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SUBSTRUCTURING PRECONDITIONERS FOR PARABOLIC PROBLEMS BY THE MORTAR METHOD

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Abstract. We study substructuring preconditioners for the linear system arising from the discretization of parabolic problems when the mortar method is applied. By using a suitable non standard norm equivalence we build an efficient edge block preconditioner and we prove a polylogarithmic bound for the condition number of the preconditioned matrix.

Key Words. Domain decomposition, iterative substructuring, mortar methods, parabolic equations.

1. Introduction

We deal with the efficient construction of preconditioners for the linear system associated to the discretization of parabolic problems when a domain decomposition method is applied. Different domain decomposition methods for parabolic problems can be found in literature, see e.g. [14, 11, 12, 25] but here we focus on the mortar method which is a nonconforming domain decomposition method that allows different discretization and/or methods in different subdomains and that weakly enforces the matching of discretizations on adjacent subdomains (see [3, 4, 8, 23]).

Implicit schemes in the time variable, such as the backward Euler and Crank-Nicolson, are considered hence, at a fixed time level, we have to solve an elliptic problem depending on the time step parameter. Consequently, we might apply the methods originally proposed for elliptic equations (see [13, 24, 21, 22]) but here we propose a preconditioner that takes into account the parabolic structure of the original problem. More specifically, after elimination of the degrees of freedom internal to the subdomains, we have to find the traces of the solution on the subdomain boundaries, i.e. to solve the Schur complement system. The approach considered here is the substructuring one, proposed in [9] for conforming domain decomposition and already applied to the mortar method in [1] for the case of order one finite elements and then generalized to a general class of discretization spaces in [7, 6]. A suitable splitting of the nonconforming discretization space in terms of "edge" and "vertex" degrees of freedom is considered and then the related block-Jacobi type preconditioners are used.

In order to design a convenient and inexpensive preconditioner, the edge and vertex blocks have to be replaced in a suitable way; indeed they are not explicitly constructed but it is important to compute efficiently the action of their inverse. For elliptic problems an efficient approximation of the edge block was built by using a norm equivalence for the space $H^{1/2}_{00}$ (see [9]). Analogously here, we propose an equivalent but cheaper to implement edge block preconditioner for parabolic

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problems by proving a suitable non standard norm equivalence. We show that the edge block can be built by adding to the known preconditioners for elliptic problems a new term that can be easily computed and that was suggested by the norm equivalence proved.

Following the abstract formulation presented in [7, 6] we prove that the condition number of the preconditioned matrix grows at most polylogarithmically with the number of degrees of freedom per subdomain, analogously to what happens for the elliptic case and it remains bounded independently of the time step parameter.

The outline of the paper is the following. In sections 2 and 3 we introduce the parabolic problem and we briefly review the mortar method and its main properties. In section 3 we define suitable norms for the trace space that will be crucial for the construction of the preconditioner. The substructuring preconditioner is proposed and studied in section 4. The main theorem of the paper (Theorem 4.1) stating the convergence of the method and the polylogarithmic bound for the condition number of the preconditioned matrix is presented in the same section. Numerical experiments that validate the theory are shown in Section 5. Finally, to help the reader, the Appendix collects some lemmas used in the paper.

For convenience, the symbols $\lesssim$, $\gtrsim$ and $\simeq$ will be used in the paper, i.e. $x_1 \lesssim y_1$, $x_2 \gtrsim y_2$ and $x_3 \simeq y_3$ mean that $x_1 \leq c_1 y_1$, $x_2 \geq c_2 y_2$ and $c_3 x_3 \leq y_3 \leq C_3 x_3$ for some constants c_1, c_2, c_3, C_3 independent of the mesh and time step parameters.

2. A parabolic problem

We consider the following parabolic problem:

find $u(x,t)$ *such that*:

$$
\begin{cases} \dfrac{\partial u}{\partial t} - div(A(\mathbf{x})\nabla u) = f & \text{in } \Omega \times]0,T[\\ u(x,t) = 0 & \text{on } \partial\Omega \times]0,T[\\ u(x,0) = u_0(x) & \text{in } \Omega \end{cases} \tag{1}
$$

where $\Omega \subset \mathbb{R}^2$ is a polygonal domain with boundary $\partial\Omega$, $f \in L^2(\Omega)$ and the matrix $A(\mathbf{x}) = (a_{ij}(\mathbf{x}))_{i,j=1,2}$ is assumed to be, for almost all $\mathbf{x} \in \Omega$, symmetric positive definite with smallest eigenvalue $\geq \alpha > 0$ and largest eigenvalue $\leq \alpha'$, α, α' independent of $\mathbf{x}$. The weak formulation of Problem (1) is:

for $t \in]0,T[$, *find* $u(x,t) \in H_0^1(\Omega)$, $u(x,0) = u_0(x)$ *in* Ω, *such that*

$$
\left(\frac{\partial u}{\partial t}, v\right) + a(u,v) = (f,v),
$$

with the bilinear form $a(\cdot,\cdot)$ defined as

$$
a(u,v) := \sum_{i,j} a_{ij}(\mathbf{x}) \frac{\partial u}{\partial \mathbf{x}_i} \frac{\partial v}{\partial \mathbf{x}_j}\, d\mathbf{x} \tag{2}
$$

assumed to be bounded and elliptic and the linear functional

$$
(f,v) = \int_\Omega f v \, dx.
$$

We consider two types of time discretization, namely, the backward Euler scheme and the Crank-Nicolson scheme. Both scheme are absolutely stable (see [18]). Let τ_n be the n-th time step, then the two schemes lead to the following problems:

for a given $g \in L^2(\Omega)$, *find* $u \in H_0^1(\Omega)$ *such that*

$$
(u,v) + \tau a(u,v) = (\tau g, v), \qquad \forall v \in H_0^1(\Omega) \tag{3}
$$

where τ is the time step parameter and $u = u^n - u^{n-1}$. For the back Euler scheme

$$\tau = \tau_n, \qquad (g, v) = (f, v) - a(u^{n-1}, v)$$

and for the Crank-Nicolson scheme

$$\tau = \tau_n/2, \qquad (g, v) = 2\left((f, v) - a(u^{n-1}, v)\right).$$

3. Mortar Method

In order to obtain space discrete approximation of problem (3) we consider the mortar method. Following the notation of [8], let $\{\Omega_\ell\}_{l=1}^L$ be a partition of Ω into L non–overlapping subdomains Ω_ℓ:

$$\Omega = \cup_{l=1}^L \Omega_\ell \qquad \text{where} \qquad \Omega_k \cap \Omega_\ell = \emptyset \quad \text{if} \quad k \neq l.$$

We denote by $\gamma_\ell^{(i)}$ $(i = 1, \ldots, 4)$ the i-th side of the l-domain, so that $\partial\Omega_\ell = \cup_{i=1}^4 \gamma_\ell^{(i)}$, and setting $\Gamma_{lk} = \partial\Omega_k \cap \partial\Omega_\ell$ then the so–called skeleton of the decomposition is

$$\mathcal{S} = \cup \Gamma_{lk}.$$

Definition 3.1. *We say that a decomposition is* geometrically conforming *if each edge $\gamma_\ell^{(i)}$ coincides with $\Gamma_{\ell n}$ for some n. If the decomposition is not geometrically conforming, then each interior edge $\gamma_\ell^{(i)}$ will be in general split as the union of several segments $\Gamma_{\ell n}$:*

$$\gamma_\ell^{(i)} = \bigcup_{n \in I_\ell^{(i)}} \Gamma_{\ell n}, \qquad \textit{where} \quad I_\ell^{(i)} = \{n : \ |\partial\Omega_n \cap \gamma_\ell^{(i)}| \neq 0\}.$$

On the subdomains Ω_ℓ we will make the same regularity assumptions of [8, 7, 6]:

(G1) the subdomains are regular in shape and the geometrical decomposition is graded, that is
  - (a) there exists a positive constant c_0 such that, for all k, Ω_ℓ contains a ball of diameter $c_0 H_k$, it is contained in a ball of diameter H_k, and the length of each side is bounded from below by $c_0 H_k$; moreover any interior angle ω satisfies $0 < c_1 < \omega < c_2 < \pi$ (c_0, c_1, and c_2 independent of k);
  - (b) there exists a positive constant c_3 such that, if ℓ, k are such that $|\Gamma_{\ell k} \cap \partial\Omega_\ell| > 0$, then it holds $H_k/H_\ell \leq c_3$;

(G2) the following bound holds $\max_{(\ell,i)} \left(\frac{|\gamma_\ell^{(i)}|}{\min_{n \in I_\ell^{(i)}} |\Gamma_{\ell n}|} \right) \leq \rho$.

The constants appearing in the estimates of the following sections will in general depend on the bound ρ.

The Mortar Method is applied by choosing a splitting of the skeleton S as the disjoint union of a certain number of subdomain sides $\gamma_\ell^{(i)}$, called *mortar* or *slave* sides: we fix an index set $I \subset \{1, \ldots, L\} \times \{1, \ldots, 4\}$ such that

$$\mathcal{S} = \bigcup_{(l,i) \in I} \gamma_\ell^{(i)}$$

and $I^* \subset \{1, \ldots, L\} \times \{1, \ldots, 4\}$ will denote the index–set corresponding to *trace* sides or *master* sides defined as

$$I^* \cap I = \emptyset \quad \text{and} \quad \mathcal{S} = \bigcup_{(l,i) \in I^*} \gamma_\ell^{(i)}.$$

A nonconforming domain decomposition method for the solution of our parabolic problem will be considered based on the above splitting of the domain Ω. We start introducing the functional setting: let

$$X = \prod_{\ell} \{u_\ell \in H^1(\Omega_\ell) |\ u_\ell = 0 \text{ on } \partial\Omega \cap \partial\Omega_\ell\}, \qquad \text{and} \qquad T = \prod_{\ell} H_*^{1/2}(\partial\Omega_\ell),$$

with

$$\begin{aligned} H_*^{1/2}(\partial\Omega_\ell) &= H^{1/2}(\partial\Omega_\ell) && \text{if } \partial\Omega_\ell \cap \partial\Omega = \emptyset \\ H_*^{1/2}(\partial\Omega_\ell) &= \{\eta \in H^{1/2}(\partial\Omega_\ell),\ \eta|_{\partial\Omega_\ell\cap\partial\Omega} \equiv 0\} \sim H_{00}^{1/2}(\partial\Omega_\ell \setminus \partial\Omega) && \text{otherwise} \end{aligned}$$

equipped with the norms and seminorms

$$\|u\|_X^2 = \sum_\ell \|u\|_{1,\Omega_\ell}^2, \qquad |u|_X^2 = \textstyle\sum_\ell |u|_{1,\Omega_\ell}^2,$$

$$\|\eta\|_T^2 = \sum_\ell \|\eta_\ell\|_{1/2,\partial\Omega_\ell}^2, \qquad |\eta|_T^2 = \textstyle\sum_\ell |\eta_\ell|_{1/2,\partial\Omega_\ell}^2.$$

Then we define a composite bilinear form $a_X : X \times X \longrightarrow \mathbb{R}$:

$$a_X(u,v) = \sum_\ell a_\ell(u_\ell, v_\ell) = \sum_\ell \int_{\Omega_\ell} \sum_{i,j} a_{ij}(\mathbf{x}) \frac{\partial u_\ell}{\partial \mathbf{x}_i} \frac{\partial v_\ell}{\partial \mathbf{x}_j}\, d\mathbf{x} \tag{4}$$

which is clearly not coercive on X. To obtain a well posed problem we have to consider proper subspaces of X, consisting of functions satisfying a suitable *weak continuity* constraint. More precisely, for any subspace $M \subset L^2(\mathcal{S})$ let the *constrained* approximation and trace spaces $\mathcal{X}$ and $\mathcal{T}$ be defined as follows:

$$\mathcal{X} = \{v \in X, \int_{\mathcal{S}} [v]\lambda\, ds = 0,\ \forall \lambda \in M\}, \quad \mathcal{T} = \{\eta \in T, \int_{\mathcal{S}} [\eta]\lambda\, ds = 0,\ \forall \lambda \in M\}$$

and let $\mathcal{X}_0$ be the subspace of $\mathcal{X}$ of functions vanishing on the skeleton.

The multiplier space M considered satisfies the assumptions stated in [8], i.e. a "broken Poincaré" inequality; this will imply the coercivity of the bilinear form a_X over $\mathcal{X}$.

Finally we introduce the following composite bilinear form $a_\tau : X \times X \longrightarrow \mathbb{R}$:

$$a_\tau(u,v) = (u,v) + \tau a_X(u,v), \tag{5}$$

and we formulate the problem:

Problem P: *find* $u \in \mathcal{X}$ *such that for all* $v \in \mathcal{X}$*:*

$$a_\tau(u,v) = (\tau g, v). \tag{6}$$

To deal with **Problem P** we consider the following norm defined for all $u \in X$ as:

$$\|u\|_\tau^2 = \sum_\ell (\|u\|_{0,\Omega_\ell}^2 + \tau\, |u|_{1,\Omega_\ell}^2). \tag{7}$$

The boundedness and positive definiteness of $a_\tau(\cdot,\cdot)$ can be easily verified; indeed it holds:

Lemma 3.1. *There exist positive constants C and c, independent of τ, H, h, such that:*

a) $|a_\tau(u,v)| \leq C\, \|u\|_\tau\, \|v\|_\tau, \qquad \forall u, v \in \mathcal{X}$
b) $|a_\tau(u,u)| \geq c\, \|u\|_\tau^2, \qquad \forall u \in \mathcal{X}.$

We will use another norm on X defined as:

$$\|u\|_{a_\tau} = \sqrt{a_\tau(u,u)}; \tag{8}$$

lemma (3.1) implies that the a_τ-norm is equivalent to the τ-norm.

3.1. Norms on T. We now introduce a suitable norm on T that will suggest how to properly construct the preconditioner. The natural norm that we can define for all $\eta = (\eta_\ell)_\ell \in T$ is:

$$\|\eta\|_{T,a_\tau} \quad := \quad \inf_{\substack{u \in X: \\ u_{|\mathcal{S}} = \eta}} \|u\|_{a_\tau} \tag{9}$$

but working with it may be difficult. Thus, we now consider another equivalent norm but easier to deal with. Moreover, the structure of the preconditioner proposed in this paper will follow this norm. Let us define:

$$\|\eta\|^2_{T,\tau} \quad := \quad \sum_\ell \left(\tau^{1/2} \, \|\eta_\ell\|^2_{0,\Gamma_\ell} + \tau \, |\eta_\ell|^2_{1/2,\Gamma_\ell}\right) \tag{10}$$

then, it can be proved that the two norms defined on T are equivalent, i.e. it holds:

Lemma 3.2. *The following norm equivalence holds for all $\eta \in \mathcal{T}$:*

$$\|\eta\|_{T,\tau} \simeq \|\eta\|_{T,a_\tau}.$$

Proof. *i)* Let us prove that $\|\eta\|_{T,\tau} \lesssim \|\eta\|_{T,a_\tau}$.

We recall that if Ω is a bounded domain of $\mathbb{R}^2$ with a Lipschitz boundary Γ then it holds (see [15] Theorem 1.5.1.10)

$$\|\phi\|^2_{0,\partial\Omega} \lesssim \varepsilon^{-1/2}\|\phi\|^2_{0,\Omega} + \varepsilon^{1/2}|\phi|^2_{1,\Omega} \qquad \forall \phi \in H^1(\Omega), \quad \varepsilon \in (0,1). \tag{11}$$

Now let $\eta = (\eta_\ell)_{\ell=1,\dots,L} \in \mathcal{T}$ and let $u = (u_\ell)_\ell$ such that $u_{|\mathcal{S}} = \eta$, then thanks to (11) and (48) we have

$$\|\eta_\ell\|^2_{0,\partial\Omega_\ell} + \varepsilon^{1/2}|\eta_\ell|^2_{1/2,\Gamma_\ell} \lesssim \varepsilon^{-1/2}\|u_\ell\|^2_{0,\Omega_\ell} + \varepsilon^{1/2}|u_\ell|^2_{1,\Omega_\ell}$$

that is

$$\varepsilon^{1/2}\|\eta_\ell\|^2_{0,\partial\Omega_\ell} + \varepsilon|\eta_\ell|^2_{1/2,\Gamma_\ell} \lesssim \|u_\ell\|^2_{0,\Omega_\ell} + \varepsilon|u_\ell|^2_{1,\Omega_\ell}.$$

Thus taking $\varepsilon = \tau$ and summing on all the subdomains Ω_ℓ we get the thesis.

ii) We now want to prove that $\|\eta\|_{T,a_\tau} \lesssim \|\eta\|_{T,\tau}$. Let $\eta = (\eta_\ell)_{\ell=1,\dots 1,L} \in \mathcal{T}$ then we have to build an extension u of η such that $u_{|\mathcal{S}} = \eta$ and

$$\|\eta\|_{T,a_\tau} \lesssim \|\eta\|_{T,\tau}. \tag{12}$$

We start by considering a subdomain $\hat{\Omega}$ of unitary diameter and we show that given a function $\hat{\eta} \in H^{1/2}(\hat{\Gamma})$ and a real parameter ε, with $0 < \varepsilon \le 1$, there exists a function $\hat{u} \in H^1(\hat{\Omega})$ such that $\hat{u}_{|\hat{\Gamma}} = \hat{\eta}$ and

$$\|\hat{u}\|^2_{0,\hat{\Omega}} + \varepsilon^2|\hat{u}|^2_{1,\hat{\Omega}} \lesssim \varepsilon\|\hat{\eta}\|^2_{0,\hat{\Gamma}} + \varepsilon^2|\hat{\eta}|^2_{1/2,\hat{\Gamma}}. \tag{13}$$

We note that the function $\hat{u}$ minimizing the left hand side of (13), among those admitting the trace $\hat{\eta}$, is the solution of the following auxiliary problem:

$$\left\{ \begin{array}{ll} -\varepsilon^2 \Delta \hat{u} + \hat{u} = 0 & \in \hat{\Omega} \\ \hat{u} = \hat{\eta} & \text{on } \hat{\Gamma} = \partial \hat{\Omega}. \end{array} \right. \tag{14}$$

Thus (13) can be obtained by using the usual properties of the elliptic problem (14), see e.g. [10, 19, 15]. See also [20] and [2] for details where a similar result is proved.

If we now scale (13) on a subdomain Ω_ℓ with diameter H_ℓ and we consider $\varepsilon = \tau^{1/2} H_\ell^{-1}$ we get

$$\|u\|_{0,\Omega_\ell}^2 + \tau |u|_{1,\Omega_\ell}^2 \lesssim \tau^{1/2} \|\eta\|_{0,\Gamma_\ell}^2 + \tau |\eta|_{1/2,\Gamma_\ell}^2 \tag{15}$$

hence (12) summing on all the subdomains.

□

3.2. Discrete Mortar Problem. To obtain a fully discrete problem, we discretize (6) in space by introducing, for each subdomain Ω_ℓ, a family $\mathcal{V}_h^\ell$ of finite dimensional subspaces of $H^1(\Omega_\ell) \cap C^0(\bar{\Omega}_\ell)$. We set

$$X_h = \prod_{l=1}^{L} \mathcal{V}_h^\ell, \quad X_h \subset X \qquad T_h = \prod_{l=1}^{L} T_h^\ell \subset T.$$

For each $m = (\ell, i) \in I$ let a finite dimensional multiplier space M_h^m (also depending on the parameter h) on γ_m, be given and let

$$M_h = \{\eta \in H^{-1/2}(\mathcal{S}),\ \forall m \in I\ \eta|_{\gamma_m} \in M_h^m\} \sim \prod_{m \in I} M_m. \tag{16}$$

The *constrained* approximation and trace spaces $\mathcal{X}_h$ and $\mathcal{T}_h$ are then defined as follows:

$$\mathcal{X}_h = \{v_h \in X_h, \int_S [v_h] \lambda \, ds = 0,\ \forall \lambda \in M_h\} \quad \mathcal{T}_h = \{\eta \in T_h, \int_{\mathcal{S}} [\eta] \lambda \, ds = 0,\ \forall \lambda \in M_h\}.$$

Moreover we will denote by $\mathcal{V}_h^{l,0} \subset \mathcal{V}_h^\ell$ and $\mathcal{X}_h^0 \subset \mathcal{X}$ the subspaces of functions vanishing on the skeleton:

$$\mathcal{V}_h^{l,0} = \mathcal{V}_h^\ell \cap H_0^1(\Omega_\ell).$$

Remark that any element of $\mathcal{X}_h^0$ have null jump, hence they trivially satisfy the jump constraint ($\mathcal{X}_h^0 \subset \mathcal{X}_h$). Let $T_h^\ell = \mathcal{V}_h^\ell|_{\partial\Omega_\ell}$ and, for any $\gamma_\ell^{(i)}$ of the subdomains Ω_ℓ,

$$T_{l,i} := \left\{ \eta \,:\, \eta \text{ is the trace on } \gamma_\ell^{(i)} \text{ of some } u_\ell \in \mathcal{V}_h^\ell \right\} \tag{17}$$

$$T_{l,i}^0 := \left\{ \eta \in T_{l,i};\, \eta = 0 \text{ at the vertices of } \gamma_\ell^{(i)} \right\}. \tag{18}$$

Then, the mortar element approximation of problem (6) is

Problem ($\mathbf{P}_h$): *find $u_h \in \mathcal{X}_h$ such that for all $v_h \in \mathcal{X}_h$:*

$$a_\tau(u_h, v_h) = (\tau g, v_h). \tag{19}$$

The class M_h of multipliers is chosen to guarantee ellipticity uniformly with respect to the mesh-size parameter h and to the number L of subdomains, see [8] for a detailed analysis.

We will make the following quite typical assumptions on the spaces considered [8].

(A1) $\forall m = (\ell, i) \in I$ ($\gamma_\ell^{(i)}$ multiplier side), there exists a bounded projection $\pi_h^m : L^2(\gamma_m) \longrightarrow T_m^0$, such that for all $\eta \in L^2(\gamma_m)$ and for all $\lambda \in M_h^m$ $\int_{\gamma_m}(\eta - \pi_h^m \eta)\lambda\, ds = 0$, and for all $\eta \in H_{00}^{1/2}(\gamma_m) \|\pi_h^m \eta\|_{H_{00}^{1/2}(\gamma_m)} \lesssim \|\eta\|_{H_{00}^{1/2}(\gamma_m)}$;

(A2) for all $\ell = 1, \dots, L$, the following inverse inequalities hold:
for all elements $\eta \in T_h^\ell$ and for all s, r $0 \le s < r \le 1$
$|\eta|_{r,\Gamma_\ell} \lesssim h_\ell^{s-r} |\eta|_{s,\Gamma_\ell}, \qquad |\eta|_{r,\gamma_\ell^{(i)}} \lesssim h_\ell^{s-r} |\eta|_{r,\gamma_\ell^{(i)}} \qquad i = 1, \dots, 4;$

(A3) $\forall \ell$ and $\forall \eta \in T_h^\ell$ there exists a function $w_h \in \mathcal{V}_h^\ell$ such that
$w_h = \eta \quad \text{on } \Gamma_\ell, \qquad \|w_h\|_{1,\Omega_\ell} \lesssim \|\eta\|_{H^{1/2}(\Gamma_\ell)}.$

By space interpolation, assumption (A1) implies that the projection operator π_h^m verifies for all s, $0 < s < 1/2$:

$$\|\pi_h^m \eta\|_{H_0^s(\gamma_m)} \lesssim \|\eta\|_{H_0^s(\gamma_m)}, \tag{20}$$

uniformly in s. Following [8], we introduce a global linear operator

$$\pi_h : \prod_{\ell=1}^{L} L^2(\partial\Omega_\ell) \longrightarrow \prod_{\ell=1}^{L} L^2(\partial\Omega_\ell), \qquad \pi_h(\eta) = (\eta_\ell^*)_{\ell=1,\cdots,L} \tag{21}$$

which is defined as π_h^m applied to the jump of η on multiplier sides, and zero elsewhere, that is

$$\begin{aligned} \eta_\ell^*|_{\gamma_m} &= \pi_h^m([\eta]|_{\gamma_m}), \text{ for } m = (\ell, i) \in I \\ \eta_\ell^*|_{\gamma_m} &= 0, \text{ for } m = (\ell, i) \in I^*, \qquad \eta_\ell^* \equiv 0 \text{ on } \partial\Omega_\ell \cap \partial\Omega. \end{aligned} \tag{22}$$

Writing conventionally

$$\frac{H}{h} = \min_\ell \left\{ \frac{H_\ell}{h_\ell} \right\}, \tag{23}$$

the following property holds (see [8]):

$$\|\pi_h(\eta)\|_T \lesssim (1 + \log(H/h)) \|\eta\|_T. \tag{24}$$

If η is linear on each γ_ℓ^i a better estimate can be proven [7]:

Lemma 3.3. *If assumptions (A1–A3) hold then, for any $\eta = (\eta_\ell)_{\ell=1,\cdots,L}$ in the trace space T such that η is linear on each $\gamma_\ell^{(i)}$, it holds*

$$\|\pi_h(\eta)\|_T \lesssim (1 + \log(H/h))^{1/2} \|\eta\|_T. \tag{25}$$

4. Substructuring Preconditioners

In this section we propose a substructuring preconditioner, in terms of sums of bilinear forms, for the problem described in the previous sections. We will consider the "substructuring" approach (see [9] and [1, 7, 6] for the case of the Mortar Finite Element method). The principle of these preconditioners consists in distinguishing three types of degrees of freedom: *interior* degrees of freedom (corresponding to basis functions vanishing on the skeleton and supported on one sub-domain), *edge* degrees of freedom, and *vertex* degrees of freedom. Then any function $u \in \mathcal{X}_h$ can be split as the sum of three suitably defined components: $u = u^0 + u^E + u^V$.

Let us detail this operator splitting. Given any discrete function $w = (w_\ell)_{\ell=1,\cdots,L} \in X_h$ we can split it in a unique way as the sum of an *interior* function $w^0 \in \mathcal{X}_h^0$ and a discrete lifting, performed subdomainwise of its trace $\eta(w) = (w^\ell|_{\partial\Omega_\ell})_{\ell=1,\cdots,L}$ which, with some abuse of notation, we will denote by $R_h(w)$ (rather than by the heavier notation $R_h(\eta(w))$):

$$w = w^0 + R_h(w), \qquad w^0 \in \mathcal{X}_h^0$$

with $R_h(w) = (R_h^\ell(w_\ell))_{\ell=1,\dots,L}$, and $R_h^\ell(w_\ell)$ the unique element in $\mathcal{V}_h^\ell$ satisfying

$$R_h^\ell(w_\ell) = w_\ell \text{ on } \Gamma_\ell, \qquad a_{\tau,\ell}(R_h^\ell(w_\ell), v_h^\ell) = 0, \quad \forall v_h \in \mathcal{V}_h^{\ell,0}.$$

Both spaces X_h (of unconstrained functions) and $\mathcal{X}_h$ (of constrained functions) can be split as direct sums of an interior and a trace component, that is

$$X_h = X_h^0 \oplus R_h(T_h), \qquad \mathcal{X}_h = \mathcal{X}_h^0 \oplus R_h(\mathcal{T}_h).$$

Therefore, it is not difficult to verify that the form a_τ in (6) satisfies

$$a_\tau(w, v) = a_\tau(w^0, v^0) + a_\tau(R_h(w), R_h(v)) := a_\tau(w^0, v^0) + s_\tau(\eta(w), \eta(v)),$$

where the *discrete Steklov-Poincaré* operator $s : T_h \times T_h \to \mathbb{R}$ is defined as

$$s_\tau(\xi, \eta) := \sum_\ell a_{\tau,\ell}(R_h^\ell(\xi), R_h^\ell(\eta)). \tag{26}$$

In this paper we study efficient preconditioners for the discrete Steklov-Poincaré operator $s_\tau(\xi, \eta)$ and we assume known preconditioners for $a_\tau(w^0, v^0)$, see existing literature e.g. [22].

We will need the following lemmas that can be easily proved:

Lemma 4.1. *For all $\eta \in \mathcal{T}_h$ it holds:*

$$\|R_h(\eta)\|_\tau \simeq \|\eta\|_{T,\tau}.$$

Lemma 4.2. *For all constrained functions $\xi \in \mathcal{T}_h$ it holds*

$$s_\tau(\xi, \xi) \simeq \|\xi\|_{T,\tau}^2. \tag{27}$$

Proof. Let $\xi \in \mathcal{T}_h$ then from the definition of s_τ and applying lemmas (3.1)-(4.1) we have

$$s_\tau(\xi, \xi) := \sum_\ell a_{\tau,\ell}(R_h^\ell(\xi), R_h^\ell(\xi)) \lesssim \sum_\ell |a_{\tau,\ell}(R_h^\ell(\xi), R_h^\ell(\xi)| \lesssim \|R_h(\xi)\|_\tau^2 \lesssim \|\xi\|_{T,\tau}.$$

Similarly we can obtain the other inequality hence the thesis. □

4.1. The preconditioner. To build a preconditioner for s_τ, we follow the same approach used for elliptic problems and we split the space of constrained skeleton functions $\mathcal{T}_h$ as the sum of *vertex* and *edge* functions. If we denote by $\mathfrak{L} \subset \prod_{\ell=1}^L H_*^{1/2}(\partial\Omega_\ell)$ the space

$$\mathfrak{L} = \{(\eta_\ell)_{\ell=1,\cdots,L},\ \eta_\ell \text{ is linear on each edge of } \Omega_\ell\} \tag{28}$$

then, the space of constrained *vertex* functions can be defined as

$$\mathcal{T}_h^V = (Id - \pi_h)\mathfrak{L}. \tag{29}$$

In the following we will make the (non-restrictive) assumption $\mathfrak{L} \subset T_h$, which yields $\mathcal{T}_h^V \subset \mathcal{T}_h$. We then introduce the space of constrained *edge* functions $\mathcal{T}_h^E \subset \mathcal{T}_h$ defined by

$$\mathcal{T}_h^E = \{\eta = (\eta_\ell)_{\ell=1,\cdots,L} \in \mathcal{T}_h,\ \eta_\ell(A) = 0,\ \forall \text{ vertex } A \text{ of } \Omega_\ell\} \tag{30}$$

from which it follows

$$\mathcal{T}_h = \mathcal{T}_h^V \oplus \mathcal{T}_h^E. \tag{31}$$

It can be easily verified that a function in $\mathcal{T}_h^E$ is uniquely defined by its value on trace edges, the value on multiplier edges being forced by the constraint.

The preconditioner $\hat{s}_\tau$ that we consider is built by introducing an edge and a vertex block diagonal global bilinear form

$$\hat{s}_\tau^E : \mathcal{T}_h^E \times \mathcal{T}_h^E \longrightarrow \mathbb{R} \qquad \hat{s}_\tau^V : \mathcal{T}_h^V \times \mathcal{T}_h^V \longrightarrow \mathbb{R} \tag{32}$$

and defining $\hat{s}_\tau : \mathcal{T}_h \times \mathcal{T}_h \longrightarrow \mathbb{R}$ as

$$\hat{s}_\tau(\eta, \xi) = \hat{s}_\tau^V(\eta^V, \xi^V) + \hat{s}_\tau^E(\eta^E, \xi^E). \tag{33}$$

We will focus mainly on the edge bilinear form following known results available in literature for the vertex one.

The edge bilinear form. The purpose of this paper is to propose an efficient edge block of the preconditioner. To this end we define an edge bilinear form following the norm $\|\cdot\|_{T,\tau}$ defined in (10). More specifically, we introduce a global edge block diagonal bilinear form $\hat{s}_\tau^E : \mathcal{T}_h^E \times \mathcal{T}_h^E \longrightarrow \mathbb{R}$

$$\hat{s}_\tau^E(\eta^E, \xi^E) = \tau^{1/2} p^E(\eta^E, \xi^E) + \tau b^E(\eta^E, \xi^E) \tag{34}$$

where the bilinear forms $p^E(\cdot,\cdot)$ and $b^E(\cdot,\cdot)$ are defined for any trace side $\gamma_\ell^{(i)}$, $m = (\ell, i) \in I^*$, as

$$p^E(\eta, \eta) = \sum_{m=(\ell,i)\in I^*} \|\eta_\ell\|^2_{L^2(\gamma_\ell^{(i)})} \tag{35}$$

$$b^E(\eta, \eta) = \sum_{m=(\ell,i)\in I^*} \|\eta_\ell\|^2_{H_{00}^{1/2}(\gamma_\ell^{(i)})}. \tag{36}$$

Note that the bilinear form $b^E(\cdot,\cdot)$ is the same used for elliptic problems (see e.g. [7, 6]) whereas $p^E(\cdot,\cdot)$ is related to the parabolic structure of our problem. The presence of τ before b^E in (34) comes from the definition of the form a_τ whereas $\tau^{1/2}$ before p^E from the norm $\|\cdot\|_{T,\tau}$. If we consider only p^E without $\tau^{1/2}$, as we naturally could think to do, we build a preconditioner that does not satisfy all the convergence properties required. The norm equivalence given by Lemma (3.2) justifies the use of the norm $\|\cdot\|_{T,\tau}$ and implies $\tau^{1/2}$ before p^E. Numerical experiments validate the theory, see Section 5.

Remark 4.1. *By using (34) the edge block of the preconditioner can be easily implemented. Indeed, for b^E we can use the same efficient approximations proposed for elliptic problems (see e.g.* [9, 13, 21, 22]*) whereas p^E simply requires to assemble a mass matrix for each master side of the decomposition.*

For the vertex block here we consider a block diagonal global bilinear form

$$\hat{s}_\tau^V : \mathcal{T}_h^V \times \mathcal{T}_h^V \longrightarrow \mathbb{R} \qquad \text{such that} \qquad \hat{s}_\tau^V(\eta^V, \eta^V) \simeq s_\tau(\eta^V, \eta^V). \tag{37}$$

Several choices are available in the literature [22]; an efficient one is proposed in [6] where as vertex preconditioner is used the vertex block of the Schur complement matrix on a fixed auxiliary coarse mesh independent of the space discretization.

Finally we can state the main theorem of the paper:

Theorem 4.1. *Let $\eta \in \mathcal{T}_h$ then we have:*

$$(1+\log(H/h))^{-2}\, s_\tau(\eta,\eta) \lesssim \hat{s}_\tau(\eta,\eta) \lesssim (1+\log(H/h))^2\, s_\tau(\eta,\eta). \tag{38}$$

Moreover, if the decomposition is geometrically conforming then

$$s_\tau(\eta,\eta) \lesssim \hat{s}_\tau(\eta,\eta) \lesssim (1+\log(H/h))^2\, s_\tau(\eta,\eta). \tag{39}$$

Proof. The proof of Theorem 4.1 follows the abstract formulation proposed in [7, 6] for elliptic problems. We now generalize this formulation to our case starting from the non geometrically conforming case. Let $\eta \in \mathcal{T}_h$, then $\eta = \eta^V + \eta^E$, and by applying (27) and (37)

$$s_\tau(\eta,\eta) \lesssim \|\eta^E\|^2_{T,\tau} + s_\tau(\eta^V,\eta^V) \tag{40}$$

Thanks to (10), (52) and (34)

$$\begin{aligned}
\|\eta^E\|^2_{T,\tau} &= \tau^{1/2}\|\eta\|_{0,T} + \tau|\eta|_T \lesssim \\
&\lesssim \tau^{1/2}\|\eta\|_{0,T} + (1+\log(H/h))^2 \sum_{(\ell,i)\in I^*} \|\eta_\ell\|_{H^{1/2}_{00}(\gamma^{(i)}_\ell)} \lesssim \\
&\lesssim (1+\log(H/h))^2 \left(\tau^{1/2} p^E(\eta^E,\eta^E) + \tau b^E(\eta^E,\eta^E)\right) \\
&\lesssim (1+\log(H/h))^2\, \hat{s}^E_\tau(\eta^E,\eta^E),
\end{aligned}$$

hence it holds

$$s_\tau(\eta,\eta) \lesssim (1+\log(H/h))^2\, \hat{s}_\tau(\eta,\eta).$$

To prove the left inequality in (38) we note that

$$\hat{s}^V_\tau(\eta^V,\eta^V) \quad \lesssim \quad s_\tau(\eta,\eta), \tag{41}$$

whereas if we consider $\hat{s}^E_\tau(\eta^E,\eta^E)$

$$\sum_{m\in I^*} \|\eta^E\|^2_{H^{1/2}_{00}(\gamma_m)} \lesssim \sum_{m\in I^*} \|\eta^E\|^2_{H^{1/2}_{00}(\gamma_m)} + \sum_{m\in I} \|\eta^E - L\eta^E\|^2_{H^{1/2}_{00}(\gamma_m)}.$$

On "trace sides" ($m \in I^*$) it can be easily verified that $\eta^E = \eta - L\eta$. Then, we can apply lemma 5.4 obtaining

$$\sum_{m\in I^*} \|\eta^E\|^2_{H^{1/2}_{00}(\gamma_m)} \lesssim \sum_\ell (1+\log(H/h))^2\, \|\eta\|^2_{H^{1/2}(\partial\Omega_\ell)}$$

that is

$$b^E(\eta^E,\eta^E) \lesssim (1+\log(H/h))^2\, \|\eta\|^2_T.$$

Thus

$$\tau^{1/2} p^E(\eta^E,\eta^E) + \tau b^E(\eta^E,\eta^E) \lesssim \tau^{1/2}\|\eta^E\|_{0,T} + (1+\log(H/h))^2\, \|\eta\|^2_T$$

and

$$\hat{s}^E_\tau(\eta^E,\eta^E) \lesssim (1+\log(H/h))^2\, s_\tau(\eta,\eta) \tag{42}$$

which, with (33)(41) and (41), concludes the proof of the first part of Theorem 4.1.

Concerning the geometrically conforming case we still have

$$s_\tau(\eta,\eta) \lesssim \|\eta^E\|^2_{T,\tau} + s_\tau(\eta^V,\eta^V) \tag{43}$$

but now we observe that, in the geometrically conforming case, letting $\eta \in \mathcal{T}^E_h$ and $\gamma_m = \Gamma_{\ell,\ell'}$ with ℓ master side and ℓ' slave side we have

$$\|\eta^{\ell'}\|_{H^{1/2}_{00}(\gamma_m)} = \|\pi_m \eta^\ell\|_{H^{1/2}_{00}(\gamma_m)} \lesssim \|\eta^\ell\|_{H^{1/2}_{00}(\gamma_m)},$$

which implies

$$|\eta|_T = \sum_{\ell} |\eta^\ell|_{H^{1/2}(\Gamma_\ell)} = \sum_{(\ell,i)\in I\cup I^*} \|\eta^\ell\|_{H^{1/2}_{00}(\gamma_\ell^{(i)})} \lesssim \sum_{(\ell,i)\in I^*} \|\eta^\ell\|_{H^{1/2}_{00}(\gamma_\ell^{(i)})},$$

whence $\qquad |\eta^E|_T^2 \lesssim b^E(\eta^E,\eta^E), \qquad \|\eta^E\|_{T,\tau}^2 \lesssim \hat{s}_\tau^E(\eta^E,\eta^E)$

and finally

$$s_\tau(\eta,\eta) \lesssim \hat{s}_\tau(\eta,\eta).$$

□

Corollary 4.1. *Let S_τ and P be the matrices obtained by discretizing respectively the bilinear forms s_τ and $\hat{s}_\tau$. Then it holds*

$$Cond(P^{-1}S_\tau) \lesssim (1+\log(H/h))^4\,. \tag{44}$$

Moreover, if the decomposition is geometrically conforming then

$$Cond(P^{-1}S_\tau) \lesssim (1+\log(H/h))^2\,. \tag{45}$$

Remark 4.2. *It is known that for elliptic problems, if (29) is not considered for the vertex space, then a factor $\frac{1}{H^2}$ appears in the estimate. Analogously, in the parabolic case, we get the factor $\frac{\tau}{H^2}$ and the convergence depends on the time step τ (see* [9, 13, 14]*).*

5. Numerical results

To test the preconditioner proposed we consider the model problem (1) with $\Omega = [0,1]^2$ and the time interval $[0, 0.5]$, i.e. $T = 0.5$. The right hand side function f and the initial data were chosen as: $f = x(1-x)y(1-y) + 2ty(1-y) + 2tx(1-x)$ and $u_0 = 0$.

We decompose the unit square into N^2 subdomains with the sidelength $H = 1/N$ and the time interval into n_T subintervals with time step $\tau = \Delta t = 0.5/n_T$. In each subdomain Ω_k we take a uniform mesh $\mathcal{T}^k$ composed by $n_k \times n_k$ equal square elements of size $h_k \times h_k$, $h_k = H/n_k = 1/(Nn_k)$. Then, in each subdomain Ω_k, we define V_h^k to be the space of Q_1 finite elements on the mesh $\mathcal{T}^k$:

$$V_h^k := \left\{u_h \in C^0(\Omega_k) \,:\, u_{h_{|\tau}} \in Q_1(\tau),\, \forall \tau \in \mathcal{T}^k\right\}.$$

The multiplier space was chosen based on a dual basis, see [23].

Let S_τ be the matrix associated to the discrete Steklov–Poincaré operator $s_\tau(\cdot,\cdot)$, then, after applying the change of basis corresponding to switching from the standard nodal basis to the basis corresponding to the splitting (31), and after ordering of the indices as nodes lying on the edges and on the vertex, we can write S_τ as:

$$S_\tau = \begin{pmatrix} S_{EE}^\tau & S_{EV}^\tau \\ (S_{EV}^\tau)^T & S_{VV}^\tau \end{pmatrix}.$$

We will study a block-Jacobi type preconditioner for S_τ: we drop all couplings between different edges and between edges and vertex points.

We recall that in defining a preconditioner for S_τ the action of $(S_{EE}^\tau)^{-1}$ is needed. However, in general S_{EE}^τ is a dense matrix which is also expensive to compute, and even if we had it, it would be expensive to compute its action, i.e. its inverse or a suitable factorization. Thus we look for efficiently invertible approximations to S_{EE}^{τ}.

First we consider the preconditioner introduced in [9, 1] for elliptic problems: S_{EE}^{τ} is replaced by its block diagonal part

$$S_{EE}^{diag} = \begin{pmatrix} S_{E_1,E_1}^{\tau} & 0 & 0 & 0 \\ 0 & S_{E_2,E_2}^{\tau} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & S_{E_m,E_m}^{\tau} \end{pmatrix}$$

with one block for each mortar where m is the number of mortars. This preconditioner requires the construction of the matrix S_{EE}^{τ}, thus to obtain a convenient and inexpensive preconditioner each S_{E_i,E_i}^{τ}, as well as S_{VV}^{τ}, should be replaced by efficient approximations. Here we focus only on the edge block; for the vertex block see e.g. [6].

An efficient and cheaper to implement approximation of S_{EE}^{τ} can be obtained using $\hat{s}_{\tau}^{E}$ defined in (34): we build $\hat{S}_{E}^{\tau}$ the matrix counterpart of $\hat{s}_{\tau}^{E}$ as

$$\hat{S}_{E}^{\tau} = \tau R + \tau^{1/2} K \tag{46}$$

where the matrix R is the square root of the stiffness matrix associated on each edge to the discretization of the Laplacian with homogeneous Dirichlet boundary conditions at the extrema (see [9]) and K is the mass matrix associated on each edge. Note that R can be built following the well known theory developed for elliptic problems [21]. Thus our preconditioner for parabolic problems can be obtained simply adding the new term $\tau^{1/2}K$ that can be easily computed.

The numerical tests relate the following three preconditioners for the Schur complement system:

$$P_1 = \begin{pmatrix} S_{EE}^{diag} & 0 \\ 0 & S_{VV}^{\tau} \end{pmatrix} \qquad P_2 = \begin{pmatrix} \hat{S}_{E}^{\tau} & 0 \\ 0 & S_{VV}^{\tau} \end{pmatrix} \qquad P_3 = \begin{pmatrix} S_{E}^{1} & 0 \\ 0 & S_{VV}^{\tau} \end{pmatrix}$$

where the edge block of P_3 is defined as $S_E^1 = \tau R + K$.

Remark 5.1. *The new preconditioner P_2 is significantly cheaper, both in terms of computational costs and memory requirements, then P_1. Indeed P_1 requires the matrix S_{EE}^{τ} which is a dense matrix and compute it or its action is an intensive and expensive task whereas preconditioner P_2 need only two matrices: R and K. The matrix K is the mass matrix associated on each edge hence it can be easily computed whereas matrix R can be obtained following the well known theory developed for elliptic problems; see* [21] *for details and efficient approximations.*

Note that the edge block of P_3 is not an equivalent approximation of S_{EE}^{τ} as it is $\hat{S}_{E}^{\tau}$ of P_2; indeed P_3 does not satisfy all the convergence properties as P_1 and P_2 hence we do not expect the same behaviour of the condition number of the preconditioned matrix. Preconditioner P_3, with K instead of $\tau^{1/2}K$ in the edge block, is chosen for comparison purposes; numerical experiments will show that only the presence of $\tau^{1/2}$ yields the expected convergence.

To study the dependence on H (size of the subdomains) and on h we set $n_k = n$ for all k so that $h_k = h = H/n$ and we test the preconditioners for n in the range $[5, 40]$ and N in the range $[4, 20]$. We remark that in this case $\max_k H_k/h_k =$

$H/h = n$. We start by fixing the time step $\tau = 10^{-3}$ and in Table (1) we display the number of conjugate gradient iterations for reducing the residual of a factor 10^{-5} for preconditioners P_1, P_2 and P_3 at the time instant $t = 0.5$. The results for P_1 and P_2 are in close agreement with the theory: the condition number of the preconditioned matrix grows at most polylogarithmically with the number of degrees of freedom per subdomain, as indicated by (45). Columns of Table (1) related to the preconditioner P_3 clearly show a worse convergence.

	P_1				P_2				P_3			
$N^2 \backslash n$	5	10	20	40	5	10	20	40	5	10	20	40
16	20	25	25	29	24	27	27	29	26	31	41	45
64	21	22	24	23	21	23	25	27	23	30	37	39
144	17	22	21	23	20	23	22	24	23	27	31	34
256	20	20	21	20	20	23	22	24	22	30	29	32
400	18	20	21	22	18	23	22	25	21	27	27	36

TABLE 1. Number of conjugate gradient iterations needed for reducing the residual of a factor 10^{-5} with the preconditioners P_1, P_2, P_3, for different combinations of the number $K = N^2$ of subdomains and n elements per edge (n^2 elements per subdomains).

To show the better convergence of our preconditioner we fix a decomposition made up of $N^2 = 16$ subdomains keeping fixed $\tau = 10^{-3}$ but with $n \in [5, 6, \dots, 40]$. Table (2) displays the condition number (computed as the ratio of the maximum and minimum eigenvalues) for the three preconditioners P_1, P_2, P_3.

n	P_1	P_2	P_3
5	26.85	31.68	64.84
6	31.27	39.89	82.99
7	35.47	44.14	101.16
8	39.40	47.28	118.68
9	43.02	50.25	135.45
10	46.35	53.04	151.33
20	68.84	73.14	270.05
30	81.65	85.80	348.78
40	90.57	95.14	410.50

TABLE 2. Condition numbers of preconditioners P_1, P_2, P_3 with $N = 4$, $\tau = 10^{-3}$ and increasing values of n.

The results show that conditioning of P_1 and P_2 satisfies the polylogarithmic bound of the theory whereas the condition number of P_3 increases for increasing values of n.

Finally, we show that the condition number of preconditioner P_2 is bounded independently of the time step size τ. The same behaviour can be verified by preconditioner P_1 as expected but not by P_3 which does not approximate correctly P_1.

To test the independence of P_2 from the time step parameter τ we consider a domain decomposition made up of $N^2 = 16$ subdomains with $n = 5$ elements per edge and decreasing values of τ. Table 3 shows that the condition number using P_1 and P_2 remains bounded independently of τ whereas it increases using P_3.

τ	P_1	P_2	P_3
$5.\,10^{-1}$	26.47	29.28	29.08
10^{-1}	26.74	29.21	29.03
$5.\,10^{-2}$	27.16	29.53	30.95
10^{-2}	30.33	32.73	53.58
$5.\,10^{-3}$	33.81	36.10	72.52
10^{-3}	46.35	50.04	151.33

TABLE 3. Condition number of preconditioners P_1, P_2, P_3, for $K = N^2 = 16$ subdomains, $n = 4$ elements per edge and decreasing values of the time step parameter τ.

Thus the numerical tests show that only preconditioner P_2, built using the norm $\|\cdot\|_{T,\tau}$ defined in (10), correctly approximate the edge block of S_τ and exhibits the right convergence.

Future works will be devoted to the application of the preconditioner to more realistic problems, see e.g. [17].

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Appendix

For the reader's convenience we now collect some lemmas used in the paper. Applying scaling arguments to some classical bound (see [16]), it is easy to check that:

Proposition 5.1. *The following bounds hold:*
for all $u \in H^1(\Omega_k)$ it holds

$$H_k^{-1}\|u\|^2_{0,\Gamma_k} + |u|^2_{1/2,\Gamma_k} \lesssim H_k^{-2}\|u\|^2_{0,\Omega_k} + |u|^2_{1,\Omega_k} \tag{47}$$

and

$$|u|^2_{1/2,\Gamma_k} \lesssim |u|^2_{1,\Omega_k} \tag{48}$$

We recall that the following lemma holds (see [5], Lemma 3.1(i)).

Lemma 5.1. *Let assumption (A2) holds and let $\mathfrak{L} \in T_h^\ell$, then the following bounds hold:*

(i) for all $\xi \in T_h^\ell$ such that $\xi(P) = 0$ for some $P \in \gamma_\ell^{(i)}$ it holds

$$\|\xi\|^2_{L^\infty(\gamma_\ell^{(i)})} \lesssim (1 + \log(H/h))\, |\xi|^2_{1/2,\gamma_\ell^{(i)}}; \tag{49}$$

(ii) for all $\xi \in T_h^\ell$, letting A_i and B_i denote the two extrema of the segment $\gamma_\ell^{(i)}$, we have

$$(\xi(A_i) - \xi(B_i))^2 \lesssim (1 + \log(H/h))\, |\xi|^2_{H^{1/2}(\gamma_\ell^{(i)})}; \tag{50}$$

(iii) for all $\xi \in T^0_{\ell,i}$ it holds

$$\|\xi\|^2_{H^{1/2}_{00}(\gamma_\ell^{(i)})} \lesssim \left(1+\log\left(\frac{H_\ell}{h_\ell}\right)\right)^2 |\xi|^2_{H^{1/2}(\gamma_\ell^{(i)})}. \tag{51}$$

Lemma 5.2. [7] *For all $\eta = (\eta_\ell)_{\ell=1,\cdots,L} \in \mathcal{T}_h^E$ we have*

$$\|\eta\|^2_T \lesssim (1+\log(H/h))^2 \sum_{(\ell,i)\in I^*} \|\eta_\ell\|^2_{H^{1/2}_{00}(\gamma_\ell^{(i)})}. \tag{52}$$

We also need the following two results that generalize Lemmas 3.2, 3.4, 3.5 of [9] (see [7] for the proof).

Lemma 5.3. *Let $\eta = (\eta_\ell)_\ell \in T_h$. Then it holds*

$$|L\eta|^2_T \lesssim (1+\log(H/h))\,|\eta|^2_T. \tag{53}$$

Lemma 5.4. *Let assumption (A2) hold, and let $\xi \in T_h^\ell$, $\xi(A) = 0$ for all A vertex of Ω_ℓ. Let $\zeta_L \in H^{1/2}(\partial\Omega_\ell)$, ζ_L linear on each edge of Ω_ℓ. Then it holds*

$$\sum_{k=1}^4 \|\xi\|^2_{H^{1/2}_{00}(\gamma_\ell^{(i)})} \lesssim \left(1+\log\left(\frac{H_\ell}{h_\ell}\right)\right)^2 \|\xi+\zeta_L\|^2_{H^{1/2}(\partial\Omega_\ell)}. \tag{54}$$

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