INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 5, Number 1, Pages 24–39

A UNIFORMLY CONVERGENT METHOD ON ARBITRARY MESHES FOR A SEMILINEAR CONVECTION-DIFFUSION PROBLEM WITH DISCONTINUOUS DATA

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(Communicated by Lubin Vulkov)

Abstract. This paper deals with a uniform (in a perturbation parameter) convergent difference scheme for solving a nonlinear singularly perturbed two-point boundary value problem with discontinuous data of a convection-diffusion type. Construction of the difference scheme is based on locally exact schemes or on local Green's functions. Uniform convergence with first order of the proposed difference scheme on arbitrary meshes is proven. A monotone iterative method, which is based on the method of upper and lower solutions, is applied to computing the nonlinear difference scheme. Numerical experiments are presented.

Key Words. convection-diffusion problem, discontinuous data, boundary layer, uniform convergence, monotone iterative method.

1. Introduction

We are interested in the semilinear two-point boundary-value problem with a convective dominated term and discontinuous data

$$-\varepsilon u'' + b(x)u' + c(x,u) + f(x) = 0, \quad x \in \omega = (0,1), \tag{1}$$
$$u(0) = 0,\ u(1) = 0, \quad b(x) \geq b_* = \text{const} > 0,\ c_u \geq 0,\ (c_u \equiv \partial c/\partial u),$$

where ε is a small positive parameter. Suppose that the function c is sufficiently smooth and b, f are piecewise smooth functions, i.e.

$$b(x), f(x) \in Q_p^n(\overline{\omega}),\ n \geq 0.$$

We say that $v(x) \in Q_p^n(\overline{\omega})$ if it is defined on $\overline{\omega}$ and has derivatives up to order n, the function itself and its derivatives may only have jump discontinuities at a finite set of points $p = \{p_1, \dots, p_J\}, 0 < p_j < p_{j+1}, j = 1, \dots, J-1$, i.e. $Q_p^n(\overline{\omega}) = C^n(\overline{\omega}\backslash p)$.

The solution to (1) is a function with a continuous first derivative, which satisfies the boundary conditions and the equation everywhere, with the exception of the points in p. The problem (1) has a unique solution [9]

$$u(x) \in C^1(\overline{\omega}) \cap Q_p^{n+2}(\overline{\omega}).$$

Linear versions of problem (1) with discontinuous data are investigated in [2], [5]. The solution of the linear problem possesses a strong boundary layer at $x = 1$ and weak interior layers at the points of discontinuity p. The boundary layer is strong in the sense that the solution is bounded, but the magnitude of its first derivative at $x = 1$, grows unboundedly as $\varepsilon \to 0$. The interior layers at p are weak: i.e.

Received by the editors November 23, 2004 and, in revised form, September 17, 2006.
2000 *Mathematics Subject Classification.* 65L10, 65L20, 65L70, 65H10.

the solution and the first derivative are bounded but the magnitude of the second derivative grows unboundedly as $\varepsilon \to 0$. We show (see Lemma 1) that problem (1) possesses a strong boundary layer at $x = 1$ and the solution and the first derivative are bounded at the points of discontinuity p.

Our goal is to construct an ε-uniform numerical method for solving problem (1), that is, a numerical method which generates ε-uniformly convergent numerical approximations to the solution. In [2], [5], for solving the linear version of problem (1), the uniform numerical methods are constructed by using the integral-difference method (or the method of locally exact schemes) on arbitrary nonuniform meshes [2], and by using the standard upwind finite difference method on the piecewise uniform mesh, which is fitted to boundary and interior layers [5].

In the next section, we establish some a priori estimates of the solution and its first derivative. In Section 3 we construct a numerical method by applying the integral-difference approach. Note that in the constructed numerical method, a difference operator corresponding to the linear differential operator $-\varepsilon d^2/dx^2 + bd/dx$ is equivalent to the upwind finite volume method from [6], [10]. In Section 4 we prove uniform convergence of the numerical method on arbitrary nonuniform meshes by extending in a natural way the proof of the main theoretical result from [3] (the difference scheme in the case of problem (1) with smooth data converges ε-uniformly). In Section 5 we construct a monotone iterative method for solving the nonlinear difference scheme and prove that the iterates converge ε-uniformly to the solution of problem (1). In the last section numerical results are presented, which are in agreement with the theoretical results.

2. Properties of the continuous problem

The following lemma contains *a priori* estimates of the solution to problem (1).

Lemma 1. *If $b(x), f(x) \in Q_p^n(\overline{\omega})$, $n \geq 0$, then a unique solution to (1) exists and $u(x) \in C^1(\overline{\omega}) \cap Q_p^{n+2}(\overline{\omega})$. The solution $u(x)$ satisfies the following estimates:*

$$\left|\frac{d^k u(x)}{dx^k}\right| \leq C\left[1 + \varepsilon^{-k}\exp\left(-\frac{b_*(1-x)}{\varepsilon}\right)\right], \; x \in \overline{\omega}, \quad k = 0, 1,$$

here and throughout the paper, C denotes a generic positive constant independent of ε.

Proof. The result that problem (1) with the piecewise smooth functions b and f has a unique solution can be found in [9].

Firstly, we estimate the solution $u(x)$ to (1). The transformation $u(x) = e^{\gamma x} w(x)$ with a positive constant γ yields the equation and the boundary conditions

$$-\varepsilon w^{''} + \tilde{b}(x) w^{'} + \tilde{c}(x, w) + e^{-\gamma x} f = 0, \quad w(0) = w(1) = 0,$$

$$\tilde{b} = b - 2\varepsilon\gamma, \quad \tilde{c}(x, w) = e^{-\gamma x} c(x, e^{\gamma x} w) + (b\gamma - \varepsilon\gamma^2) w.$$

If we choose $\gamma = b_*/4$ and assume that $\varepsilon \leq 1$, then

$$\tilde{b}(x) \geq \tilde{b}_* = b_*/2, \quad \tilde{c}_w \geq \tilde{c}_* = (3/16) b_*^2.$$

If $w(x)$ is the exact solution of the above problem, then by the mean-value theorem, we can represent $\tilde{c}(x, w)$ in the form

$$\tilde{c}(x, w) = \tilde{c}(x, 0) + \tilde{c}_w(x) w(x),$$

where $\tilde{c}_w(x) = \tilde{c}_w\left(x, \theta(x) w(x)\right)$, $0 < \theta(x) < 1$. Assuming that $\tilde{c}_w(x)$ is given as a function of x, then the solution $w(x)$ may be considered as a solution of the linear problem

$$\tilde{L}_\varepsilon w \equiv -\varepsilon w^{''} + \tilde{b}(x)w^{'} + \tilde{c}_w(x)w = -\tilde{f}(x), \quad x \in \omega = (0,1), \tag{2}$$

$$w(0) = w(1) = 0, \quad \tilde{f} = \tilde{c}(x,0) + e^{-\gamma x}f, \quad \tilde{b} \geq \tilde{b}_* > 0, \quad \tilde{c}_w \geq \tilde{c}_* > 0.$$

We now prove that the maximum principle for the differential operator $\tilde{L}_\varepsilon$ with the piecewise smooth coefficients holds true: if $w(x) \in C^1(\overline{\omega}) \cap Q_p^{n+2}(\overline{\omega})$ and satisfies $\tilde{L}_\varepsilon w(x) \geq 0$, $x \in \omega$, $w(0), w(1) \geq 0$, then $w(x) \geq 0$, $x \in \overline{\omega}$. Suppose to the contrary that there is a point x_* where $w(x_*) < 0$. If $x_* \notin p$, where p is the set of the points of discontinuity then from $w^{'}(x_*) = 0$ and $w^{''}(x_*) \geq 0$, it follows that $\tilde{L}_\varepsilon w(x_*) < 0$, so we get the contradiction with our assumption. Now suppose that $x_* \in p$. Since $w^{'}(x)$ is a continuous function then $w^{'}(x_*) = 0$ and $w^{'}(x) \leq 0$ in some small vicinity $[x_* - \delta, x_*]$, $\delta > 0$. In general, $w^{''}(x)$ has a jump point at x_*, but on the interval $[x_* - \delta, x_*)$, it is a continuous function. Now, if δ is small enough, then $\tilde{b}$, $\tilde{c}_w$ and $\tilde{f}$ are continuous functions and $w^{''}(x)$ does not change a sign in this interval. Representing $w^{'}(x)$ in the form $w^{'}(x) = -\int_x^{x_*-0} w^{''}(s)ds$, we conclude that $w''(x) \leq 0$, $x \in [x_* - \delta, x_*)$. Hence, $\tilde{L}_\varepsilon w(x) < 0$, $x \in [x_* - \delta, x_*)$, that contradicts our assumption. The uniform estimate on the solution $w(x)$ of problem (2) is derived by applying the maximum principle to the functions $-\max_{x\in\omega} |\tilde{f}(x)|/\tilde{c}_* \pm w(x)$. Taking into account that $u(x) = \exp(\gamma x)w(x)$, we prove the uniform estimate on $u(x)$.

We now prove the estimate on $u^{'}(x)$. Representing the differential equation from (1) in the linear form (2)

$$-\varepsilon u^{''} + b(x)u^{'} + c_u(x)u + \hat{f}(x) = 0, \quad \hat{f}(x) = c(x,0) + f(x),$$

we can prove the estimate on $u^{'}(x)$ in the same way as in [2]. $\square$

Consider the problem

$$-\varepsilon v^{''}(x) + \bar{b}(x)v^{'}(x) + c(x,v) + \bar{f}(x) = 0, \; x \in \omega = (0,1), \tag{3}$$

$$v(0) = 0, \quad v(1) = 0, \quad \bar{b}(x) \geq b_*,$$

where c and b_* are defined in (1).

Lemma 2. *In (1), (3), let* $b(x), \bar{b}(x), f(x), \bar{f}(x) \in Q_p^n(\overline{\omega})$, $n \geq 0$. *Then for* $z(x) = u(x) - v(x)$ *the following estimate holds:*

$$\max_{x\in\overline{\omega}} |z(x)| \leq C \left(\sup_{x\in\overline{\omega}} |b(x) - \bar{b}(x)| + \sup_{x\in\overline{\omega}} |f(x) - \bar{f}(x)| \right),$$

where $u(x)$, $v(x)$ *are the solutions to (1) and (3), respectively, a constant* C *is independent of* ε.

Proof. Introduce Green's function of the differential operator $L_\varepsilon^b = -\varepsilon d^2/dx^2 + bd/dx$:

$$G(x,s) = \frac{1}{-\varepsilon w(s)} \begin{cases} \varphi^{II}(x)\varphi^I(s), & 0 \leq x \leq s \leq 1, \\ \varphi^{II}(s)\varphi^I(x), & 0 \leq s \leq x \leq 1, \end{cases}$$

$$\varphi^I(x) = l(x)/l(0), \quad \varphi^{II}(x) = 1 - \varphi^I(x), \quad l(x) = \int_x^1 e(s)ds,$$

$$e(s) = \exp\left(-\varepsilon^{-1}\int_s^1 b(\tau)d\tau\right), \quad w(s) = -e(s)/l(0).$$

The functions $\varphi^I(x), \varphi^{II}(x)$ are the solutions of the problems

$$L_\varepsilon^b \varphi^{I,II} = 0, \; x \in \omega, \; \varphi^I(0) = \varphi^{II}(1) = 1, \; \varphi^I(1) = \varphi^{II}(0) = 0.$$

From the definition of $G(x,s)$, one can conclude that $G(x,s) \geq 0$.

Now we prove the uniform in the small parameter estimate

$$\max_{x\in\overline{\omega}}\int_0^1 G(x,s)ds \le C. \tag{4}$$

Using the explicit formula for $G(x,s)$, we get

$$\begin{aligned}\int_0^1 G(x,s)ds &= \frac{1}{\varepsilon}\int_0^x \frac{l(x)(l(0)-l(s))}{l(0)e(s)}ds + \\ &\quad \frac{1}{\varepsilon}\int_x^1 \frac{(l(0)-l(x))l(s)}{l(0)e(s)}ds.\end{aligned}$$

From here, it follows that

$$\int_0^1 G(x,s)ds \le \frac{2}{\varepsilon}\int_0^1 \frac{l(0)-l(s)}{e(s)}ds = \frac{2}{\varepsilon}\int_0^1 g(s)ds,$$

$$g(s) = \exp\left(\varepsilon^{-1}\int_s^1 b(t)dt\right)\int_0^s \exp\left(-\varepsilon^{-1}\int_y^1 b(t)dt\right)dy.$$

The function $g(s)$ is the solution of the initial value problem

$$g^{'}(s) = -\frac{b(s)}{\varepsilon}g(s)+1, \quad g(0)=0.$$

From the maximum principle for this initial value problem, we obtain the estimate

$$\max_{s\in\overline{\omega}}|g(s)| \le \varepsilon/b_*.$$

From here, we conclude (4) with $C = 2/b_*$.

From (1), (3) and using the mean-value theorem, it follows that $z(x) = u(x) - v(x)$ is the solution of the following problem

$$L_\varepsilon z(x) \equiv -\varepsilon z^{''}(x) + bz^{'}(x) + c_u z(x) = -(b-\bar{b})v^{'}(x) - (f-\bar{f}), \quad x\in\omega,$$
$$z(0)=0, \quad z(1)=0.$$

Let $z^*(x)$ be the solution of the problem

$$L_\varepsilon z^*(x) = |(b-\bar{b})v^{'}(x)| + |f(x)-\bar{f}(x)|, \quad x\in\omega, \quad z^*(0)=z^*(1)=0.$$

From the maximum principle, the following inequality holds

$$|z(x)| \le z^*(x), \quad x\in\overline{\omega}.$$

Now using Green's function $G(x,s)$ of the differential operator L_ε^b, we write down $z^*(x)$ in the form

$$\begin{aligned} z^*(x) &= -\int_0^1 G(x,s)c_u(s)z^*(s)ds + \\ &\quad \int_0^1 G(x,s)\left(|(b-\bar{b})v^{'}(s)| + |f(s)-\bar{f}(s)|\right)ds.\end{aligned}$$

Since $G(x,s)\ge 0$, $z^*(x)\ge 0$ and $c_u(x) > 0$, it follows that

$$z^*(x) \le \int_0^1 G(x,s)\left(|(b-\bar{b})v^{'}(s)| + |f(s)-\bar{f}(s)|\right)ds.$$

From here, (4), Lemma 1 applied to (3), and taking into account that

$$\varepsilon^{-1}\int_0^1 \exp\left(-\varepsilon^{-1}b_*(1-x)\right)dx = b_*^{-1}(1-\exp\left(\varepsilon^{-1}b_*\right) \le 2b_*^{-1},$$

we prove Lemma 2. $\square$

3. Construction of difference scheme

On $\overline{\omega}$, introduce a nonuniform mesh

$$\overline{\omega}^h = \{0 = x_0 < x_1 < \ldots < x_{N-1} < x_N = 1,\ h_i = x_{i+1} - x_i\}, \quad p \subset \omega^h,$$

where we assume that the points of discontinuity of the functions $b(x)$ and $f(x)$ belong to ω^h.

On $\overline{\omega}$, introduce the piecewise-constant functions

$$\bar{b}(x) = b(x_i + 0), \quad \bar{f}(x) = f(x_i + 0), \quad x_i \le x \le x_{i+1},$$
$$x_i \in \overline{\omega}^h,\ i = 0, \ldots, N-1, \quad f(x_i \pm 0) = \lim_{x \to x_i \pm 0} f(x).$$

We now apply the integral-difference method from [2] to the problem

$$-\varepsilon v^{''}(x) + \bar{b}(x) v^{'}(x) + c(x, v) + \bar{f}(x) = 0, \quad x \in \omega, \tag{5}$$
$$v(0) = 0, \quad v(1) = 0,$$

where the functions $\bar{b}$ and $\bar{f}$ are defined above and the function c from (1). Let G_i be Green's function of the differential operator $-\varepsilon d^2/dx^2 + \bar{b}(x_i) d/dx$ on $[x_i, x_{i+1}]$. We represent the exact solution of the problem (5) on $[x_i, x_{i+1}]$ in the form

$$v_i\,(x) = v\,(x_i)\,\varphi_i^I(x) + v\,(x_{i+1})\,\varphi_i^{II}(x) + \int_{x_i}^{x_{i+1}} G_i(x,s)\psi_i\,(s)\,ds,$$
$$\psi_i(x) \equiv -c(x, v) - \bar{f}(x), \quad x \in [x_i, x_{i+1}],$$

where the local Green function G_i is given by

$$G_i(x,s) = \frac{1}{-\varepsilon w_i(s)} \begin{cases} \varphi_i^I(s)\varphi_i^{II}(x), & x \le s; \\ \varphi_i^I(x)\varphi_i^{II}(s), & x \ge s, \end{cases}$$
$$w_i(s) = \varphi_i^{II}(s)\left[\varphi_i^I(x)\right]'_{x=s} - \varphi_i^I(s)\left[\varphi_i^{II}(x)\right]'_{x=s},$$

and $\varphi_i^I(x),\ \varphi_i^{II}(x)$ are defined by

$$\varphi_i^I(x) = \frac{1 - \exp\left(-b_i(x_{i+1} - x)/\varepsilon\right)}{1 - \exp\left(-b_i h_i/\varepsilon\right)}, \quad \varphi_i^{II}(x) = 1 - \varphi_i^I(x), \quad x_i \le x \le x_{i+1},$$

where $b_i = b(x_i + 0)$. Equating the derivatives $dv_{i-1}(x_i - 0)/dx$ and $dv_i(x_i + 0)/dx$ calculated on the intervals $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$, respectively, we get the following integral-difference scheme

$$B_i(v_{i+1} - v_i) - A_i(v_i - v_{i-1}) = \Psi_i\left[\psi_{i-1}, \psi_i\right], \quad i = 1, \ldots, N-1, \tag{6}$$
$$v_0 = v_N = 0,$$
$$A_i = -\frac{b_{i-1}}{1 - \exp\left(-b_{i-1}h_{i-1}/\varepsilon\right)}, \quad B_i = \exp\left(-b_i h_i/\varepsilon\right) A_{i+1},$$
$$\begin{aligned} \Psi_i\left[\psi_{i-1}, \psi_i\right] \quad = \quad & -\frac{A_i}{b_{i-1}} \int_{x_{i-1}}^{x_i} \left[1 - \exp\left(-\frac{b_{i-1}(s - x_{i-1})}{\varepsilon}\right)\right] \psi_{i-1}(s) ds - \\ & \frac{B_i}{b_i} \int_{x_i}^{x_{i+1}} \left[\exp\left(\frac{b_i(x_{i+1} - s)}{\varepsilon}\right) - 1\right] \psi_i(s) ds. \end{aligned}$$

Now we approximate $c(s, v)$ on $[x_{i-1}, x_{i+1}]$ by the value at x_i. Doing this, we obtain the following nonlinear difference scheme

$$B_i(V_{i+1} - V_i) - A_i(V_i - V_{i-1}) = -D_i c\,(x_i, V_i) - \left(D_i^{(l)} f_{i-1} + D_i^{(r)} f_i\right), \tag{7}$$
$$i = 1, \ldots, N-1, \quad V_0 = V_N = 0, \quad D_i = \Psi_i[1,1], \quad f_i = f(x_i + 0),$$

$$D_i = D_i^{(l)} + D_i^{(r)}, \quad D_i^{(l)} = \frac{|A_i| h_{i-1}}{b_{i-1}} - \frac{\varepsilon}{b_{i-1}} > 0, \quad D_i^{(r)} = \frac{\varepsilon}{b_i} - \frac{|B_i| h_i}{b_i} > 0.$$

We mention here that the coefficients of the difference scheme satisfy the inequalities $A_i < 0$, $B_i < 0$ and $D_i > 0$.

With the above assumptions on the data of the problem (5), the nonlinear problem (7) has a unique solution [7]. Under the assumption $c^* \geq c_u \geq c_* > 0$, where c_*, c^* are constants, this result will be proved in Section 5.

Remark 1. *The linear part of the nonlinear difference operator from (7) corresponding to the linear differential operator $-\varepsilon d^2/dx^2 + \bar{b}d/dx$ can be represented in the equivalent form*

$$\begin{aligned} B_i(V_{i+1} - V_i) - A_i(V_i - V_{i-1}) \quad = \quad & -\varepsilon \left(\frac{V_{i+1} - V_i}{h_i} - \frac{V_i - V_{i-1}}{h_{i-1}} \right) + \\ & \rho\left(\frac{b_i h_i}{\varepsilon}\right) b_i \left(V_{i+1} - V_i\right) + \\ & \rho\left(-\frac{b_{i-1} h_{i-1}}{\varepsilon}\right) b_{i-1} \left(V_i - V_{i-1}\right), \end{aligned}$$

where

$$\rho(\zeta) = \frac{1}{\zeta}\left(1 - \frac{\zeta}{\exp(\zeta) - 1}\right).$$

This difference operator is equivalent to the upwind finite volume method from [6], [10]*, where the weighting function $\rho(\zeta)$ is denoted by $\rho_I(\zeta)$. We mention here that the difference operator satisfies the maximum principle.*

4. Uniform convergence of the difference scheme (7)

In the following lemma, we estimate a solution of the linear difference problem

(8) $\quad B_i\left(W_{i+1} - W_i\right) - A_i\left(W_i - W_{i-1}\right) = T_i, \quad 1 \leq i \leq N-1, \quad W_0 = W_N = 0,$

where A_i and B_i from (7).

Lemma 3. *For the linear difference problem (8), the following estimate holds true*

$$|W_i| \leq \frac{1}{b_*} \sum_{j=1}^{N-1} |T_j|, \quad 1 \leq i \leq N-1, \tag{9}$$

where b_* defined in (1).

Proof. First of all, we transform the difference problem (8) to a self-adjoint form [8]. We multiply each equation in (8) by $P_i \neq 0$ and require that $A_i P_i = a_i$ and $B_i P_i = a_{i+1}$. From here, one can conclude that $A_{i+1} P_{i+1} = B_i P_i = a_{i+1}$, i.e.

$$P_{i+1} = \frac{B_i}{A_{i+1}} P_i = P_1 \prod_{k=1}^{i} \frac{B_k}{A_{k+1}}.$$

Using the relation $B_k = \exp\left(-b_k h_k/\varepsilon\right) A_{k+1}$ from (6) and choosing $P_1 = 1$, we get

$$P_{i+1} = \prod_{k=1}^{i} \exp\left(-\frac{b_k h_k}{\varepsilon}\right).$$

From here and taking into account that $a_{i+1} = A_{i+1} P_{i+1}$, the difference problem (8) is transformed to the self-adjoint form

(10) $\quad a_{i+1}(W_{i+1} - W_i) - a_i(W_i - W_{i-1}) = T_i P_i, \quad 1 \leq i \leq N-1,$

$$W_0 = W_N = 0, \quad a_{i+1} = \left(-\frac{b_i}{1 - \exp(-b_i h_i/\varepsilon)} \right) \prod_{k=1}^{i} \exp\left(-\frac{b_k h_k}{\varepsilon} \right),$$

where $a_1 = A_1$. Representing the right hand side in the form

$$T_i P_i = R_i - R_{i+1}, \quad R_i = \sum_{k=i}^{N-1} T_k P_k, \quad 1 \le i \le N-1, \quad R_N = 0,$$

we obtain

$$a_{j+1}(W_{j+1} - W_j) + R_{j+1} = a_j(W_j - W_{j-1}) + R_j = K, \quad 1 \le j \le N-1,$$

where K is a constant which will be determined below. Thus,

$$W_j = W_{j-1} + \frac{K - R_j}{a_j}.$$

Summing these expressions from $j = 1$ to $j = i$, we get

$$W_i = W_0 + K \sum_{j=1}^{i} \frac{1}{a_j} - \sum_{j=1}^{i} \frac{R_j}{a_j}.$$

From here and taking into account that $W_0 = W_N = 0$, one can conclude that

$$K = \left(\sum_{j=1}^{N} \frac{R_j}{a_j} \right) \left(\sum_{j=1}^{N} \frac{1}{a_j} \right)^{-1}.$$

We can represent W_i in the form

$$W_i = -(1 - \theta_i) \sum_{j=1}^{i} \frac{R_j}{a_j} + \theta_i \sum_{j=i+1}^{N} \frac{R_j}{a_j}, \quad 1 \le i \le N-1,$$

where

$$\theta_i = \left(\sum_{j=1}^{i} \frac{1}{a_j} \right) \left(\sum_{j=1}^{N} \frac{1}{a_j} \right)^{-1}, \quad 0 < \theta_i < 1.$$

Thus,

$$|W_i| \le (1 - \theta_i) \sum_{j=1}^{N} \frac{|R_j|}{|a_j|} + \theta_i \sum_{j=i+1}^{N} \frac{|R_j|}{|a_j|} \le \sum_{j=1}^{N-1} \frac{|R_j|}{|a_j|},$$

where we take into account that $R_N = 0$. Now we estimate

$$Q = \sum_{j=1}^{N-1} \frac{|R_j|}{|a_j|} = \sum_{j=1}^{N-1} \frac{1}{|a_j|} \left(\sum_{k=j}^{N-1} |T_k| P_k \right).$$

Changing the order of summation, write down Q as

$$Q = \sum_{j=1}^{N-1} |T_j| P_j \left(\sum_{k=1}^{j} \frac{1}{|a_k|} \right).$$

Representing $|a_{i+1}|$ in the form

$$|a_{i+1}| = \frac{b_i}{P_{i+1}^{-1} - P_i^{-1}}, \quad 1 \le i \le N-1, \quad |a_1| = -A_1,$$

and using the assumption on b from (1), we estimate

$$\sum_{k=1}^{j} \frac{1}{|a_k|} = \frac{1}{|a_1|} + \sum_{k=2}^{j-1} \frac{P_{k+1}^{-1} - P_k^{-1}}{b_k} \leq \frac{1}{b_* P_j}.$$

Thus,

$$Q \leq \frac{1}{b_*} \sum_{j=1}^{N-1} |T_j|,$$

and we prove the lemma. $\square$

Theorem 1. *The nonlinear difference scheme (7) on arbitrary meshes converges ε-uniformly to the solution of problem (1) :*

$$\max_{0 \leq i \leq N} |u(x_i) - V_i| \leq Ch, \quad h = \max_{0 \leq i \leq N-1} h_i.$$

Proof. From Lemma 2 and the construction of the functions $\bar{b}$ and $\bar{f}$ in (5), we conclude the estimate

$$\max_{x \in \overline{\omega}} |u(x) - v(x)| \leq Ch, \tag{11}$$

where u and v are the solutions to (1) and (5), respectively.

We now estimate the error $\{Z_i = v(x_i) - V_i,\ 0 \leq i \leq N\}$ in the approximation of the continuous solution of the problem (5) by the nonlinear difference scheme (7). From (5) and (7) by the mean-value theorem, we conclude that the error function $\{Z_i,\ 0 \leq i \leq N\}$ solves the following difference problem

$$B_i \left(Z_{i+1} - Z_i\right) - A_i \left(Z_i - Z_{i-1}\right) + D_i c_u Z_i = \Psi_i[\delta_{i-1}, \delta_i], \quad 1 \leq i \leq N-1, \tag{12}$$
$$Z_0 = Z_N = 0,$$

where

$$\delta_{i-1}(s) = \int_s^{x_i} \frac{dc}{dx} dx, \quad s \in [x_{i-1}, x_i], \quad \delta_i(s) = -\int_{x_i}^{s} \frac{dc}{dx} dx, \quad s \in [x_i, x_{i+1}],$$

and the functional Ψ_i is defined in (6).

Let $w_i(s) \geq 0,\ s \in [x_i, x_{i+1}],\ 0 \leq i \leq N-1$. From (6), we can represent $\Psi_i[w_{i-1}, w_i]$ in the form

$$\begin{aligned} \Psi_i \quad = \quad & \frac{1}{1 - \exp\left(-b_{i-1} h_{i-1}/\varepsilon\right)} \int_{x_{i-1}}^{x_i} \left[1 - \exp\left(-b_{i-1}(s - x_{i-1})/\varepsilon\right)\right] w_{i-1}(s) ds + \\ & \frac{1}{\exp\left(b_i h_i/\varepsilon\right) - 1} \int_{x_i}^{x_{i+1}} \left[\exp\left(b_i(x_{i+1} - s)/\varepsilon\right) - 1\right] w_i(s) ds. \end{aligned}$$

Taking into account the inequalities

$$1 - \exp\left(-b_{i-1}(s - x_{i-1})/\varepsilon\right) \leq 1 - \exp\left(-b_{i-1} h_{i-1}/\varepsilon\right), \quad s \in [x_{i-1}, x_i],$$
$$\exp\left(b_i(x_{i+1} - s)/\varepsilon\right) - 1 \leq \exp\left(b_i h_i/\varepsilon\right) - 1, \quad s \in [x_i, x_{i+1}],$$

we prove that

$$0 \leq \Psi_i[w_{i-1}, w_i] \leq \int_{x_{i-1}}^{x_i} w_{i-1}(s) ds + \int_{x_i}^{x_{i+1}} w_i(s) ds, \quad 1 \leq i \leq N-1.$$

From here, Lemma 1 applied to (5), we obtain the estimate of the right-hand side from (12) in the form

$$|\Psi_i[\delta_{i-1}, \delta_{i+1}]| \leq T_i, \quad 1 \leq i \leq N-1, \tag{13}$$

$$T_i = C \int_{x_{i-1}}^{x_{i+1}} h \left[1 + \frac{1}{\varepsilon} \exp\left(-\frac{b_*(1-s)}{\varepsilon} \right) \right] ds.$$

Using the maximum principle for the difference operator in (8) and Lemma 3, the solution of problem (8) with the above right-hand side is estimated as

$$0 \le W_i \le \frac{1}{b_*} \sum_{j=1}^{N-1} T_k \le C \sum_{j=1}^{N-1} \int_{x_{i-1}}^{x_{i+1}} h \left[1 + \frac{1}{\varepsilon} \exp\left(-\frac{b_*(1-s)}{\varepsilon} \right) \right] ds.$$

Thus,

$$0 \le W_i \le Ch, \quad 1 \le i \le N-1.$$

We now show that

$$|Z_i| \le W_i \le Ch, \quad 0 \le i \le N, \tag{14}$$

where $\{Z_i,\ 0 \le i \le N\}$ is the solution to (12) and $\{W_i,\ 0 \le i \le N\}$ is the solution to (8) with the right-hand side from (13). $\{W_i,\ 0 \le i \le N\}$ satisfies the difference problem

$$B_i \left(W_{i+1} - W_i \right) - A_i \left(W_i - W_{i-1} \right) + D_i c_u W_i = T_i + D_i c_u W_i, \quad 1 \le i \le N-1,$$
$$W_0 = W_N = 0,$$

where c_u from (12). Taking into account that $D_i > 0$, $c_u \ge 0$, $W_i \ge 0$ and $|\Psi_i[\delta_{i-1}, \delta_i]| \le T_i$, $1 \le i \le N-1$, by the maximum principle for the difference operator in (12) we conclude (14). The theorem now follows from (11) and (14). □

5. Monotone iterative method

In this section, we construct an iterative method for solving the nonlinear difference scheme (7) which possesses the monotone convergence. This method is based on the approach from [1].

Additionally, we assume that $c(x, u)$ from (1) satisfies the two-sided constraint

$$0 < c_* \le c_u \le c^*, \quad c_*, c^* = \text{const}. \tag{15}$$

We mention that the assumption $c_u \ge c_*$ can always be achieved by a transformation $u = \tilde{u} \exp(\gamma x)$, with γ chosen appropriately.

Introduce the linear version of (7)

$$\left(\mathcal{L}^h + c \right) W(x) = -F(x), \quad x \in \omega^h, \quad W(0) = w_0, \quad W(1) = w_1, \tag{16}$$
$$c(x) \ge c_0 = \text{const} > 0,\ x \in \overline{\omega}^h, \quad F(x_i) = \left(D_i^{(l)} f_{i-1} + D_i^{(r)} f_i \right) / D_i,$$
$$\mathcal{L}^h W(x_i) \equiv \left[B_i (W_{i+1} - W_i) - A_i (W_i - W_{i-1}) \right] / D_i.$$

Now we formulate the maximum principle for the difference operator $\mathcal{L}^h + c$ and give an estimate of the solution to (16).

Lemma 4. *(i) If $W(x)$ satisfies the conditions*

$$\left(\mathcal{L}^h + c \right) W(x) \ge 0 (\le 0),\ x \in \omega^h, \quad W(0), W(1) \ge 0 (\le 0),$$

then $W(x) \ge 0 (\le 0)$, $x \in \overline{\omega}^h$.

(ii) The following estimate of the solution to (16) holds true

$$\|W\|_{\overline{\omega}^h} \le \max \left[|W(0)|, |W(1)|, \|f\|_{\omega^h} / c_0 \right], \tag{17}$$

where

$$\|W\|_{\overline{\omega}^h} \equiv \max_{x \in \omega^h} |W(x)|, \quad \|f\|_{\omega^h} \equiv \max_{x \in \omega^h} |f(x+0)|.$$

Proof. Taking into account that $D_i = D_i^{(l)} + D_i^{(r)}$, we conclude

$$\|F\|_{\omega^h} \leq \|f\|_{\omega^h}. \tag{18}$$

Now, the proof of the lemma can be found in [8]. $\square$

The iterative method is constructed in the following way. Choose an initial mesh function $V^{(0)}$ satisfying the boundary conditions $V^{(0)}(0) = V^{(0)}(1) = 0$. The iterative sequence $\{V^{(n)}\}$, $n \geq 1$, is defined by the recurrence formulae

$$\left(\mathcal{L}^h + c^*\right) Z^{(n)}(x) = -\mathcal{R}^h\left(x, V^{(n-1)}\right), \quad x \in \omega^h, \tag{19}$$

$$Z^{(n)}(0) = Z^{(n)}(1) = 0,$$
$$V^{(n)}(x) = V^{(n-1)}(x) + Z^{(n)}(x),\ x \in \overline{\omega}^h.$$
$$\mathcal{R}^h\left(x, V^{(n-1)}\right) \equiv \mathcal{L}^h V^{(n-1)}(x) + c\left(x, V^{(n-1)}\right) + F(x),$$

where $\mathcal{L}^h$ and F are defined in (16) and $\mathcal{R}^h\left(x, V^{(n-1)}\right)$ is the residual of the difference scheme (7) on $V^{(n-1)}$.

We say that $\overline{V}(x)$ is an upper solution of (7) if it satisfies the inequalities

$$\mathcal{L}^h\overline{V}(x) + c(x, \overline{V}) + F(x) \geq 0,\ x \in \omega^h, \quad \overline{V}(x) \geq 0,\ x = 0, 1.$$

Similarly, $\underline{V}(x)$ is called a lower solution if it satisfies the reversed inequalities. Upper and lower solutions satisfy the inequality

$$\underline{V}(x) \leq \overline{V}(x), \quad x \in \overline{\omega}^h.$$

Indeed, by the definition of lower and upper solutions and the mean-value theorem, for $\delta V = \overline{V} - \underline{V}$ we have

$$\mathcal{L}^h \delta V + c_u(x)\delta V(x) \geq 0, \quad x \in \omega^h, \quad \delta V(x) \geq 0, \quad x = 0, 1,$$

where $c_u(x) \equiv c_u\left[x, \underline{V}(x) + \vartheta(x)\delta V(x)\right],\ 0 < \vartheta(x) < 1$. In view of the maximum principle in Lemma 4, we conclude the required inequality.

The following theorem gives the monotone property of the iterative method (19).

Theorem 2. *Let $\overline{V}^{(0)}, \underline{V}^{(0)}$ be upper and lower solutions of (7), and let $c(x,u)$ satisfy (15). Then the upper sequence $\{\overline{V}^{(n)}\}$ generated by (19) converges monotonically from above to the unique solution V of (7), the lower sequence $\{\underline{V}^{(n)}\}$ generated by (19) converges monotonically from below to V:*

$$\underline{V}^{(0)} \leq \underline{V}^{(n)} \leq \underline{V}^{(n+1)} \leq V \leq \overline{V}^{(n+1)} \leq \overline{V}^{(n)} \leq \overline{V}^{(0)}, \quad \text{on } \overline{\omega}^h,$$

and the sequences converge at the linear rate $q = 1 - c_*/c^*$.

Proof. We consider only the case of the upper sequence. If $\overline{V}^{(0)}$ is an upper solution, then from (19) we conclude that

$$\left(\mathcal{L}^h + c^*\right) Z^{(1)}(x) \leq 0, \quad x \in \omega^h, \quad Z^{(1)}(0) = Z^{(1)}(1) = 0.$$

From Lemma 4, by the maximum principle for the difference operator $\mathcal{L}^h + c^*$, it follows that $Z^{(1)}(x) \leq 0$, $x \in \overline{\omega}^h$. Using the mean-value theorem and the equation for $Z^{(1)}$, we represent $\mathcal{R}^h\left(x, V^{(1)}\right)$ in the form

$$\mathcal{R}^h\left(x, V^{(1)}\right) = -(c^* - c_u^{(1)}(x))Z^{(1)}(x), \quad x \in \omega^h, \tag{20}$$

where $c_u^{(1)}(x) \equiv c_u[x, \overline{V}^{(0)}(x) + \vartheta^{(1)}(x)Z^{(1)}(x)], 0 < \vartheta^{(1)}(x) < 1$. Since the mesh function $Z^{(1)}$ is non-positive on ω^h and taking into account (15), we conclude that $\overline{V}^{(1)}$ is an upper solution. By induction we obtain that $Z^{(n)}(x) \leq 0$, $x \in \overline{\omega}^h$,

$n = 1, 2, \ldots$, and prove that $\{\overline{V}^{(n)}\}$ is a monotonically decreasing sequence of upper solutions.

We now prove that the monotone sequence $\{\overline{V}^{(n)}\}$ converges to the solution of (7). Similar to (20), we obtain

$$\mathcal{R}^h(x, \overline{V}^{(n)}) = -(c^* - c_u^{(n)}(x))Z^{(n)}(x), \quad x \in \omega^h,$$

and from (19), it follows that $Z^{(n+1)}$ satisfies the difference equation

$$\left(\mathcal{L}^h + c^*\right) Z^{(n+1)}(x) = (c^* - c_u^{(n)}(x))Z^{(n)}(x), \quad x \in \omega^h.$$

Using (15) and (17), we have

$$\|Z^{(n+1)}\|_{\overline{\omega}^h} \leq q^n \|Z^{(1)}\|_{\overline{\omega}^h}. \tag{21}$$

This proves convergence of the upper sequence at the linear rate q. Now by linearity of the operator $\mathcal{L}^h$ and the continuity of c, we have also from (19) that the mesh function V defined by

$$V(x) = \lim_{n\to\infty} \overline{V}^{(n)}(x), \quad x \in \overline{\omega}^h,$$

is an exact solution to (7). The uniqueness of the solution to (7) follows from estimate (17). Indeed, if by contradiction, we assume that there exist two solutions V_1 and V_2 to (7), then by the mean-value theorem, the difference $\delta V = V_1 - V_2$ satisfies the difference problem

$$\mathcal{L}^h \delta V + c_u \delta V = 0, \quad x \in \omega^h, \quad \delta V(0) = \delta V(1) = 0.$$

By (17), $\delta V = 0$ which leads to the uniqueness of the solution to (7). This proves the theorem. $\square$

Remark 2. *Consider the following approach for constructing initial upper and lower solutions $\overline{V}^{(0)}$ and $\underline{V}^{(0)}$. Suppose that a mesh function $T(x)$ is defined on $\overline{\omega}^h$ and satisfies the boundary condition $T(0) = T(1) = 0$. Introduce the difference problems*

$$\left(\mathcal{L}^h + c_*\right) Z_\nu^{(0)} = \nu|\mathcal{R}^h(x, T)|, \quad x \in \omega^h, \tag{22}$$

$$Z_\nu^{(0)}(0) = Z_\nu^{(0)}(1) = 0, \quad \nu = 1, -1,$$

where c_* from (15). Then the functions $\overline{V}^{(0)} = T + Z_1^{(0)}$, $\underline{V}^{(0)} = T + Z_{-1}^{(0)}$ are upper and lower solutions, respectively. We check only that $\overline{V}^{(0)}$ is an upper solution. From the maximum principle in Lemma 4, it follows that $Z_1^{(0)} \geq 0$ on $\overline{\omega}^h$. Now using the difference equation for $Z_1^{(0)}$ and the mean-value theorem, we have

$$\mathcal{R}^h\left(x, \overline{V}^{(0)}\right) = \mathcal{R}^h(x, T) + |\mathcal{R}^h(x, T)| + \left(c_u^{(0)} - c_*\right) Z_1^{(0)}.$$

Since $c_u^{(0)} \geq c_*$ and $Z_1^{(0)}$ is nonnegative, we conclude that $\overline{V}^{(0)}$ is an upper solution.

Remark 3. *Since the initial iteration in method (19) is either an upper or lower solution, which can be constructed directly from the difference equation without any knowledge of the solution as we have suggested in the previous remark, this algorithm eliminates the search for the initial iteration as is often needed in Newton's method. This gives a practical advantage in the computation of numerical solutions.*

Let the initial function $V^{(0)}$ be chosen in the form of (22) with $T(x)=0$, i.e. $V^{(0)}$ is the solution of the difference problem

$$\left(\mathcal{L}^h + c_*\right) V^{(0)} = \nu \left| c(x,0) + F(x) \right|, \quad x \in \omega^h, \tag{23}$$

$$V^{(0)}(0) = V^{(0)}(1) = 0, \quad \nu = 1, -1.$$

Then the functions $\overline{V}^{(0)}(x)$, $\underline{V}^{(0)}(x)$ corresponding to $\nu=1$ and $\nu=-1$ are upper and lower solutions, respectively.

Theorem 3. *If the initial upper or lower solution $V^{(0)}$ is chosen in the form of (23), then the monotone iterative method (19) converges ε-uniformly to the solution of problem (1):*

$$\left\| V^{(n)} - u \right\|_{\overline{\omega}^h} \le Ch + \frac{c_0\,(q)^n}{(1-q)} \left(\|c(x,0)\|_{\overline{\omega}^h} + \|f\|_{\overline{\omega}^h} \right),$$

$$q = 1 - \frac{c_*}{c^*} < 1, \quad c_0 = \frac{3c_* + c^*}{c_* c^*},$$

where constant C is independent of ε and h.

Proof. Using (21), we have

$$\begin{aligned}
\left\| V^{(n+k)} - V^{(n)} \right\|_{\overline{\omega}^h} &\le \sum_{i=n}^{n+k-1} \left\| V^{(i+1)} - V^{(i)} \right\|_{\overline{\omega}^h} = \sum_{i=n}^{n+k-1} \left\| Z^{(i+1)} \right\|_{\overline{\omega}^h} \\
&\le \frac{q}{1-q} \left\| Z^{(n)} \right\|_{\overline{\omega}^h} \le \frac{(q)^n}{1-q} \left\| Z^{(1)} \right\|_{\overline{\omega}^h}.
\end{aligned}$$

Taking into account that $\lim V^{(n+k)} = V$ as $k \to \infty$, where V is the solution to (7), we conclude the estimate

$$\left\| V^{(n)} - V \right\|_{\overline{\omega}^h} \le \frac{(q)^n}{1-q} \left\| Z^{(1)} \right\|_{\overline{\omega}^h}. \tag{24}$$

From (18), (19), (23), the definition of $\mathcal{R}^h$ in (19) and the mean-value theorem

$$\begin{aligned}
\left\| Z^{(1)} \right\|_{\overline{\omega}^h} &\le \frac{1}{c^*} \left\| \mathcal{L}^h V^{(0)} \right\|_{\omega^h} + \frac{1}{c^*} \left\| c(x, V^{(0)}) \right\|_{\omega^h} + \frac{1}{c^*} \|f\|_{\omega^h} \\
&\le \frac{1}{c^*} \left(c_* \left\| V^{(0)} \right\|_{\omega^h} + \|c(x,0)\|_{\omega^h} + \|f\|_{\omega^h} \right) + \\
&\quad \frac{1}{c^*} \|c(x,0)\|_{\omega^h} + \left\| V^{(0)} \right\|_{\omega^h} + \frac{1}{c^*} \|f\|_{\omega^h}.
\end{aligned}$$

From here and estimating $V^{(0)}$ from (23) by (17) and (18),

$$\left\| V^{(0)} \right\|_{\omega^h} \le \frac{1}{c_*} \|c(x,0)\|_{\omega^h} + \frac{1}{c_*} \|f\|_{\omega^h},$$

we conclude the estimate on $Z^{(1)}$ in the form

$$\left\| Z^{(1)} \right\|_{\overline{\omega}^h} \le c_0 \left(\|c(x,0)\|_{\omega^h} + \|f\|_{\omega^h} \right),$$

where c_0 is defined in the theorem. Thus, from here, (24) and Theorem 1, we prove the theorem. □

6. Numerical experiments

We solve the nonlinear difference scheme (7) on uniform meshes by the monotone iterative method (19). The stopping criterion is

$$\max_{x\in\overline{\omega}^h} |V^{(n)}(x) - V^{(n-1)}(x)| \leq \sigma,$$

where σ is the required accuracy. If at step $n = n_*$ we satisfy the stopping criterion, then $V(x) = V^{(n_*)}(x)$, $x \in \omega^h$, where $V(x)$ is the corresponding numerical solution.

In the absence of an exact solution for test problems, for fixed value of ε, the nonlinear difference scheme (7) with $N = 8192$ is solved by the monotone iterative method (19) with the stopping criterion $\sigma = 10^{-5}$. This generates a reference solution $V_{ref}(x)$.

The basic feature of monotone convergence of the upper and lower sequences is observed in all the numerical experiments. In fact, the monotone property of the sequences holds at every mesh point in the domain. Of course, this is expected from the analytical considerations.

Test problem 1. Consider the following test problem:

$$-\varepsilon u'' + b(x)u' + c(x,u) + f(x) = 0, \quad u(0) = 1, \quad u(1) = 1,$$
$$c(x,u) = 1 - \exp(-u),$$
$$b(x) = 1, \quad f(x) = \begin{cases} 1, & x \leq 0.5, \\ -0.5, & x > 0.5. \end{cases}$$

It is easily seen that

$$\overline{V}^{(0)}(x) = 1, \quad \underline{V}^{(0)}(x) = 0, \quad x \in \overline{\omega}^h, \tag{25}$$

are upper and lower solutions to (7). From Theorem 2, we conclude that

$$c_* = \min_{0\leq u\leq 1} c_u = e^{-1}, \quad c^* = \max_{0\leq u\leq 1} c_u = 1, \tag{26}$$

where c_* and c^* are defined in (15). In our numerical experiments, the upper solution $\overline{V}^{(0)}(x) = 1$, $x \in \overline{\omega}^h$ is used as an initial iteration.

$N\backslash\varepsilon$	0.1	0.01	0.001	≤ 0.0001
32	2.52×10^{-3}	5.04×10^{-3}	5.32×10^{-3}	5.33×10^{-3}
64	1.23×10^{-3}	2.49×10^{-3}	2.64×10^{-3}	2.65×10^{-3}
128	6.07×10^{-4}	1.23×10^{-3}	1.31×10^{-3}	1.31×10^{-3}
256	2.97×10^{-4}	6.03×10^{-4}	6.42×10^{-4}	6.45×10^{-4}
512	1.44×10^{-4}	2.91×10^{-4}	3.10×10^{-4}	3.12×10^{-4}
1024	6.70×10^{-5}	1.36×10^{-4}	1.45×10^{-4}	1.46×10^{-4}
2048	2.87×10^{-5}	5.82×10^{-5}	6.20×10^{-5}	6.24×10^{-5}

TABLE 1. Maximal approximate error $\overline{E}_{N,\varepsilon}$ for the monotone iterative method (19) applied to the test problem 1.

In Table 1 for various values of ε and N, we present the maximal approximate error

$$\overline{E}_{N,\varepsilon} = \max_{x\in\overline{\omega}^h_N} E_{N,\varepsilon}(x), \quad E_{N,\varepsilon}(x) \equiv |V_{N,\varepsilon}(x) - V_{ref,\varepsilon}(x)|,$$

where $V_{N,\varepsilon}(x)$ is the numerical solution of the nonlinear difference scheme (7) by the monotone iterative method (19). For $\varepsilon \leq 10^{-4}$, the error is independent of ε and decreases with N. This table verifies our convergent results from Theorems 1 and 3, that is the nonlinear difference scheme by the monotone iterative method converges ε-uniformly.

The numerical order of convergence $\overline{\alpha}_{N,\varepsilon}$ and the uniform numerical order of convergence $\overline{\alpha}_N^*$ are calculated as in [4],

$$\overline{R}_{N,\varepsilon} = \max_{x \in \overline{\omega}_N^h} |V_N(x;\varepsilon) - V_{2N}(x;\varepsilon)|, \quad \overline{R}_N^* = \max_{\varepsilon} \overline{R}_{N,\varepsilon},$$

$$\overline{\alpha}_{N,\varepsilon} = \log_2 \left(\frac{\overline{R}_{N,\varepsilon}}{\overline{R}_{2N,\varepsilon}} \right), \quad \overline{\alpha}_N^* = \log_2 \left(\frac{\overline{R}_N^*}{\overline{R}_{2N}^*} \right),$$

and are close to one. This confirms the theoretical result from Theorem 1.

The iteration counts are presented in Table 2. For $\varepsilon \leq 10^{-3}$, the convergence iteration count is 6. This result confirms the theoretical result from Theorems 2 and 3, that the convergence factor q of the monotone iterative method (19) is independent of ε.

The approximate error $E_{N,\varepsilon}(x)$ with $N = 128$ and $\varepsilon = 10^{-2}, 10^{-3}$ is depicted in Fig. 1. The maximum of the approximate error ia attained in the vicinity of the point of discontinuity $x = 0.5$ of $f(x)$.

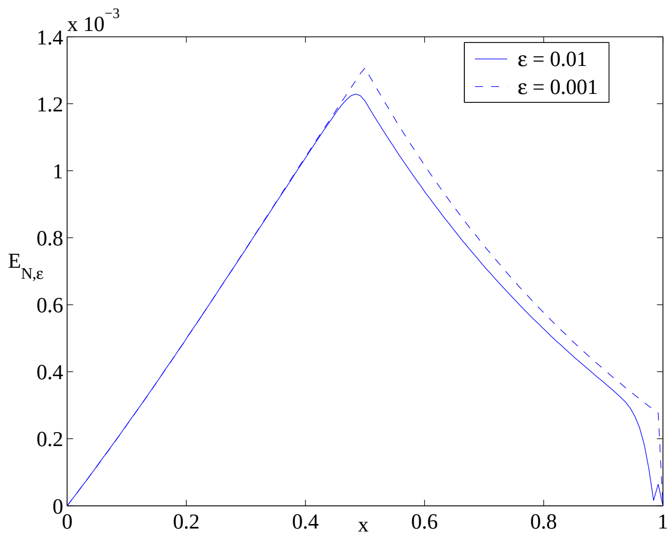

FIGURE 1. $E_{N,\varepsilon}(x)$ with N=128 and $\varepsilon = 10^{-2}, 10^{-3}$ for the test problem 1.

Test problem 2. The second test problem is defined by

$$-\varepsilon u'' + b(x)u' + c(x,u) + f(x) = 0, \quad u(0) = 1, \quad u(1) = 1,$$

$$c(x,u) = 1 - \exp(-u),$$

$$b(x) = \begin{cases} 2, & x \leq 0.5, \\ 1, & x > 0.5, \end{cases} \quad f(x) = -0.5.$$

$N\backslash\varepsilon$	1	0.1	0.01	≤ 0.001
32	5	7	6	6
64	5	7	6	6
128	5	7	6	6
256	5	7	6	6
≥ 512	5	7	7	6

TABLE 2. Iteration counts for the monotone iterative method (19) applied to the test problem 1.

Similar to the test problem 1, the functions from (25) are upper and lower solutions to (7) and c_*, c^* are defined by (26). The upper solution $\overline{V}^{(0)} = 1$, $x \in \overline{\omega}^h$ is used as an initial iteration.

In Table 3, the maximal approximate error is presented for various values of ε and N. For $\varepsilon \leq 10^{-4}$, the error is independent of ε and decreases with N. This table verifies our convergent results from Theorems 1 and 3.

$N\backslash\varepsilon$	0.1	0.01	0.001	≤ 0.0001
32	3.97×10^{-4}	8.25×10^{-4}	8.69×10^{-4}	8.69×10^{-4}
64	1.95×10^{-4}	4.08×10^{-4}	4.40×10^{-4}	4.40×10^{-4}
128	9.64×10^{-5}	2.02×10^{-4}	2.19×10^{-4}	2.20×10^{-4}
256	4.76×10^{-5}	9.92×10^{-5}	1.08×10^{-4}	1.09×10^{-4}
512	2.33×10^{-5}	4.80×10^{-5}	5.22×10^{-5}	5.27×10^{-5}
1024	1.12×10^{-5}	2.24×10^{-5}	2.44×10^{-5}	2.46×10^{-5}
2048	5.23×10^{-6}	9.63×10^{-6}	1.05×10^{-5}	1.06×10^{-5}

TABLE 3. Maximal approximate error $\overline{E}_{N,\varepsilon}$ for the monotone iterative method (19) applied to the test problem 2.

The numerical order of convergence $\overline{\alpha}_{N,\varepsilon}$ and the uniform numerical order of convergence $\overline{\alpha}_N^*$ are close to one, this confirms the theoretical result from Theorem 1.

The iteration counts are the same as for the test problem 1 presented in Table 2. For $\varepsilon \leq 10^{-3}$, the convergence iteration count is 6. This result confirms the theoretical result from Theorems 2 and 3 that the convergence factor q of the monotone iterative method (19) is independent of ε.

The approximate error $E_{N,\varepsilon}(x)$ with $N = 128$ and $\varepsilon = 10^{-2}, 10^{-3}$ is depicted in Fig. 2. The maximum of the approximate error is attained in the boundary layer at $x = 1$.

From the numerical evidence, the following observations are made:

- The numerical experiments confirm our theoretical result that the scheme (7) is first order ε-convergent.
- The numerical experiments confirm that the monotone iterative method (19) converges ε-uniformly.

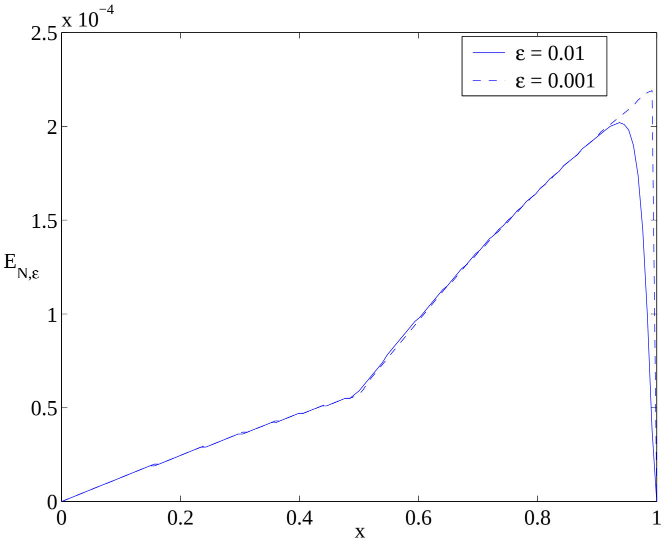

FIGURE 2. $E_{N,\varepsilon}(x)$ with N=128 and $\varepsilon = 10^{-2}, 10^{-3}$ for the test problem 2.

- Convergence of the iterative method (19) is monotonic.

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