of unknowns N to (a) maintain the number of nodes per wavelength as k increases; or (b) increase the accuracy of the solution for fixed k. We show that in case (a) the wavelet Galerkin scheme has approximate complexity $O(N^2)$, which is in agreement with Wagner and Chew [18]. In contrast, in case (b) the wavelet Galerkin scheme has optimal complexity $O(N)$, which is the complexity observed for the Laplace problem [6, 10].

We write $a(k,N) = O(b(k,N))$ when there exist constants $C, M \geq 0$ such that $a(k,N) \leq Cb(k,N)$ for all $N > M$. We write $a(k,N) \lesssim b(k,N)$ when there exists constant $C > 0$ such that $a(k,N) \leq Cb(k,N)$ for all N. We write $a(k,N) \sim b(k,N)$ when $a(k,N) \lesssim b(k,N)$ and $b(k,N) \lesssim a(k,N)$.

In Section 2 we state the Laplace and Helmholtz problems and review the relevant boundary integral operators for these problems. In Section 3 we review the relevant details of the wavelet Galerkin scheme [6, 10]. In Section 4 we review the compression theory for the wavelet Galerkin scheme applied to the Laplace problem. In Section 5 we state the compression theory for the wavelet Galerkin scheme applied to the Helmholtz problem. In Section 6 we present numerical results.

2. Integral Operators and Equations

We consider here the exterior Laplace and Helmholtz problems. Both problems can be reformulated as boundary integral equations and we list the component boundary integral operators. The treatment for the interior problem is similar, and the same boundary integral operators appear.

Let $D \subseteq \mathbb{R}^3$ be a bounded region with surface Γ and exterior $D_+ = \mathbb{R}^3 \setminus (D \cup \Gamma)$. The exterior Laplace equation is

$$\nabla^2 u(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in D_+ \cup \Gamma. \tag{1}$$

The single and double layer potentials for the Laplace equation are

$$(\mathcal{L}u)(\boldsymbol{x}) = \int_\Gamma G(\boldsymbol{x},\boldsymbol{y})u(\boldsymbol{y})ds_{\boldsymbol{y}}, \quad (\mathcal{M}u)(\boldsymbol{x}) = \int_\Gamma \frac{\partial G}{\partial \boldsymbol{n}_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y})u(\boldsymbol{y})ds_{\boldsymbol{y}},$$

where $G(\boldsymbol{x},\boldsymbol{y}) = 1/4\pi\|\boldsymbol{x}-\boldsymbol{y}\|$ is the free space Green's function for the Laplace equation.

Differentiating leads to the operators

$$(\mathcal{M}^T u)(\boldsymbol{x}) = \frac{\partial}{\partial \boldsymbol{n}_{\boldsymbol{x}}}(\mathcal{L}u)(\boldsymbol{x}), \quad (\mathcal{N}u)(\boldsymbol{x}) = \frac{\partial}{\partial \boldsymbol{n}_{\boldsymbol{x}}}(\mathcal{M}u)(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma.$$

The operators $\mathcal{L}, \mathcal{M}, \mathcal{M}^T$ and $\mathcal{N}$ appear in several boundary integral equations for (1).

The exterior Helmholtz equation with wavenumber $k > 0$ is

$$\nabla^2 u(\boldsymbol{x}) + k^2 u(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in D_+ \cup \Gamma. \tag{2}$$

Radiating solutions of (2) additionally satisfy the Sommerfeld radiation condition

$$\lim_{|\boldsymbol{x}|\to\infty} |\boldsymbol{x}| \left(\frac{\partial u}{\partial \boldsymbol{x}} - iku \right) = 0.$$

The single and double layer potentials for the Helmholtz equation are

$$(\mathcal{L}_k u)(\boldsymbol{x}) = \int_\Gamma G_k(\boldsymbol{x}, \boldsymbol{y}) u(\boldsymbol{y}) ds_{\boldsymbol{y}}, \quad (\mathcal{M}_k u)(\boldsymbol{x}) = \int_\Gamma \frac{\partial G_k}{\partial \boldsymbol{n}_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y}) u(\boldsymbol{y}) ds_{\boldsymbol{y}},$$

where $G_k(\boldsymbol{x}, \boldsymbol{y}) = e^{ik\|\boldsymbol{x}-\boldsymbol{y}\|}/4\pi\|\boldsymbol{x} - \boldsymbol{y}\|$ is the free space Green's function for the Helmholtz equation.

Differentiating leads to the operators

$$(\mathcal{M}_k^T u)(\boldsymbol{x}) = \frac{\partial}{\partial \boldsymbol{n}_{\boldsymbol{x}}}(\mathcal{L}_k u)(\boldsymbol{x}), \quad (\mathcal{N}_k u)(\boldsymbol{x}) = \frac{\partial}{\partial \boldsymbol{n}_{\boldsymbol{x}}}(\mathcal{M}_k u)(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma.$$

The operators $\mathcal{L}_k, \mathcal{M}_k, \mathcal{M}_k^T$ and $\mathcal{N}_k$ appear in several boundary integral equations for (2). We consider the general BIE

$$\mathcal{A}u(\boldsymbol{x}) = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma,$$

where $\mathcal{A}$ involves one or more of the operators above, and f is derived from the boundary condition. Traditional boundary element methods seek an approximation to $u(\boldsymbol{x})$ in a space spanned by piecewise polynomial basis functions. We shall seek an approximation to $u(\boldsymbol{x})$ in a space spanned by wavelet basis functions.

3. The Wavelet Galerkin Scheme

In this section we review a robust wavelet Galerkin scheme of Dahmen, Harbrecht and Schneider [6]. In Section 4 we will review the theory for the application of this scheme to the Laplace problem. In Section 5 we will present theory for the application of this scheme to the Helmholtz problem.

Wavelets are easily defined on a simple domain such as the unit square, which can be used as a reference domain for computations on a general surface. In the context of operator compression in BIEs, the key property of wavelets is that they can transform smoothness of the kernel on the reference domain into small, or even negligible, coefficients. For BIEs defined on a general surface it can be difficult to smoothly map the reference domain to the surface. It is necessary to split the domain into patches so that the mappings from the reference domain to each patch are smooth.

We split Γ into patches with $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_N$ and each patch Γ_i the image of $\square = [0,1] \times [0,1]$ under some diffeomorphism $\boldsymbol{\gamma}_i$. Adjoining patches are required to match at their common boundaries, so that when $\Gamma_i \cap \Gamma_j \neq \emptyset$ there exists an affine mapping $\boldsymbol{\theta}_{ij} : \square \to \square$ such that, whenever $\boldsymbol{\xi} \in \Gamma_i \cap \Gamma_j$ then $\boldsymbol{\xi} = \boldsymbol{\gamma}_i(\boldsymbol{s})$ and $\boldsymbol{\xi} = \boldsymbol{\gamma}_j(\boldsymbol{\theta}_{ij}(\boldsymbol{s}))$ for some $\boldsymbol{s} \in \square$.

Define the inner product $\langle \cdot, \cdot \rangle$ by

$$\begin{aligned} \langle f, g \rangle &= \int_\Gamma f(\boldsymbol{x}) g(\boldsymbol{x}) ds_{\boldsymbol{x}} \\ &= \sum_{j=1}^N \int_\square f(\boldsymbol{\gamma}_j(\boldsymbol{u})) g(\boldsymbol{\gamma}_j(\boldsymbol{u})) \left\| \frac{\partial \boldsymbol{\gamma}_j}{\partial u_1}(\boldsymbol{u}) \times \frac{\partial \boldsymbol{\gamma}_j}{\partial u_2}(\boldsymbol{u}) \right\| ds_{\boldsymbol{u}}. \end{aligned}$$

Given a coarsest refinement level j_0 and a finest refinement level J, let

$$\begin{aligned} \Phi_{j_0} &= \{\phi_{j_0,k}\}_{k \in S_{j_0}}, \\ \Psi_j &= \{\psi_{j,k}\}_{k \in T_j}, \quad (j = j_0, \ldots, J) \end{aligned}$$

denote scaling functions and biorthogonal wavelets defined on Γ, where S_j and T_j denote index sets for scaling functions and wavelets respectively at level j. Denote by d the degree of approximation of the wavelets and by $\widetilde{d}$ the number of vanishing moments of the wavelets. For convenience write $\Psi_{j_0-1} = \Phi_{j_0}$, $T_{j_0-1} = S_{j_0}$ and $\psi_{j_0-1,k} = \phi_{j_0,k}$.

Let $\Theta_{j,k} = \operatorname{supp} \psi_{j,k}$ and $\widetilde{\Theta}_{j,k} = \operatorname{sing\,supp} \psi_{j,k}$. We assume $\operatorname{diam} \Theta_{j,k} \sim 2^{-j}$ for $j \geq j_0$. The basis used in the Galerkin scheme is $\Psi^J = \cup_{j=j_0-1}^J \Psi_j$. The number of unknowns is then $N_J = |\Psi^J|$. We have in mind the piecewise linear continuous biorthogonal wavelets used by Dahmen, Harbrecht and Schneider [6] and Harbrecht [10] for the Laplace equation.

Let $\mathcal{A}$ be an integral operator on Γ with order $2q$ and kernel K, so

$$(\mathcal{A}\phi)(\boldsymbol{x}) = \int_\Gamma K(\boldsymbol{x}, \boldsymbol{y}) \phi(\boldsymbol{y}) ds_{\boldsymbol{y}}.$$

The transported kernels of $\mathcal{A}$ are

$$\widetilde{K}_{ii'}(\boldsymbol{u}, \boldsymbol{v}) = K(\boldsymbol{\gamma}_i(\boldsymbol{u}), \boldsymbol{\gamma}_{i'}(\boldsymbol{v})) \left\| \frac{\partial \boldsymbol{\gamma}_i}{\partial u_1}(\boldsymbol{u}) \times \frac{\partial \boldsymbol{\gamma}_i}{\partial u_2}(\boldsymbol{u}) \right\| \left\| \frac{\partial \boldsymbol{\gamma}_{i'}}{\partial v_1}(\boldsymbol{v}) \times \frac{\partial \boldsymbol{\gamma}_{i'}}{\partial v_2}(\boldsymbol{v}) \right\|,$$

for $1 \leq i, i' \leq N$. We use the shorthand $\partial_{\boldsymbol{u}}^\alpha$ in place of $\partial_{u_1}^{\alpha_1} \partial_{u_2}^{\alpha_2}$ and write $|\alpha| = \alpha_1 + \alpha_2$.

Definition 1. *The kernel K is said to be analytically standard of order r if its transported kernels $\widetilde{K}_{ii'}$ $(1 \leq i, i' \leq N)$ satisfy*

$$|\partial_{\boldsymbol{u}}^\alpha \partial_{\boldsymbol{v}}^\beta \widetilde{K}_{ii'}(\boldsymbol{u}, \boldsymbol{v})| \leq C \frac{1}{\|\boldsymbol{\gamma}_i(\boldsymbol{u}) - \boldsymbol{\gamma}_{i'}(\boldsymbol{v})\|_2^{2+r+|\alpha|+|\beta|}} \tag{3}$$

for $2 + r + |\alpha| + |\beta| > 0$ and some constant C.

The matrix representation A_J of $\mathcal{A}$ is given by

$$(A_J)_{(j,k)(j',k')} = \langle \mathcal{A}\psi_{j',k'}, \psi_{j,k} \rangle.$$

It is well known that there are some operators $\mathcal{A}$ whose matrix representation A_J has many small entries that may be discarded without compromising the convergence of the discretisation scheme. In Sections 4 and 5 we consider a particular class of these operators for which the locations of the small entries can be predicted in advance [6], so that these small entries need not be computed at all. This is called matrix compression. In the scheme presented by Dahmen, Harbrecht and Schneider [6] the matrix compression is split into a first compression and a second compression.

The first and second compression make use of the following conditions:

$$\text{dist}(\Theta_{j,k}, \Theta_{j',k'}) > aB_{j,j'} \tag{4}$$

$$\text{dist}(\Theta_{j,k}, \Theta_{j',k'}) \lesssim 2^{-\min\{j,j'\}} \tag{5}$$

$$\text{dist}(\widetilde{\Theta}_{j,k}, \Theta_{j',k'}) > a'B'_{j,j'} \tag{6}$$

$$\text{dist}(\Theta_{j,k}, \widetilde{\Theta}_{j',k'}) > a'B'_{j,j'} \tag{7}$$

where

$$\begin{aligned} B_{j,j'} &= \max\left\{2^{-\min\{j,j'\}}, 2^{\frac{2J(d'-q)-(j+j')(d'+\widetilde{d})}{2(\widetilde{d}+q)}}\right\} \\ B'_{j,j'} &= \max\left\{2^{-\max\{j,j'\}}, 2^{\frac{2J(d'-q)-(j+j')d'-\max\{j,j'\}\widetilde{d}}{\widetilde{d}+2q}}\right\} \end{aligned} \tag{8}$$

with $d < d' < \widetilde{d} + 2q$ and a, a' are constants.

The first compression yields $A_J^{\epsilon_1}$ given by

$$(A_J^{\epsilon_1})_{(j,k)(j',k')} = \begin{cases} 0, & \text{when } j, j' \geq j_0 \text{ and (4) satisfied,} \\ (A_J)_{(j,k)(j',k')} & \text{otherwise.} \end{cases} \tag{9}$$

The second compression yields A_J^{ϵ} given by

$$(A_J^{\epsilon})_{(j,k)(j',k')} = \begin{cases} 0, & \text{when } j' > j \geq j_0 - 1 \text{ and (5), (6) satisfied,} \\ 0, & \text{when } j > j' \geq j_0 - 1 \text{ and (5), (7) satisfied,} \\ (A_J^{\epsilon_1})_{(j,k)(j',k')} & \text{otherwise.} \end{cases} \tag{10}$$

The matrix A_J comprises block matrices $A_{j,j'} = (A_J)_{(j,T_j)(j',T_{j'})}$. Similarly $A^{\epsilon_1}_{j,j'} = (A_J^{\epsilon_1})_{(j,T_j)(j',T_{j'})}$ and $A^{\epsilon}_{j,j'} = (A_J^{\epsilon})_{(j,T_j)(j',T_{j'})}$. The first compression and second compression respectively induce block matrices

$$\begin{aligned} R_{j,j'} &= A_{j,j'} - A^{\epsilon_1}_{j,j'}, \\ S_{j,j'} &= A^{\epsilon_1}_{j,j'} - A^{\epsilon}_{j,j'}. \end{aligned}$$

Assumptions must be made on the operator $\mathcal{A}$ to ensure that this compression strategy preserves the accuracy of the discretisation scheme. These assumptions are specified in the next section.

4. Matrix Compression for the Laplace Problem

In this section we review the compression theory of Dahmen, Harbrecht and Schneider [6] and Harbrecht [10] for the Laplace problem.

Theorem 2 (Harbrecht [10])**.** *The kernels of the boundary integral operators $\mathcal{L}$, $\mathcal{M}$, $\mathcal{M}^T$ and $\mathcal{N}$ are analytically standard of the same order as their operator order.*

Henceforth we denote by $\mathcal{A}$ a general operator with an analytically standard kernel of order $2q$. The following theorem establishes estimates for the entries of A_J and the block perturbation matrices $R_{j,j'}$ and $S_{j,j'}$.

Theorem 3 (Dahmen, Harbrecht and Schneider [6])**.** *Let $2 + 2\widetilde{d} + 2q > 0$. Then,*

(*i*)

$$|\langle \mathcal{A}\psi_{j',k'}, \psi_{j,k}\rangle| \lesssim \frac{2^{-(j+j')(\widetilde{d}+1)}}{\text{dist}(\Theta_{j,k}, \Theta_{j',k'})^{2+2q+2\widetilde{d}}} \tag{11}$$

uniformly with respect to j when $j, j' \geq j_0$.

(*ii*)

$$|\langle \mathcal{A}\psi_{j,k}, \psi_{j',k'}\rangle|, |\langle \mathcal{A}\psi_{j',k'}, \psi_{j,k}\rangle| \lesssim \frac{2^j 2^{-j'(\tilde{d}+1)}}{\operatorname{dist}(\widetilde{\Theta}_{j,k}, \Theta_{j',k'})^{2q+\tilde{d}}} \tag{12}$$

uniformly with respect to j *provided that* $\operatorname{dist}(\widetilde{\Theta}_{j,k}, \Theta_{j',k'}) \gtrsim 2^{-j'}$ *and* $j' > j \geq j_0 - 1$.

(*iii*)

$$\|R_{j,j'}\|_2 \lesssim a^{-2(\tilde{d}+q)}\ 2^{2Jq} 2^{-2d'(J-(j+j')/2)}. \tag{13}$$

(*iv*)

$$\|S_{j,j'}\|_2 \lesssim a'^{-(\tilde{d}+2q)}\ 2^{2Jq} 2^{-2d'(J-(j+j')/2)}. \tag{14}$$

Proof. The idea is to find a Taylor polynomial for $\widetilde{K}$. Then (11) follows from (3) and the vanishing moment properties of the wavelets. Substituting the distance threshold (4) into (11) establishes (13). A similar argument yields (12) and (14). See Dahmen, Harbrecht and Schneider [6] for a complete proof. □

The following theorem quantifies the sparsity of the compressed matrix A_J^ϵ.

Theorem 4 (Dahmen, Harbrecht and Schneider [6])**.** *The compressed matrix* A_J^ϵ *has* $O(N_J)$ *nonzero entries.*

Proof. There are several cases to consider. We sketch the argument for the first case. The other arguments are similar. A complete proof is given by Dahmen, Harbrecht and Schneider [6].

Consider the first compression (9) for those blocks $A_{j,j'}$ for which

$$2^{\frac{2J(d'-q)-(j+j')(d'+\tilde{d})}{2(\tilde{d}+q)}} > 2^{-\min\{j,j'\}},$$

so that from (8)

$$B_{j,j'} = 2^{\frac{2J(d'-q)-(j+j')(d'+\tilde{d})}{2(\tilde{d}+q)}}.$$

Then the nonzero entries in the block $A_{j,j'}^{\epsilon_1}$ come from wavelets whose supports satisfy

$$\operatorname{dist}(\Theta_{j,k}, \Theta_{j',k'}) \leq aB_{j,j'}. \tag{15}$$

The block has $O([2^{j'} aB_{j,j'}]^2)$ entries in each row that satisfy (15), and the block has $O([2^{j+j'} aB_{j,j'}]^2)$ entries in total that satisfy (15). The number of all such entries in all of the blocks has order bounded by

$$\begin{aligned} C^1 &= \sum_{j,j'=0}^{J} a^2 [2^{j+j'} B_{j,j'}]^2 \\ &= a^2 \sum_{j,j'=0}^{J} [2^{j+j'} B_{j,j'}]^2. \end{aligned} \tag{16}$$

The sum in (16) is shown in [6] to be of order N_J.

Similar arguments apply for the other cases, each having either $O(a^2 N_J)$ or $O((a')^2 N_J)$ nonzeros. □

Remark 5. *The complexity of the wavelet Galerkin scheme can be established by including in (16) a cost factor for each nonzero entry. When the cost factor is of order* $[J-(j+j')/2]^\alpha$ *for some* $\alpha \geq 0$ *the linear complexity in* N_J *is preserved* [6].

5. Matrix Compression for the Helmholtz Problem

The results of the previous section establish the decay properties of A_J when $\mathcal{A}$ has an analytically standard kernel. It is well known that the Helmholtz integral operators $\mathcal{L}_k, \mathcal{M}_k, \mathcal{M}_k^T$ and $\mathcal{N}_k$ have analytically standard kernels. However, the results of the previous section do not quantify the influence of the wavenumber on the decay properties of A_J when $\mathcal{A}$ is one of these operators. In this section we give new decay estimates for A_J that quantify the influence of the wavenumber on operator compression.

We now state our fundamental result, which establishes the dependency of the partial derivatives of the kernel on the wavenumber k.

Theorem 6. *The kernels of the boundary integral operators $\mathcal{L}_k, \mathcal{M}_k, \mathcal{M}_k^T$ and $\mathcal{N}_k$ are analytically standard of the same order as their operator order. The constant C in (3) has a dependence on k with*

$$C \lesssim k^{|\alpha|+|\beta|+2q+1}$$

where $2q$ is the operator order. Thus the transported kernels $\widetilde{K}_{ii'}(\boldsymbol{x}, \boldsymbol{y})$ satisfy

$$|\partial_{\boldsymbol{u}}^{\alpha}\partial_{\boldsymbol{v}}^{\beta}\widetilde{K}_{ii'}(\boldsymbol{u}, \boldsymbol{v})| \lesssim k^{|\alpha|+|\beta|+2q+1}\frac{1}{\|\boldsymbol{\gamma}_i(\boldsymbol{u}) - \boldsymbol{\gamma}_{i'}(\boldsymbol{v})\|_2^{2+2q+|\alpha|+|\beta|}} \tag{17}$$

for $2 + 2q + |\alpha| + |\beta| > 0$.

Proof. The kernel of the operator $\mathcal{L}_k, \mathcal{M}_k, \mathcal{M}_k^T$ or $\mathcal{N}_k$ can be written

$$K(\boldsymbol{x}, \boldsymbol{y}) = k^{2q+1}\frac{e^{ik\|\boldsymbol{x}-\boldsymbol{y}\|}}{\|\boldsymbol{x} - \boldsymbol{y}\|^{2+2q}} f(\|\boldsymbol{x} - \boldsymbol{y}\|, k) g(\boldsymbol{x}, \boldsymbol{y}) \tag{18}$$

where $g(\boldsymbol{x}, \boldsymbol{y})$ is analytic, and $f(r, k)$ is analytic as a function of $r = \|\boldsymbol{x} - \boldsymbol{y}\|$ and depends on k only through terms $k^{-\delta}$ for $\delta < 2q + 1$.

An induction argument using the product rule, the chain rule and the analyticity of $\boldsymbol{\gamma}_i$ and $\boldsymbol{\gamma}_{i'}$ then gives

$$|\partial_{\boldsymbol{u}}^{\alpha}\partial_{\boldsymbol{v}}^{\beta}\widetilde{K}_{ii'}(\boldsymbol{u}, \boldsymbol{v})| \lesssim C(k)\frac{1}{\|\boldsymbol{\gamma}_i(\boldsymbol{u}) - \boldsymbol{\gamma}_{i'}(\boldsymbol{v})\|_2^{2+2q+|\alpha|+|\beta|}}.$$

Here $C(k)$ is independent of $\boldsymbol{u}$ and $\boldsymbol{v}$.

Note that (excepting terms in f) k appears in $\partial_{\boldsymbol{u}}^{\alpha}\partial_{\boldsymbol{v}}^{\beta}K(\boldsymbol{u}, \boldsymbol{v})$ only through the factors

$$k^{2q+1}\frac{\partial^j}{\partial r^j}e^{ikr} = k^{2q+1}(ik)^j e^{ikr}, \quad j = 1, \ldots, |\alpha| + |\beta|.$$

The highest order term in k is $k^{|\alpha|+|\beta|+2q+1}$. It follows that $C(k) \lesssim k^{|\alpha|+|\beta|+2q+1}$. □

Theorem 7. *Let $2 + 2\widetilde{d} + 2q > 0$. Then,*

(i)

$$|\langle \mathcal{A}\psi_{j',k'}, \psi_{j,k}\rangle| \lesssim k^{2\widetilde{d}+2q+1}\frac{2^{-(j+j')(\widetilde{d}+1)}}{\operatorname{dist}(\Theta_{j,k}, \Theta_{j',k'})^{2+2q+2\widetilde{d}}} \tag{19}$$

uniformly with respect to j when $j, j' \geq j_0$.

(ii)

$$|\langle \mathcal{A}\psi_{j,k}, \psi_{j',k'}\rangle|, |\langle \mathcal{A}\psi_{j',k'}, \psi_{j,k}\rangle| \lesssim k^{\widetilde{d}+2q+1}\frac{2^j 2^{-j'(\widetilde{d}+1)}}{\operatorname{dist}(\widetilde{\Theta}_{j,k}, \Theta_{j',k'})^{2q+\widetilde{d}}} \tag{20}$$

uniformly with respect to j provided that $\operatorname{dist}(\widetilde{\Theta}_{j,k}, \Theta_{j',k'}) \gtrsim 2^{-j'}$ *and* $j' > j \geq j_0 - 1$.

(*iii*)

$$\|R_{j,j'}\|_2 \lesssim a^{-2(\tilde{d}+q)} k^{2\tilde{d}+2q+1} 2^{2Jq} 2^{-2d'(J-(j+j')/2)}. \tag{21}$$

(*iv*)

$$\|S_{j,j'}\|_2 \lesssim a'^{-(\tilde{d}+2q)} k^{\tilde{d}+2q+1} 2^{2Jq} 2^{-2d'(J-(j+j')/2)}. \tag{22}$$

Proof. The proof follows the proof of Theorem 3 (see Dahmen, Harbrecht and Schneider [6]) on substitution of (17) for (3). The dependency of the partial derivatives of the kernel on k is carried into the matrix estimates (19) and (20), and into the perturbation bounds (21) and (22). □

Discarding entries in the matrix according to the compression strategy (9) and (10) leads to a finger sparsity pattern in A_J^ϵ. The parameters a and a' control the width of the fingers.

We would like bounds for $\|R_{j,j'}\|_2$ and $\|S_{j,j'}\|_2$ that are independent of k, as is the case for the Laplace problem. Then the error introduced into A_J^ϵ by the thresholding would be independent of the wavenumber. We may choose the parameters a and a' to cancel out the k terms in (21) and (22). In this way we make the width of the fingers a function of the wavenumber. We write $a = a(k)$ and $a' = a'(k)$.

Theorem 8. *Let*

$$\begin{aligned} a(k) &= b\, k^{1+1/(2\tilde{d}+2q)}, \\ a'(k) &= b'\, k^{1+1/(\tilde{d}+2q)} \end{aligned} \tag{23}$$

for some constants b and b'. Then

$$\|R_{j,j'}\|_2 \lesssim b^{-2(\tilde{d}+q)} 2^{2Jq} 2^{-2d'(J-(j+j')/2)}, \tag{24}$$

$$\|S_{j,j'}\|_2 \lesssim b'^{-(\tilde{d}+2q)} 2^{2Jq} 2^{-2d'(J-(j+j')/2)}. \tag{25}$$

Proof. Let $a(k) = b\, k^{1+1/(2\tilde{d}+2q)}$. Note $1 + 1/(2\tilde{d}+2q) = (2\tilde{d}+2q+1)/(2\tilde{d}+2q)$. Then

$$\begin{aligned} \|R_{j,j'}\|_2 &\lesssim a(k)^{-2(\tilde{d}+q)} k^{2\tilde{d}+2q+1} 2^{2Jq} 2^{-2d'(J-(j+j')/2)} \\ &\lesssim b^{-2(\tilde{d}+q)} k^{-2(\tilde{d}+q)(2\tilde{d}+2q+1)/(2\tilde{d}+2q)} k^{2\tilde{d}+2q+1} 2^{2Jq} 2^{-2d'(J-(j+j')/2)} \\ &\lesssim b^{-2(\tilde{d}+q)} k^{-(2\tilde{d}+2q+1)} k^{2\tilde{d}+2q+1} 2^{2Jq} 2^{-2d'(J-(j+j')/2)} \\ &\lesssim b^{-2(\tilde{d}+q)} 2^{2Jq} 2^{-2d'(J-(j+j')/2)}. \end{aligned}$$

The proof for $\|S_{j,j'}\|_2$ is similar. □

The following theorem is the analogue of Theorem 4 for the Helmholtz problem.

Theorem 9. *Let α be chosen so that $k^\alpha = \max\{k^{2+2/(2\tilde{d}+2q)}, k^{2+2/(\tilde{d}+2q)}\}$. Then the compressed matrix A_J^ϵ has $O(k^\alpha N_J)$ nonzero entries.*

Proof. Let $a(k)$ and $a'(k)$ be given by (23). The proof follows the proof of Theorem 4.

Replacing a by $a(k)$ in (16) gives

$$C^1 = a(k)^2 \sum_{j,j'=0}^{J} [2^{j+j'} B_{j,j'}]^2$$

operator	order $2q$	$a(k)$	$a'(k)$	nonzeros
$\mathcal{L}_k$	-1	$k^{4/3}$	k^2	$O(k^4 N_J)$
$\mathcal{M}_k$	0	$k^{5/4}$	$k^{3/2}$	$O(k^3 N_J)$
$\mathcal{M}_k^T$	0	$k^{5/4}$	$k^{3/2}$	$O(k^3 N_J)$
$\mathcal{N}_k$	1	$k^{6/5}$	$k^{4/3}$	$O(k^{8/3} N_J)$

TABLE 1. Dependency of the parameters $a(k), a'(k)$ and the number of nonzeros in the compressed matrix A_J^ϵ on the wavenumber k for operators discretised using wavelets with $\widetilde{d} = 2$ vanishing moments.

which, by the same argument as Theorem 4, is of order $a(k)^2 N_J$.

Similar arguments apply for other cases, each having either $O(a(k)^2 N_J)$ or $O(a'(k)^2 N_J)$ nonzeros. Thus there are $O(\max\{a(k)^2, a'(k)^2\} N_J) = O(k^\alpha N_J)$ nonzeros in A_J^ϵ. □

Table 1 summarises the precise influence of the wavenumber k on the parameters $a(k)$ and $a'(k)$, and on the number of nonzeros in the compressed matrix A_J^ϵ, for the operators $\mathcal{L}_k, \mathcal{M}_k, \mathcal{M}_k^T$ and $\mathcal{N}_k$ when they are discretised using wavelets with $\widetilde{d} = 2$ vanishing moments.

Remarks 10.
(*i*) *For a given operator (with q fixed), the complexity of the wavelet Galerkin scheme tends to $O(k^2 N_J)$ as the number of vanishing moments $\widetilde{d}$ of the wavelets increases.*
(*ii*) *$N^{1/2} \sim k$ when the number of nodes per wavelength is fixed. In this case $O(k^2 N_J)$ complexity becomes $O(N_J^2)$ complexity.*
(*iii*) *When the wavenumber k is fixed the wavelet Galerkin scheme has complexity $O(N_J)$.*

In summary, the scheme has optimal compression when the wavenumber is fixed. When the number of nodes per wavelength is fixed the scheme is less useful.

6. Numerical Results

In this section we demonstrate numerically the compression obtained for the operator $\mathcal{M}_k$, while solving the Surface Helmholtz Equation

$$(-\frac{1}{2}I + \mathcal{M}_k)u(\boldsymbol{x}) = \mathcal{L}_k \frac{\partial u}{\partial \boldsymbol{n}_{\boldsymbol{x}}}(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma, \tag{26}$$

for the Helmholtz problem exterior to the unit sphere with a Neumann boundary condition. Exact Neumann and Dirichlet data for the test problem are generated from a point source in the interior of the sphere.

The Surface Helmholtz Equation (26) does not have a unique solution when k is an eigenvalue of the interior Dirichlet problem. An alternative to the Surface Helmholtz Equation is the Burton and Miller formulation [5],

$$(-\frac{1}{2}I + \mathcal{M}_k + \alpha\mathcal{N}_k)u(\boldsymbol{x}) = (\alpha\frac{1}{2}I + \alpha\mathcal{M}_k^T + \mathcal{L}_k)\frac{\partial u}{\partial \boldsymbol{n}_{\boldsymbol{x}}}(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma,$$

which has a unique solution for all k, provided the coupling parameter $\alpha(k)$ has positive imaginary part. In our experiments we use the Surface Helmholtz Equation (26) instead of the Burton and Miller formulation so that the compression of $\mathcal{M}_k$ can be isolated and observed. We assume that k is not an eigenvalue of the interior Dirichlet problem.

In this exploratory work we compute A_J by applying a certain discrete wavelet transform to a dense matrix obtained from a single scale Galerkin scheme [12, 11]. This is mathematically equivalent to the Galerkin scheme in Section 3. To obtain A_J in this way requires $O(N_J^2)$ operations and storage. The storage requirement restricts the maximum number of unknowns (and wavenumber). However, the scheme in Section 3 requires only $O(N_J)$ operations and storage. In future work we will present results obtained with the scheme in Section 3 for large numbers of unknowns and large wavenumber. To simplify the experiments presented we adopt two *a-posteriori* compression strategies.

The first *a-posteriori* compression strategy applies thresholding with the threshold parameter chosen independently of the wavenumber. This mimics the compression strategy in Section 3, which is independent of the magnitude of the entries in A_J. Define the thresholded matrix A_J^η by

$$(A_J^\eta)_{(j,k)(j',k')} = \begin{cases} (A_J)_{(j,k)(j',k')} & \text{when } |(A_J)_{(j,k)(j',k')}| > \eta \times 1538/N_J, \\ 0 & \text{otherwise,} \end{cases}$$

with η fixed as k varies. The scaling factor 1538 is included only to simplify the presentation.

In the first experiment we fix $N_J = 1538$ and consider the operator $\mathcal{M}_k$ with $k = 1, 2, \ldots, 10$. The ratio of wavelength to maximum mesh spacing ranges from approximately 50 to approximately 5. The range of k is constrained because the oscillatory nature of the problem cannot be accurately represented when this ratio is small.

Table 2 lists sparsity $\text{nnz}(A_J^\eta)/N_J$ for a range of threshold parameters η. The matrix is dense when $\text{nnz}(A_J^\eta)/N_J = 1538$. For small η the matrix becomes almost dense and $\text{nnz}(A_J^\eta)/N_J \to 1538$ as k increases.

Table 3 lists the l_2 norm of the error in the solution of the compressed matrix system for the model problem. There is a peak in the error for $k = 7$. We ascribe this to $k = 7$ being close to an eigenvalue of the interior Dirichlet problem. When k is an eigenvalue of the interior Dirichlet problem the Surface Helmholtz Equation does not have a unique solution.

The sparsity and error for $\eta = 10^{-6}$ are plotted against k in Figure 1. The error remains approximately constant against k. We expect from Table 1 that $\text{nnz}(A_J^\eta)/N_J \sim (k^{5/4})^2 = k^{5/2}$ because for this small test problem $a(k)$ will dominate. Figure 1 shows that this exponent is approached but is not attained for the low values of k demonstrated.

In the second experiment we consider again the operator $\mathcal{M}_k$ and vary N_J with the wavenumber so that the wavelength is ten times the maximum element width.

Table 4 lists sparsity $\text{nnz}(A_J^\eta)/N_J$ for a range of threshold parameters η. Table 5 lists the l_2 norm of the error in the solution of the compressed matrix system for the model problem.

The sparsity and error for $\eta = 10^{-6}$ are plotted against k in Figure 2. The error decreases with k due to the increasing number of unknowns. As before, we expect from Table 1 that $\text{nnz}(A_J^\eta)/N_J \sim (k^{5/4})^2 = k^{5/2}$ because for this small test problem $a(k)$ will dominate. Figure 2 shows that this exponent is approached but is not attained for the low values of k demonstrated.

The second *a-posteriori* compression strategy is widely used [2, 18, 13]. The threshold parameter η is given by $\eta = \mu \times \|F_J\|_\infty / 1538$ where F_J denotes the matrix from the single scale Galerkin scheme. It follows from (18) that $\|F_J\|_\infty \sim k^{2q+1}$. This allows the number of nonzeros in A_J^η to increase more slowly with k than with the first *a-posteriori* compression strategy, at the cost of increasing error.

k	$\eta = 10^{-8}$	$\eta = 10^{-7}$	$\eta = 10^{-6}$
1	337.24	199.65	123.00
2	451.98	222.07	128.20
3	638.64	274.30	137.03
4	815.07	361.64	151.90
5	971.36	474.09	179.77
6	1101.12	580.37	218.91
7	1199.10	690.71	260.94
8	1281.16	810.03	312.53
9	1347.01	915.12	368.55
10	1397.07	1003.59	427.28

TABLE 2. Compressed matrix sparsity $\text{nnz}(A_J^\eta)/N_J$ for the operator $\mathcal{M}_k$. The number of unknowns N_J is fixed at 1538.

k	$\eta = 0$	$\eta = 10^{-8}$	$\eta = 10^{-7}$	$\eta = 10^{-6}$
1	4.52×10^{-4}	1.42×10^{-3}	1.21×10^{-2}	5.74×10^{-2}
2	7.14×10^{-4}	8.27×10^{-4}	1.38×10^{-2}	6.14×10^{-2}
3	1.82×10^{-3}	1.83×10^{-3}	9.40×10^{-3}	4.81×10^{-2}
4	1.66×10^{-3}	1.69×10^{-3}	5.95×10^{-3}	5.58×10^{-2}
5	2.26×10^{-3}	2.26×10^{-3}	6.44×10^{-3}	6.94×10^{-2}
6	3.48×10^{-3}	3.48×10^{-3}	6.15×10^{-3}	6.95×10^{-2}
7	3.49×10^{-2}	3.49×10^{-2}	3.52×10^{-2}	9.20×10^{-2}
8	4.59×10^{-3}	4.59×10^{-3}	7.16×10^{-3}	7.28×10^{-2}
9	7.34×10^{-3}	7.34×10^{-3}	9.20×10^{-3}	7.88×10^{-2}
10	5.39×10^{-3}	5.39×10^{-3}	8.27×10^{-3}	7.29×10^{-2}

TABLE 3. Error in solution of compressed system for the operator $\mathcal{M}_k$. The number of unknowns N_J is fixed at 1538.

k	N_J	$\eta = 10^{-8}$	$\eta = 10^{-7}$	$\eta = 10^{-6}$
2.86	386	306.54	185.36	78.11
5.62	1538	1054.36	538.88	202.82
11.24	6146	3789.31	1973.11	683.30

TABLE 4. Compressed matrix sparsity $\text{nnz}(A_J^\eta)/N_J$ for the operator $\mathcal{M}_k$. The number of unknowns varies so that the wavelength is ten times the maximum element width.

As before, in the first experiment we fix $N_J = 1538$ and consider the operator $\mathcal{M}_k$ with $k = 1, 2, \ldots, 10$. Table 6 lists sparsity $\text{nnz}(A_J^\eta)/N_J$ for a range of threshold parameters μ. The matrix is dense when $\text{nnz}(A_J^\eta)/N_J = 1538$. Table 7 lists the l_2 norm of the error in the solution of the compressed matrix system, for the model problem. As before there is a peak in the error for $k = 7$.

The sparsity and error for $\mu = 10^{-2}$ are plotted against k in Figure 3. The error increases with k. We expect from Table 1 that $\text{nnz}(A_J^\eta)/N_J \sim (k^{5/4})^2 = k^{5/2}$. The observed exponent is much smaller. This is because the threshold parameter increases with k.

k	N_J	$\eta = 0$	$\eta = 10^{-8}$	$\eta = 10^{-7}$	$\eta = 10^{-6}$
2.86	386	4.26×10^{-3}	4.26×10^{-3}	1.12×10^{-2}	1.18×10^{-1}
5.62	1538	3.79×10^{-3}	3.79×10^{-3}	6.36×10^{-3}	6.96×10^{-2}
11.24	6146	1.73×10^{-3}	1.74×10^{-3}	6.50×10^{-3}	3.09×10^{-2}

TABLE 5. Error in solution of compressed system for the operator $\mathcal{M}_k$. The number of unknowns varies so that the wavelength is ten times the maximum element width.

k	$\|F_J\|_\infty$	$\mu = 10^{-3}$	$\mu = 10^{-2}$	$\mu = 10^{-1}$
1	1.9×10^{-2}	316.31	191.69	115.17
2	2.6×10^{-2}	383.47	198.23	106.05
3	3.3×10^{-2}	487.79	218.16	100.20
4	4.0×10^{-2}	606.62	249.69	100.50
5	4.7×10^{-2}	712.72	294.07	104.52
6	5.4×10^{-2}	821.31	354.35	110.56
7	6.1×10^{-2}	926.95	415.03	118.82
8	6.7×10^{-2}	1008.91	471.25	128.24
9	7.3×10^{-2}	1081.86	522.96	139.13
10	7.9×10^{-2}	1144.29	580.14	149.64

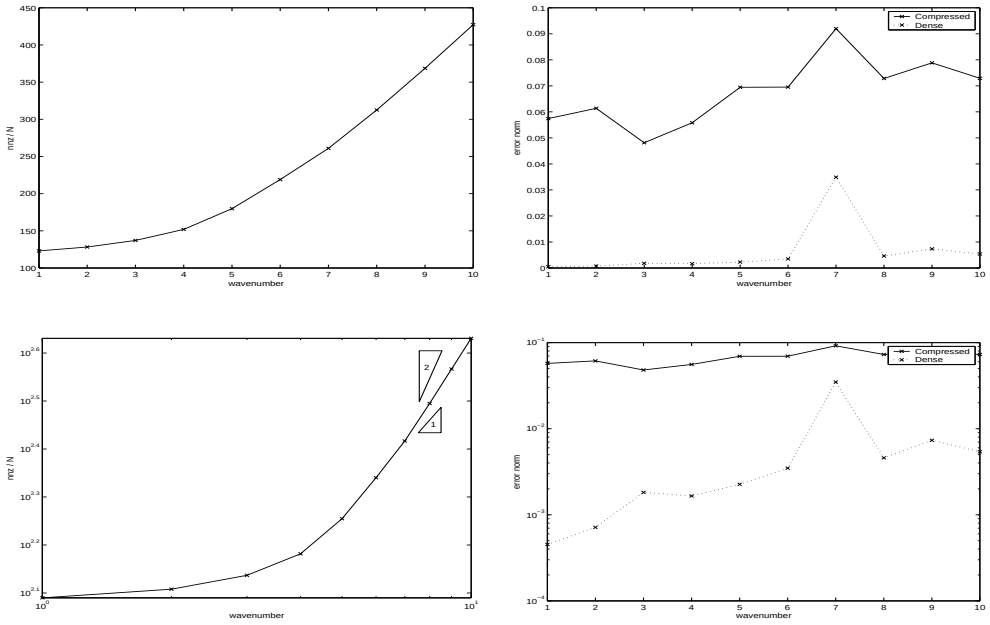
TABLE 6. Compressed matrix sparsity $\mathrm{nnz}(A_J^\eta)/N_J$ for the operator $\mathcal{M}_k$ with $\eta = \mu \, \|F_J\|_\infty/1538$. The number of unknowns N_J is fixed at 1538.

k	$\mu = 0$	$\mu = 10^{-3}$	$\mu = 10^{-2}$	$\mu = 10^{-1}$
1	4.52×10^{-4}	1.68×10^{-3}	1.24×10^{-2}	5.71×10^{-2}
2	7.14×10^{-4}	1.43×10^{-3}	1.98×10^{-2}	1.03×10^{-1}
3	1.82×10^{-3}	2.04×10^{-3}	1.90×10^{-2}	1.10×10^{-1}
4	1.66×10^{-3}	1.74×10^{-3}	2.18×10^{-2}	2.03×10^{-1}
5	2.26×10^{-3}	2.45×10^{-3}	2.15×10^{-2}	1.10×10^{-1}
6	3.48×10^{-3}	3.55×10^{-3}	2.38×10^{-2}	1.25×10^{-1}
7	3.49×10^{-2}	3.49×10^{-2}	4.43×10^{-2}	1.79×10^{-1}
8	4.59×10^{-3}	4.65×10^{-3}	3.53×10^{-2}	1.98×10^{-1}
9	7.34×10^{-3}	7.43×10^{-3}	4.12×10^{-2}	3.66×10^{-1}
10	5.39×10^{-3}	5.62×10^{-3}	4.39×10^{-2}	5.73×10^{-1}

TABLE 7. Error in solution of compressed system for the operator $\mathcal{M}_k$ with $\eta = \mu \, \|F_J\|_\infty/1538$. The number of unknowns N_J is fixed at 1538.

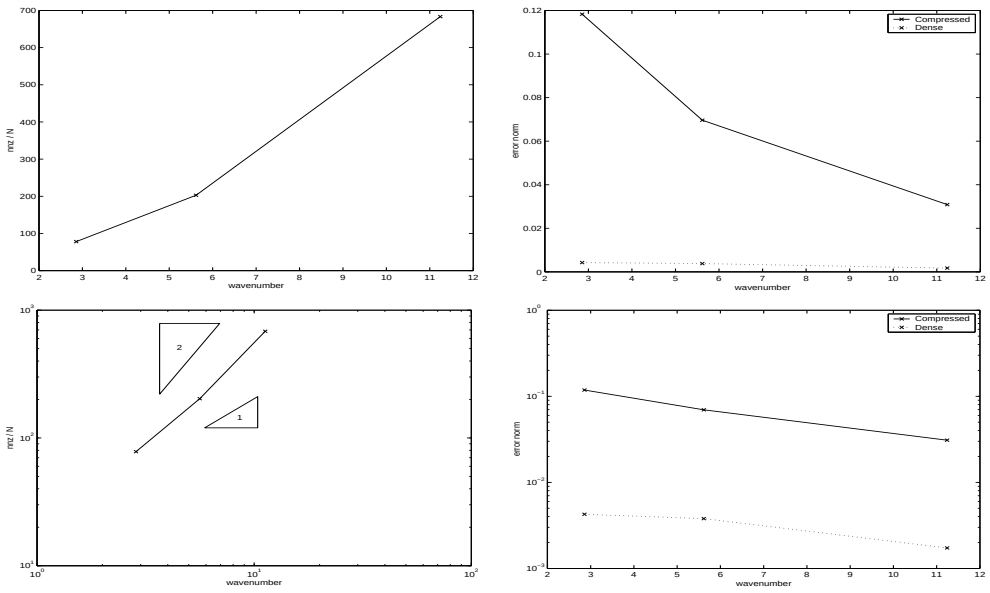
7. Conclusions

We have applied the wavelet Galerkin scheme [6, 10] to the Helmholtz problem and examined the precise dependence of the compression on the wavenumber k. For wavelets with high vanishing moments the scheme produces a compressed matrix with approximately $O(k^2 N_J)$ nonzeros. The factor k^2 appears through the width of the fingers in the sparsity pattern of the compressed matrix, which get wider as k increases.



(a) (b)

FIGURE 1. (a) $\mathrm{nnz}(A_J^\eta)/N_J$ and (b) error norm plotted against wavenumber k for the operator $\mathcal{M}_k$ with $\eta = 10^{-6}$. The number of unknowns N_J is fixed at 1538.



(a) (b)

FIGURE 2. (a) $\mathrm{nnz}(A_J^\eta)/N_J$ and (b) error norm plotted against wavenumber k for the operator $\mathcal{M}_k$ with $\eta = 10^{-6}$. The number of unknowns varies so that the wavelength is ten times the maximum element width.

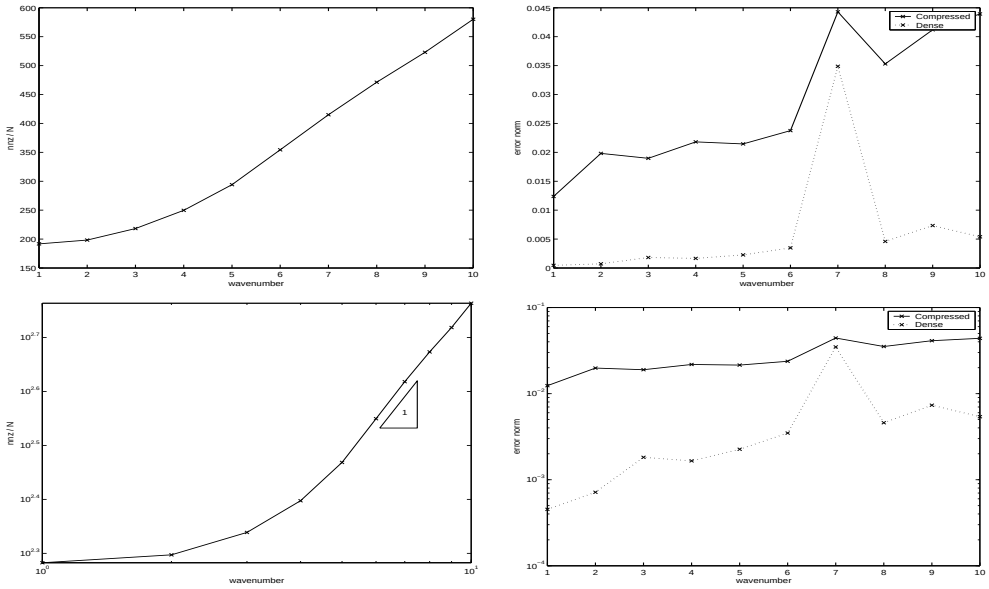


FIGURE 3. (a) $\mathrm{nnz}(A_J^\eta)/N_J$ and (b) error norm plotted against wavenumber k for the operator $\mathcal{M}_k$ with $\eta = 10^{-2}\ \|F_J\|_\infty/1538$. The number of unknowns N_J is fixed at 1538.

When the number of nodes per wavelength is preserved the compressed matrix has approximately $O(N_J^2)$ nonzeros because on a two dimensional surface the number of unknowns must then satisfy $N_J^{1/2} \sim k$.

When the wavenumber is fixed the compressed matrix has $O(N_J)$ nonzeros because the number of unknowns is then independent of k. Although the number of nonzeros depends on k, the complexity is of the same order for the Helmholtz problem as for the Laplace problem.

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