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CELL CENTERED FINITE VOLUME METHODS USING TAYLOR SERIES EXPANSION SCHEME WITHOUT FICTITIOUS DOMAINS

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Abstract. The goal of this article is to study the stability and the convergence of cell-centered finite volumes (FV) in a domain $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$ with non-uniform rectangular control volumes. The discrete FV derivatives are obtained using the Taylor Series Expansion Scheme (TSES), (see [4] and [10]), which is valid for any quadrilateral mesh. Instead of using compactness arguments, the convergence of the FV method is obtained by comparing the FV method to the associated finite differences (FD) scheme. As an application, using the FV discretizations, convergence results are proved for elliptic equations with Dirichlet boundary condition.

Key Words. finite volume methods, finite difference methods, Taylor series expansion scheme (TSES), convergence and stability, elliptic equations.

1. Introduction.

Finite volumes (FV) are widely used both in Engineering (see e.g. [4], [10] and [13]) and in Geophysical Fluid Dynamics (GFD) (see e.g. [11], [1] and [8]), because of their local conservation property on each control volume. From the mathematical and numerical analysis points of view, these methods are well studied for their stability and convergence, using a variety of methods to compute the fluxes (see e.g. [5], [6], [7], [9] and [14]). On a control volume in $\mathbb{R}^2$, one simple way to compute the flux along a boundary is to start with the difference of the given data at two cell centers divided by the length of the vector connecting those cell centers and then, taking the flux as the product of that quantity and the length of the boundary, which is the analog of the one dimensional case (see [5], [6], [7] and [9]). However this is not the best choice when the unit normal on the boundary is not parallel to the vector connecting the two cell centers; to deal with complicated meshes in $\mathbb{R}^2$, more efficient ways to compute the fluxes are needed. In this article, we consider the cell centered FV by Taylor Series Expansion Scheme (TSES), which permits to compute the fluxes on a general quadrilateral mesh in $\mathbb{R}^2$ (see [4] and [10]), and apply them to quasi- (but, non-) uniform meshes on Ω; we also intend to consider more general meshes in the future. For the mathematical analysis of the FV method, one specific difficulty is due to the "weak consistency" of FV. Indeed the companion discrete FV derivative arising in the discrete integration by parts does not usually converge strongly to the corresponding derivative of the limit function (see e.g. [6] or [9]). To overcome this difficulty, discrete compactness arguments have been used

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as in e.g. [6]. But here instead we consider the finite differences (FD) associated with the FV and compare the FV and FD spaces by defining a map between them. The convergence of the FV method is then inferred.

Our work is organized as follows. In Section 2, we describe the cell centered FV setting by TSES without using fictitious domains, but using instead "flat" domains at the boundary. In Section 3, we introduce an external approximation of $H^1_0(\Omega)$ using FV spaces V_h (see [3] and [15]), and show that the truncation error between a function in $H^1_0(\Omega)$ and its projection onto the FV space V_h tends to zero as the mesh sizes decrease. Due to the weak consistency of the FV, we are not able at this point to show that the external approximation of $H^1_0(\Omega)$ by the FV spaces is convergent. Instead, in Section 4, we present the FD method associated with this FV method and prove the stability and convergence of the external approximation of $H^1_0(\Omega)$ by the FD spaces $\widetilde{V}_h$ in Section 5. In Section 6, comparing the FV and FD spaces and thanks to the convergence of the FD, we obtain the convergence of the FV in the end. Finally, in Section 7, as an application, we demonstrate how one can use the FV method to approximate the solution of some typical elliptic equations with Dirichlet boundary condition, and, using our results, show the convergence of such an approximation via finite volumes to the solution of the original problem.

2. The Finite Volume Setting.

The domain is $\Omega = (0,1)\times(0,1)$ in $\mathbb{R}^2$. We set $x_0 = x_{\frac{1}{2}} = 0$, $x_{M+\frac{1}{2}} = x_{M+1} = 1$, $y_0 = y_{\frac{1}{2}} = 0$, $y_{N+\frac{1}{2}} = y_{N+1} = 1$ and we choose the nodal points $x_{i+\frac{1}{2}}$, $y_{j+\frac{1}{2}}$ for $1 \le i \le M-1$, $1 \le j \le N-1$,

$$\begin{aligned} 0 = (x_0 =)x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{M+\frac{1}{2}} (= x_{M+1}) = 1, \\ 0 = (y_0 =)y_{\frac{1}{2}} < y_{\frac{3}{2}} < \cdots < y_{N+\frac{1}{2}} (= y_{N+1}) = 1. \end{aligned} \tag{2.1}$$

We define the control volumes on Ω which appear on Fig. 1,

$$K_{i,j} = \begin{cases} (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}), & 1 \le i \le M, \ \ 1 \le j \le N, \\ (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times \{y_j\}, & 1 \le i \le M, \ \ j = 0, N+1, \\ \{x_i\} \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}), & i = 0, M+1, \ \ 1 \le j \le N. \end{cases} \tag{2.2}$$

Here, we have chosen flat control volumes at the boundary to handle and enforce the boundary conditions.
For $1 \le i \le M$, $1 \le j \le N$, the center of $K_{i,j}$ is

$$(x_i, y_j) = \Big(\frac{x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}}{2}, \frac{y_{j-\frac{1}{2}} + y_{j+\frac{1}{2}}}{2} \Big). \tag{2.3}$$

We set

$$\begin{aligned} h_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}, \quad k_j = y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}, \quad 1 \le i \le M, \quad 1 \le j \le N, \\ h_{i+\frac{1}{2}} = x_{i+1} - x_i, \quad k_{j+\frac{1}{2}} = y_{j+1} - y_j, \quad 0 \le i \le M, \quad 0 \le j \le N, \end{aligned} \tag{2.4}$$

and, for convenience, we also set

$$h_0 = h_{M+1} = k_0 = k_{N+1} = 0. \tag{2.5}$$

Then we infer from (2.3)-(2.5) that

$$h_{i+\frac{1}{2}} = \frac{1}{2}\left(h_i + h_{i+1}\right), \ k_{j+\frac{1}{2}} = \frac{1}{2}\left(k_j + k_{j+1}\right), \ 0 \le i \le M, \ 0 \le j \le N, \tag{2.6}$$

and write the nodal points $x_{i+\frac{1}{2}}$, $y_{j+\frac{1}{2}}$ as proper weighted averages of the points x_i, x_{i+1}, y_j and y_{j+1} (see Fig. 2):

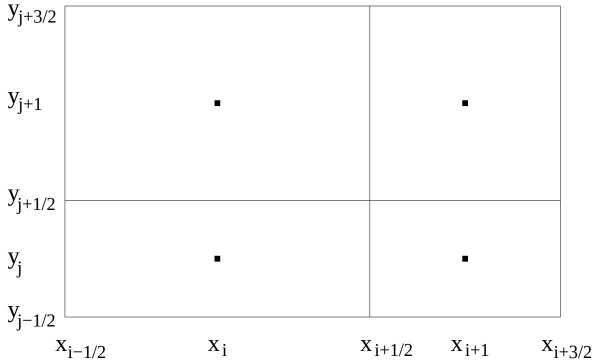

FIGURE 1. Control volumes $K_{i,j}$ and those centers (x_i, y_j) on Ω.

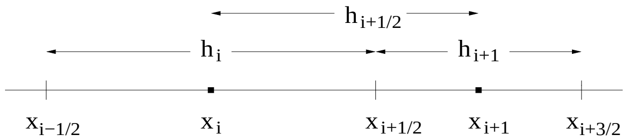

FIGURE 2. The order of the FV points in the x direction.

$$\begin{aligned} x_{i+\frac{1}{2}} &= \frac{h_{i+1}x_i + h_i x_{i+1}}{h_i + h_{i+1}}, \quad 1 \le i \le M-1, \\ y_{j+\frac{1}{2}} &= \frac{k_{j+1}y_j + k_j y_{j+1}}{k_j + k_{j+1}}, \quad 1 \le j \le N-1. \end{aligned} \tag{2.7}$$

It is interesting to notice and emphasize the sequential order of the FV points x (or y):

$$\begin{aligned} &x_i = x_{i-\frac{1}{2}} + \frac{1}{2}h_i, \qquad x_{i+\frac{1}{2}} = x_{i-\frac{1}{2}} + h_i, \\ &x_{i+1} = x_{i-\frac{1}{2}} + h_i + \frac{1}{2}h_{i+1}, \qquad x_{i+\frac{3}{2}} = x_{i-\frac{1}{2}} + h_i + h_{i+1}. \end{aligned} \tag{2.8}$$

We now introduce the following function space:

$$V_h := \left\{ \begin{array}{l} \text{step functions } u_h \text{ on } \overline{\Omega} \text{ such that} \\ u_h|_{K_{i,j}} = u_{i,j},\ 0 \le i \le M+1,\ 0 \le j \le N+1, \\ \text{and } u_{i,j} = 0,\ \text{if } i = 0, M+1,\ \text{or } j = 0, N+1 \end{array} \right\}, \tag{2.9}$$

and, for any $u_h \in V_h$, we write

$$u_h = \sum_{\substack{1 \le i \le M \\ 1 \le j \le N}} u_{i,j} \chi_{K_{i,j}}, \tag{2.10}$$

where $\chi_{K_{i,j}}$ is the characteristic function of $K_{i,j}$.
For $1 \le i \le M$, $0 \le j \le N$, we define the quadrilateral $K_{i,j+\frac{1}{2}}$ (solid line in Fig.

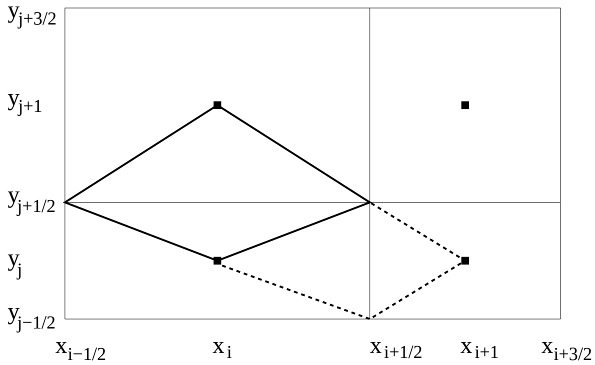

FIGURE 3. $K_{i+\frac{1}{2},j}$ (dashed line) and $K_{i,j+\frac{1}{2}}$ (thick solid line) as domains of constancy for the FV derivative.

3):
(2.11)

$$K_{i,j+\frac{1}{2}} \text{ is the quadrilateral connecting } (x_{i-\frac{1}{2}}, y_{j+\frac{1}{2}}),\ (x_i, y_{j+1}),\ (x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}), \text{ and } (x_i, y_j),$$

and, for $0 \le i \le M, 1 \le j \le N$, we also define the quadrilateral $K_{i+\frac{1}{2},j}$[1] (dashed line in Fig. 3):

$$\text{(2.12)} \qquad K_{i+\frac{1}{2},j} \text{ is the quadrilateral connecting } (x_i, y_j),\ (x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}),\ (x_{i+1}, y_j), \text{ and } (x_{i+\frac{1}{2}}, y_{j-\frac{1}{2}}).$$

The discrete FV derivative $\nabla_h u_h = (\nabla_h^x u_h, \nabla_h^y u_h)$ for $u_h \in V_h$ is obtained by TSES; see [4] and [10].
Here we slightly modify the original TSES of [4] and [10] to ensure the consistency (see (3.4) and (3.16) below) and we set:

$$\text{(2.13)} \qquad \nabla_h^x u_h = \begin{cases} \dfrac{u_{i+1,j} - u_{i,j}}{h_{i+\frac{1}{2}}} \text{ on } K_{i+\frac{1}{2},j}, & 0 \le i \le M, \ 1 \le j \le N, \\ \dfrac{u_{i+\frac{1}{2},j+\frac{1}{2}} - u_{i-\frac{1}{2},j+\frac{1}{2}}}{h_i} \text{ on } K_{i,j+\frac{1}{2}}, & 1 \le i \le M, \ 0 \le j \le N, \end{cases}$$

where we define the term $u_{i+\frac{1}{2},j+\frac{1}{2}}$ by a weighted average between the four neighbors (i,j), $(i+1,j)$, $(i,j+1)$ and $(i+1,j+1)$: for $0 \le i \le M$, $0 \le j \le N$,

$$\text{(2.14)} \qquad u_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{h_i k_j u_{i+1,j+1} + h_i k_{j+1} u_{i+1,j} + h_{i+1} k_j u_{i,j+1} + h_{i+1} k_{j+1} u_{i,j}}{(h_i + h_{i+1})(k_j + k_{j+1})};$$

note that, due to (2.5) and (2.9), $u_{i+1/2,j+1/2}$ is equal to 0 when $i = 0, M$ or $j = 0, N$.
The definition of $\nabla_h^y u_h$ is similar; we obtain $\nabla_h^y u_h$ from (2.13) by replacing x and h by y and k, and interchanging the indices i and j. We define on V_h, the scalar

[1] Because $x_0 = x_{1/2} = 0$, $K_{1/2,j}$ is in fact a triangle, $1 \le j \le N$. The same is true of $K_{M+1/2,j}$, $K_{i,1/2}$ and $K_{i,N+1/2}$.

products $(\cdot,\cdot)_{V_h}$ and $((\cdot,\cdot))_{V_h}$ that mimic those of $L^2(\Omega)$ and $H_0^1(\Omega)$: for $u_h, v_h \in V_h$,

$$\begin{aligned}(u_h, v_h)_{V_h} &= (u_h, v_h)_{L^2(\Omega)} = \sum_{\substack{1\le i\le M\\1\le j\le N}} u_{i,j} v_{i,j} h_i k_j,\\ ((u_h, v_h))_{V_h} &= \left(\nabla_h^x u_h, \nabla_h^x v_h\right)_{L^2(\Omega)} + \left(\nabla_h^y u_h, \nabla_h^y v_h\right)_{L^2(\Omega)},\end{aligned}\tag{2.15}$$

with

$$\begin{aligned}&\left(\nabla_h^x u_h, \nabla_h^x v_h\right)_{L^2(\Omega)}\\ &\quad= \sum_{\substack{0\le i\le M\\1\le j\le N}} \frac{k_j}{2h_{i+\frac12}} \left(u_{i+1,j} - u_{i,j}\right)\left(v_{i+1,j} - v_{i,j}\right)\\ &\quad+ \sum_{\substack{1\le i\le M\\0\le j\le N}} \frac{k_{j+\frac12}}{2h_i} (u_{i+\frac12,j+\frac12} - u_{i-\frac12,j+\frac12})(v_{i+\frac12,j+\frac12} - v_{i-\frac12,j+\frac12}).\end{aligned}\tag{2.16}$$

The corresponding norms $|\cdot|_{V_h}$ and $||\cdot||_{V_h}$ are defined as usual.
We will need to impose some restrictions on the mesh sizes h_i and k_j. Here we begin with the "uniformity" assumptions; further hypotheses appear below in (5.9) and (6.11). We set

$$\begin{aligned}\overline{h} &= \max_{1\le i\le M} h_i, \quad \underline{h} = \min_{1\le i\le M} h_i,\\ \overline{k} &= \max_{1\le j\le N} k_j, \quad \underline{k} = \min_{1\le j\le N} k_j,\\ \overline{\rho} &= \max(\overline{h}, \overline{k}), \quad \underline{\rho} = \min(\underline{h}, \underline{k}),\end{aligned}\tag{2.17}$$

and assume that, as $\overline{\rho} \to 0$, there exit $0 < \alpha_x, \alpha_y < 1$ such that

$$\underline{h} \ge \alpha_x \overline{h}, \quad \underline{k} \ge \alpha_y \overline{k},\tag{2.18}$$

and, furthermore, comparing the x and y directions, we also assume that,

$$\underline{k} \ge \alpha_y \overline{h}, \quad \underline{h} \ge \alpha_x \overline{k}.\tag{2.19}$$

If we set

$$\underline{\alpha} = \min\left(\alpha_x, \alpha_y\right), \quad \overline{\alpha} = \max\left(\alpha_x, \alpha_y\right),\tag{2.20}$$

then we infer from (2.18) that

$$\begin{aligned}M\overline{h} &\le M\frac{1}{\alpha_x}\underline{h} \le \frac{1}{\alpha_x}\sum_{1\le i\le M} h_i \le \frac{1}{\alpha_x} \le \frac{1}{\underline{\alpha}},\\ N\overline{k} &\le N\frac{1}{\alpha_y}\underline{k} \le \frac{1}{\alpha_y}\sum_{1\le j\le N} k_j \le \frac{1}{\alpha_y} \le \frac{1}{\underline{\alpha}}.\end{aligned}\tag{2.21}$$

We start with the following easy lemma which provides the discrete Poincaré inequality for the FV space.

Lemma 2.1. *For every* $u_h \in V_h$,

$$|u_h|_{V_h} \le \sqrt{2}\underline{\alpha}^{-1}||u_h||_{V_h}.\tag{2.22}$$

Proof. We consider u_h as in (2.10). For any $1 \le i \le M$, $1 \le j \le N$, since $u_{0,j} = 0$, we have

$$u_{i,j} = \left(u_{i,j} - u_{i-1,j}\right) + \left(u_{i-1,j} - u_{i-2,j}\right) + \cdots + \left(u_{1,j} - u_{0,j}\right).$$

Therefore, by the Schwarz inequality,

$$(u_{i,j})^2 \leq i \sum_{l=0}^{i-1} (u_{l+1,j} - u_{l,j})^2 \leq M \sum_{l=0}^{M} (u_{l+1,j} - u_{l,j})^2 .$$

Then, using (2.21), we find

$$\begin{aligned}
|u_h|^2_{V_h} &= \sum_{\substack{1\leq i\leq M\\ 1\leq j\leq N}} u^2_{i,j} h_i k_j \\
&\leq \sum_{1\leq j\leq N} \left\{ \sum_{1\leq i\leq M} M \sum_{0\leq l\leq M} (u_{l+1,j} - u_{l,j})^2 h_i \right\} k_j \\
&\leq M^2 \sum_{\substack{1\leq j\leq N\\ 0\leq l\leq M}} (u_{l+1,j} - u_{l,j})^2 \overline{h} k_j \\
&\leq \left(M\overline{h}\right)^2 \sum_{\substack{1\leq j\leq N\\ 0\leq l\leq M}} \frac{(u_{l+1,j} - u_{l,j})^2}{h_{l+\frac{1}{2}}} k_j \\
&\leq \frac{2}{\alpha_x^2} |\nabla^x_h u_h|^2_{L^2(\Omega)} \leq 2\underline{\alpha}^{-2} |\nabla^x_h u_h|^2_{L^2(\Omega)}
\end{aligned}$$

Similarly, we find

$$|u_h|^2_{V_h} \leq 2\underline{\alpha}^{-2} |\nabla^y_h u_h|^2_{L^2(\Omega)} ,$$

and hence, we obtain (2.22). □

3. External approximation of $H^1_0(\Omega)$ by FV.

As we briefly mentioned in the introduction, here we introduce an external approximation of a normed space V as a set consisting of a normed space F, an isomorphism $\overline{\omega}$ of V into F and a family of triples $\{W_h, p_h, r_h\}$, in which, for each h, W_h is a normed space, p_h is a linear prolongation operator of W_h into F and r_h is a restriction operator of V into W_h; see [3], [2], [15] and Fig. 4. Here we set $V = H^1_0(\Omega)$, $F = L^2(\Omega)^3$ and $W_h = V_h$, and define the maps $\overline{\omega}$, p_h and r_h as follows:

(3.1)
$$\begin{aligned}
&\overline{\omega}(u) = (u, D_x u, D_y u) , \quad \forall u \in H^1_0(\Omega), \\
&p_h(u_h) = (u_h, \nabla^x_h u_h, \nabla^y_h u_h) , \quad \forall u_h \in V_h,
\end{aligned}$$

and, for all $v \in \mathcal{V} = \mathcal{C}^\infty_0(\Omega)$,

(3.2)
$$r_h(v)(x,y) = \begin{cases} \dfrac{1}{h_i k_j} \displaystyle\int_{K_{i,j}} v(x',y')dx'dy', \ (x,y) \in K_{i,j}, \ 1 \leq i \leq M, \ 1 \leq j \leq N, \\ 0, \ (x,y) \in K_{i,j}, \ i = 0 \text{ or } M+1, \text{ or } j = 0 \text{ or } N+1. \end{cases}$$

Thanks to the Poincaré inequality (2.22), we have

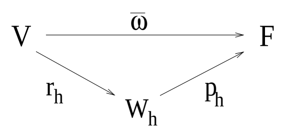

FIGURE 4

$$\begin{aligned} ||p_h(u_h)||_F^2 &= |u_h|^2_{L^2(\Omega)} + |\nabla_h u_h|^2_{L^2(\Omega)} \\ &= |u_h|^2_{V_h} + ||u_h||^2_{V_h} \\ &\leq \left(1+c_0^2\right)||u_h||^2_{V_h}, \end{aligned} \tag{3.3}$$

where $c_0 = \sqrt{2}\underline{\alpha}$ is independent of h_i or k_j; the stability of the p_h follows.
To prove the convergence of our FV scheme, we need to prove the two following properties (see [3] and [15]):

$$\begin{aligned} &(C1) \quad \forall u \in \mathcal{V}, \;\; p_h r_h u \to \overline{\omega} u \text{ in } F \text{ as } \overline{\rho} \to 0, \\ &(C2) \quad \text{If } u_h \in V_h \text{ and } p_h u_h \to \phi \text{ in } F \text{ weakly as } \overline{\rho} \to 0, \text{ then } \phi \in \overline{\omega} V. \end{aligned} \tag{3.4}$$

Along the proof of these properties, we will use repeatedly the following lemmas:

Lemma 3.1. *For any quadrilateral K with barycenter (x_G, y_G) and area $|K|$, and for any function $\phi \in \mathcal{C}^2(\overline{K})$, we have*

$$\frac{1}{|K|}\int_K \phi\left(x', y'\right) dx'dy' = \phi\left(x_G, y_G\right) + O'\left(|K|\right), \tag{3.5}$$

where

$$O'(|K|) \leq |\phi|_{\mathcal{C}^2(K)}|K|. \tag{3.6}$$

We also introduce the useful interpolation lemmas:

Lemma 3.2. *For any function $\phi \in \mathcal{C}^2(l)$ where l is the line connecting the points ξ_1 and ξ_2 in $\mathbb{R}^2$, and for any point $\xi \in l$, we have*

$$\phi(\xi_2) - \phi(\xi_1) = \nabla\phi(\xi)\cdot(\xi_2 - \xi_1) + O'(|\xi_2 - \xi_1|^2), \tag{3.7}$$

where

$$O'(|\xi_2 - \xi_1|^2) \leq |\phi|_{\mathcal{C}^2(l)}|\xi_2 - \xi_1|^2. \tag{3.8}$$

Lemma 3.3. *For any two-dimensional convex polygon K with p vertices ξ_i, $1 \leq i \leq p$, $p \geq 2$, we consider a point ξ in K, $\xi = \sum_{i=1}^p \lambda_i \xi_i$ where $\sum_{i=1}^p \lambda_i = 1$ and $\lambda_i \geq 0$. Then, for any function $\phi \in \mathcal{C}^2(K)$, we have*

$$\sum_{1\leq i\leq p} \lambda_i \phi(\xi_i) = \phi(\xi) + O'\big(\max_{1\leq i,j\leq p}|\xi_i - \xi_j|^2\big). \tag{3.9}$$

where

$$O'\big(\max_{1\leq i,j\leq p}|\xi_i - \xi_j|^2\big) \leq |\phi|_{\mathcal{C}^2(K)} \max_{1\leq i,j\leq p}|\xi_i - \xi_j|^2. \tag{3.10}$$

For any other point η in K,

$$\sum_{1\leq i\leq p} \lambda_i \phi(\xi_i) = \phi(\eta) + O'\big(\max_{1\leq i,j\leq p}|\xi_i - \xi_j|\big). \tag{3.11}$$

We omit the proofs of these elementary lemmas; using the Taylor expansions, one can easily verify (3.5), (3.7) and (3.9). We obtain (3.11) by combining (3.7) and (3.9).

3.1. Proof of $(C1)$ for FV. To prove $(C1)$, we first show that, for $u \in \mathcal{V}$,

$$r_h u \to u \text{ strongly in } L^2(\Omega) \text{ as } \overline{\rho} \to 0, \tag{3.12}$$

We consider $(x, y) \in K_{i,j} \subset \Omega$ and, using (3.5) for the set $K_{i,j}$ with center (x_i, y_j) and area $h_i k_j$, and also using (3.7), we write

$$\begin{aligned} |r_h u(x,y) - u(x,y)| &= \left| \frac{1}{h_i k_j} \int_{K_{i,j}} u(x', y') dx' dy' - u(x,y) \right| \\ &\leq \left| u(x_i, y_j) + O'(\overline{\rho}^2) - u(x,y) \right| \\ &\leq \sup_{\Omega} |Du| \overline{\rho} + O'(\overline{\rho}^2) \to 0 \text{ as } \overline{\rho} \to 0, \end{aligned} \tag{3.13}$$

where $O'(\overline{\rho}^2)$ means, throughout this article, $O'(\overline{\rho}^2) \leq c\overline{\rho}^2$, c independent of the mesh sizes; here, of course, c depends on the $\mathcal{C}^2$ norm of u.
Hence, from (3.13), $r_h u \to u$ in $L^\infty(\Omega)$ as $\overline{\rho} \to 0$ and (3.12) holds.
To show that, for $u \in \mathcal{V}$,

$$\nabla_h^x r_h u \to D_x u \text{ strongly in } L^2(\Omega) \text{ as } \overline{\rho} \to 0, \tag{3.14}$$

we consider two cases. Firstly, if $(x, y) \in K_{i+\frac{1}{2},j}$ for some $0 \leq i \leq M$, $1 \leq j \leq N$, then, using (2.4), (2.13) and (3.7) in the x direction along (x_i, x_{i+1}) at $x_{i+\frac{1}{2}}$, we have

$$\begin{aligned} \nabla_h^x r_h u(x,y) &= D_x u(x_{i+\frac{1}{2}}, y_j) + O'(\overline{\rho}) \\ &= D_x u(x,y) + O'(\overline{\rho}). \end{aligned} \tag{3.15}$$

Secondly, if $(x, y) \in K_{i,j+\frac{1}{2}}$ for some i, j, we first notice that, using (3.2) and (3.5), the term $(r_h u)_{i+\frac{1}{2},j+\frac{1}{2}}$ is obtained by the same average as in (2.14). Then, applying (3.9) to u where K is the quadrilateral connecting (x_i, y_j), (x_{i+1}, y_j), (x_{i+1}, y_{j+1}) and (x_i, y_{j+1}), with the weighted average $(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}})$ in (2.7), we find that, for $1 \leq i \leq M-1$, $1 \leq j \leq N-1$,

$$(r_h u)_{i+\frac{1}{2},j+\frac{1}{2}} = u\big(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}\big) + O'(\overline{\rho}^2). \tag{3.16}$$

Then, using also (2.13) for $r_h u$ and (3.7) again in the x direction along $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$ at x_i, we obtain that, for $(x, y) \in K_{i,j+\frac{1}{2}}$,

$$\begin{aligned} \nabla_h^x r_h u(x,y) &= D_x u(x_i, y_{j+\frac{1}{2}}) + O'(\overline{\rho}) \\ &= D_x u(x,y) + O'(\overline{\rho}). \end{aligned} \tag{3.17}$$

Hence, from (3.15) and (3.17), we see that in both cases

$$|\nabla_h^x r_h u(x,y) - D_x u(x,y)| \quad \leq c\overline{\rho}, \tag{3.18}$$

where the constant c related to the $\mathcal{C}^2$ norm of u is independent of x, y and $\overline{\rho}$.
Therefore, $\nabla_h^x r_h u \to D_x u$ in $L^\infty(\Omega)$ as $\overline{\rho} \to 0$ and (3.14) holds. The proof being the same for the y derivative, $(C1)$ is now proven.

Remark 3.1. *As we mentioned in the introduction, due to the weak consistency of the FV scheme, we cannot prove $(C2)$ for FV directly (see* [6] *and* [9]*). Instead we introduce the corresponding (associated) FD scheme and prove $(C2)$ for it (as well as the stability and $(C1)$ properties). Then, comparing the FV space and the FD space, we will finally prove $(C2)$ for FV.*

4. The corresponding Finite Difference Setting.

To define the FD mesh associated with the previous FV mesh, we set $\widetilde{x}_0 = \widetilde{x}_{\frac{1}{2}} = \widetilde{y}_0 = \widetilde{y}_{\frac{1}{2}} = 0$ and $\widetilde{x}_{M+1} = \widetilde{x}_{M+\frac{1}{2}} = \widetilde{y}_{N+1} = \widetilde{y}_{N+\frac{1}{2}} = 1$. We also set $\widetilde{x}_i = x_i$, $\widetilde{y}_j = y_j$ for $1 \le i \le M$, $1 \le j \le N$. Then, we define the FD nodal points $\widetilde{x}_{i+\frac{1}{2}}$, $\widetilde{y}_{j+\frac{1}{2}}$ (see Fig. 5):

$$\begin{aligned} \widetilde{x}_{i+\frac{1}{2}} &= \frac{1}{2}\left(\widetilde{x}_i + \widetilde{x}_{i+1}\right), 1 \le i \le M-1, \\ \widetilde{y}_{j+\frac{1}{2}} &= \frac{1}{2}\left(\widetilde{y}_j + \widetilde{y}_{j+1}\right), 1 \le j \le N-1. \end{aligned} \tag{4.1}$$

Together with the order of the FV points in (2.8), it is also interesting to notice the sequential order of the FD points $\widetilde{x}$ (or $\widetilde{y}$):

$$\begin{aligned} \widetilde{x}_i &= x_i, \\ \widetilde{x}_{i+\frac{1}{2}} &= \frac{1}{2}\left(x_i + x_{i+1}\right) = (\text{with } (2.8)) = x_{i-\frac{1}{2}} + \frac{3}{4}h_i + \frac{1}{4}h_{i+1}. \end{aligned} \tag{4.2}$$

Hence, comparing with (2.8), we see that

$$x_i = \widetilde{x}_i < x_{i+\frac{1}{2}},\ \widetilde{x}_{i+\frac{1}{2}} < x_{i+1} = \widetilde{x}_{i+1},\ 1 \le i \le M-1, \tag{4.3}$$

but the respective orders of $x_{i+\frac{1}{2}}$ and $\widetilde{x}_{i+\frac{1}{2}}$ may vary with i.
We set

$$\begin{aligned} \widetilde{h}_i &= \widetilde{x}_{i+\frac{1}{2}} - \widetilde{x}_{i-\frac{1}{2}}, \quad \widetilde{k}_j = \widetilde{y}_{j+\frac{1}{2}} - \widetilde{y}_{j-\frac{1}{2}}, \quad 1 \le i \le M, \quad 1 \le j \le N, \\ \widetilde{h}_{i+\frac{1}{2}} &= \widetilde{x}_{i+1} - \widetilde{x}_i, \quad \widetilde{k}_{j+\frac{1}{2}} = \widetilde{y}_{j+1} - \widetilde{y}_j, \quad 0 \le i \le M, \quad 0 \le j \le N, \end{aligned} \tag{4.4}$$

and compare the FV mesh sizes h (or k) and the FD mesh sizes $\widetilde{h}$ (or $\widetilde{k}$):

$$\begin{aligned} \widetilde{h}_i &= \widetilde{x}_{i+\frac{1}{2}} - \widetilde{x}_{i-\frac{1}{2}} \\ &= \left(x_{i-\frac{1}{2}} + \frac{3}{4}h_i + \frac{1}{4}h_{i+1}\right) - \left(x_{i-\frac{3}{2}} + \frac{3}{4}h_{i-1} + \frac{1}{4}h_i\right) \\ &= \left(x_{i-\frac{1}{2}} - x_{i-\frac{3}{2}}\right) - \frac{3}{4}h_{i-1} + \frac{1}{2}h_i + \frac{1}{4}h_{i+1} \\ &= \frac{1}{4}(h_{i-1} + 2h_i + h_{i+1}), \\ \widetilde{h}_{i+\frac{1}{2}} &= h_{i+\frac{1}{2}}. \end{aligned} \tag{4.5}$$

Due to (4.1), we also observe useful relations among the FD mesh sizes: for $2 \le i \le M-2$,

$$\widetilde{h}_{i+\frac{1}{2}} + \widetilde{h}_{i-\frac{1}{2}} = (\widetilde{x}_{i+1} + \widetilde{x}_i) - (\widetilde{x}_i + \widetilde{x}_{i-1}) = 2(\widetilde{x}_{i+\frac{1}{2}} - \widetilde{x}_{i-\frac{1}{2}}) = 2\widetilde{h}_i. \tag{4.6}$$

The $\widetilde{K}_{i,j}$ are defined in the same manner as the $K_{i,j}$ in (2.2) by replacing x and y by $\widetilde{x}$ and $\widetilde{y}$; their sides are $\widetilde{h}_{i+\frac{1}{2}}$ and $\widetilde{k}_{j+\frac{1}{2}}$, and the geometric relation between the FV and FD grids appears on Fig. 6 (in which e.g. $\widetilde{x}_{i+\frac{1}{2}} < x_{i+\frac{1}{2}}$ but $\widetilde{y}_{j-\frac{1}{2}} > y_{j-\frac{1}{2}}$; see (4.3)).

The space of step functions for FD is given by,

$$\widetilde{V}_h := \left\{ \begin{array}{l} \text{step functions } \widetilde{u}_h \text{ on } \overline{\Omega} \text{ such that} \\ \widetilde{u}_h|_{\widetilde{K}_{i,j}} = \widetilde{u}_{i,j},\ 0 \le i \le M+1,\ 0 \le j \le N+1, \\ \text{and } \widetilde{u}_{i,j} = 0, \text{ if } i = 0, M+1, \text{ or } j = 0, N+1 \end{array} \right\}. \tag{4.7}$$

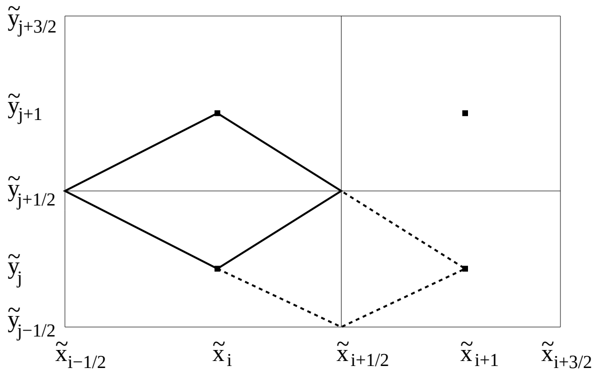

FIGURE 5. The corresponding FD mesh and sets $\widetilde{K}_{i,j}$ (rectangles), $\widetilde{K}_{i+\frac{1}{2},j}$ (dashed line) and $\widetilde{K}_{i,j+\frac{1}{2}}$ (thick solid line).

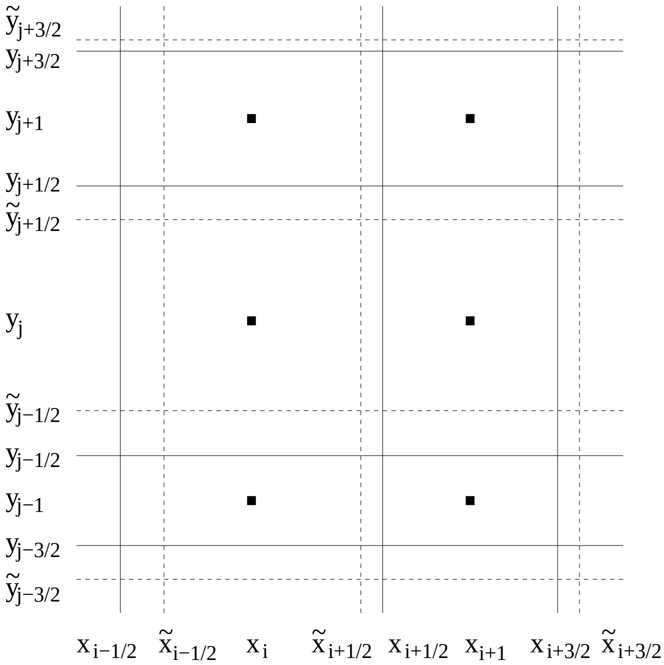

FIGURE 6. The FV(solid lines) and FD(dashed lines) meshes in Ω.

We also introduce the discrete FD derivative: for $\widetilde{u}_h \in \widetilde{V}_h$,

$$\widetilde{\nabla}_h^x \widetilde{u}_h = \begin{cases} \dfrac{\widetilde{u}_{i+1,j} - \widetilde{u}_{i,j}}{\widetilde{h}_{i+\frac{1}{2}}} \text{ on } \widetilde{K}_{i+\frac{1}{2},j}, \quad 0 \leq i \leq M, \quad 1 \leq j \leq N, \\ \dfrac{\widetilde{u}_{i+\frac{1}{2},j+\frac{1}{2}} - \widetilde{u}_{i-\frac{1}{2},j+\frac{1}{2}}}{\widetilde{h}_i} \text{ on } \widetilde{K}_{i,j+\frac{1}{2}}, \quad 1 \leq i \leq M, \quad 0 \leq j \leq N, \end{cases} \tag{4.8}$$

with

$$\widetilde{u}_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{\widetilde{u}_{i+1,j+1} + \widetilde{u}_{i+1,j} + \widetilde{u}_{i,j+1} + \widetilde{u}_{i,j}}{4},\ 0 \le i \le M,\ 0 \le j \le N.^2 \tag{4.9}$$

The discrete derivative $\widetilde{\nabla}_h^y \widetilde{u}_h$ is defined similarly by replacing $\widetilde{x}$, $\widetilde{h}$ by $\widetilde{y}$, $\widetilde{k}$ and interchanging the indices i, j in (4.8). The domains of constancy for $\widetilde{\nabla}_h \widetilde{u}_h$ are the sets $\widetilde{K}_{i,j+\frac{1}{2}}$ for $1 \le i \le M, 0 \le j \le N$ and $\widetilde{K}_{i+\frac{1}{2},j}$ for $0 \le i \le M, 1 \le j \le N$ that are defined in the similar way as in the FV case using $\widetilde{x}$, $\widetilde{y}$ instead of x, y (see (2.11), (2.12) and Fig. 5). The scalar products $(\cdot,\cdot)_{\widetilde{V}_h}$, $((\cdot,\cdot))_{\widetilde{V}_h}$ and corresponding norms $|\cdot|_{\widetilde{V}_h}$, $||\cdot||_{\widetilde{V}_h}$ are obtained from (2.15) by replacing u_h, v_h, h, k by $\widetilde{u}_h$, $\widetilde{v}_h$, $\widetilde{h}$, $\widetilde{k}$.

Exactly as in the FV case, we can prove the discrete Poincaré inequality for FD, which occurs with the same constant:

Lemma 4.1. *For every* $\widetilde{u}_h \in \widetilde{V}_h$*,*

$$|\widetilde{u}_h|_{\widetilde{V}_h} \le \sqrt{2}\underline{\alpha}^{-1} ||\widetilde{u}_h||_{\widetilde{V}_h}. \tag{4.10}$$

5. External approximation of $H_0^1(\Omega)$ by FD.

We again consider the diagram in Fig. 4 with now $V = H_0^1(\Omega)$, $F = L^2(\Omega)^3$ and $W_h = \widetilde{V}_h$. The maps $\overline{\omega}$ and p_h are the same as in (3.2), but we now define $r_h v$, for $v \in \mathcal{V}$, as follows

$$r_h(v)(x,y) = \begin{cases} v(\widetilde{x}_i, \widetilde{y}_j),\ (x,y) \in \widetilde{K}_{i,j},\ 1 \le i \le M,\ \ 1 \le j \le N, \\ 0,\ (x,y) \in \widetilde{K}_{i,j},\ i = 0 \text{ or } N+1, \text{ or } j = 0 \text{ or } M+1. \end{cases} \tag{5.1}$$

Similar to the FV case, the stability of the operators p_h follows from the Poincaré inequality in Lemma 4.1.

5.1. Proof of the property $(C1)$ for FD.

The proof is similar, and even simpler than in the FV case; we recall it briefly. To show that, for $u \in \mathcal{V}$,

$$r_h u \to u \text{ strongly in } L^2(\Omega) \text{ as } \overline{\rho} \to 0, \tag{5.2}$$

we consider $(x,y) \in \widetilde{K}_{i,j} \subset \Omega$, and, using (3.7), write

$$\begin{aligned} |r_h u(x,y) - u(x,y)| &= |u(\widetilde{x}_i, \widetilde{y}_j) - u(x,y)| \\ &\le \sup_{\Omega} |Du| \overline{\rho} + O'(\overline{\rho}^2). \end{aligned} \tag{5.3}$$

Hence $r_h u \to u$ in $L^\infty(\Omega)$ as $\overline{\rho} \to 0$ and (5.2) is valid.
To show that, for $u \in \mathcal{V}$,

$$\widetilde{\nabla}_h^x r_h u \to D_x u \text{ strongly in } L^2(\Omega) \text{ as } \overline{\rho} \to 0, \tag{5.4}$$

we let $(x,y) \in \Omega$ and consider two cases. Firstly, if $(x,y) \in \widetilde{K}_{i+\frac{1}{2},j}$ for some $0 \le i \le M$, $1 \le j \le N$, then by using (4.8), (5.1), and (3.7) along $(\widetilde{x}_i, \widetilde{x}_{i+1})$ at $\widetilde{x}_{i+\frac{1}{2}}$,

$$\begin{aligned} \widetilde{\nabla}_h^x r_h u(x,y) &= D_x u(\widetilde{x}_{i+\frac{1}{2}}, \widetilde{y}_j) + O'(\overline{\rho}) \\ &= D_x u(x,y) + O'(\overline{\rho}). \end{aligned} \tag{5.5}$$

[2] Note that $\widetilde{u}_{i+1/2,j+1/2}$ may not be zero for $i = 0, M$ or $j = 0, N$, but may be "small". This is not inconsistent with the Dirichlet boundary condition which is well enforced by (4.7).

Secondly, if $(x,y) \in \widetilde{K}_{i,j+\frac{1}{2}}$ for some i,j, we observe that, from (5.1), the term $(r_h u)_{i+\frac{1}{2},j+\frac{1}{2}}$ is given by the same average as in (4.9). Hence, applying (3.9) to u where K is the quadrilateral connecting $(\widetilde{x}_i,\widetilde{y}_j)$, $(\widetilde{x}_{i+1},\widetilde{y}_j)$, $(\widetilde{x}_{i+1},\widetilde{y}_{j+1})$ and $(\widetilde{x}_i,\widetilde{y}_{j+1})$, with barycenter $(\widetilde{x}_{i+\frac{1}{2}},\widetilde{y}_{j+\frac{1}{2}})$, we find that, for $1 \le i \le M-1$, $1 \le j \le N-1$,

$$(r_h u)_{i+\frac{1}{2},j+\frac{1}{2}} = u\big(\widetilde{x}_{i+\frac{1}{2}},\widetilde{y}_{j+\frac{1}{2}}\big) + O'(\overline{\rho}^2). \tag{5.6}$$

From (4.8), (5.1) and (5.6), we now infer that, for $(x,y) \in \widetilde{K}_{i,j+\frac{1}{2}}$, using (3.7) again along $(\widetilde{x}_{i-\frac{1}{2}},\widetilde{x}_{i+\frac{1}{2}})$ at $\widetilde{x}_i$,

$$\begin{aligned} \widetilde{\nabla}_h^x r_h u(x,y) &= D_x u(\widetilde{x}_i,\widetilde{y}_{j+\frac{1}{2}}) + O'(\overline{\rho}) \\ &= D_x u(x,y) + O'(\overline{\rho}). \end{aligned} \tag{5.7}$$

From (5.5) and (5.7), we see that in all cases

$$|\widetilde{\nabla}_h^x r_h u(x,y) - D_x u(x,y)| \quad \le O'(\overline{\rho}) \quad \to 0 \text{ as } \overline{\rho} \to 0, \tag{5.8}$$

and thus, $\widetilde{\nabla}_h^x r_h u \to D_x u$ in $L^\infty(\Omega)$ as $\overline{\rho} \to 0$ and (5.4) holds. With the same result for the y variable, the proof of $(C1)$ is complete.

5.2. Proof of $(C2)$ for FD. To prove $(C2)$, we impose another condition to our mesh sizes, namely

$$\begin{aligned} \sup_{2\le i\le M-1} \frac{h_{i+1}-h_{i-1}}{\underline{h}} &= \eta_1 \to 0 \text{ as } \overline{\rho} \to 0, \\ \sup_{2\le j\le N-1} \frac{k_{j+1}-k_{j-1}}{\underline{k}} &= \eta_2 \to 0 \text{ as } \overline{\rho} \to 0. \end{aligned} \tag{5.9}$$

Note that $\eta_1 = \eta_2 = 0$ for a uniform mesh and in the typically annoying case considered in [6] and [9] where $h_{2i} = h$, $h_{2i+1} = 2h$.
We want to verify $(C2)$; so let us assume that $\widetilde{u}_h \in \widetilde{V}_h$, $\forall h$, and that, as $\overline{\rho} \to 0$:

$$\widetilde{u}_h \rightharpoonup \phi_0, \quad \widetilde{\nabla}_h \widetilde{u}_h \rightharpoonup (\phi_1,\phi_2) \text{ weakly in } L^2(\Omega). \tag{5.10}$$

We have to show that, for $\forall \theta \in C_0^\infty(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} \big(\overline{\phi}_1,\overline{\phi}_2\big)\,\theta dxdy = -\int_{\mathbb{R}^2} \overline{\phi}_0 D\theta dxdy \tag{5.11}$$

where $\overline{\phi}$ is equal to ϕ in Ω and to 0 in $\mathbb{R}^2 \setminus \Omega$. Indeed if (5.11) is proven, then $(\overline{\phi}_1,\overline{\phi}_2) = D\overline{\phi}_0$ which implies that $\overline{\phi}_0 \in H^1(\mathbb{R}^2)$, and thus $\phi_0 \in H_0^1(\Omega)$ with $(\phi_1,\phi_2) = D\phi_0$, since $\overline{\phi}_0$ vanishes outside of Ω.
We set

$$I_h = (I_h^x, I_h^y) = \int_{\mathbb{R}^2} \overline{\widetilde{\nabla}_h \widetilde{u}_h}\theta dxdy = \int_\Omega \widetilde{\nabla}_h \widetilde{u}_h \theta dxdy. \tag{5.12}$$

By (5.10), we promptly see that I_h converges to the left-hand side of (5.11). Both directions being handled similarly, to verify (5.11) and $(C2)$, we only need to show that:

$$I_h^x = \int_\Omega \widetilde{\nabla}_h^x \widetilde{u}_h \theta dxdy \to -\int_\Omega \phi_0 D_x\theta dxdy = -\int_{\mathbb{R}^2} \overline{\phi}_0 D_x \theta dxdy, \text{ as } \overline{\rho} \to 0. \tag{5.13}$$

The regions of constancy $\widetilde{K}_{i+\frac{1}{2},j}$ and $\widetilde{K}_{i,j+\frac{1}{2}}$ of the discrete FD derivatives $\widetilde{\nabla}_h \widetilde{u}_h$ are handled differently. Therefore, to obtain (5.13), we write

(5.14)

$$I_h^x = I_1 + I_2,$$

$$I_1 = \int_\Omega \Big(\sum_{\substack{0\le i\le M\\1\le j\le N}} \widetilde{\nabla}_h^x \widetilde{u}_h \chi_{\widetilde{K}_{i+\frac{1}{2},j}} \Big) \theta dxdy, \quad I_2 = \int_\Omega \Big(\sum_{\substack{1\le i\le M\\0\le j\le N}} \widetilde{\nabla}_h^x \widetilde{u}_h \chi_{\widetilde{K}_{i,j+\frac{1}{2}}} \Big) \theta dxdy,$$

and, after we simplify each of I_1 and I_2, we will show that under the assumptions (2.18), (2.19) and (5.9), I_h^x converges to the right-hand side of (5.13) as $\overline{\rho} \to 0$.

We observe that, for $0 \le i \le M$, $1 \le j \le N$, the areas of the quadrilaterals $\widetilde{K}_{i+\frac{1}{2},j}$ are $2^{-1}\widetilde{h}_{i+\frac{1}{2}}\widetilde{k}_j$ and their centers are $(\widehat{x}_{i+\frac{1}{2}}, \widehat{y}_j)$, where

$$\begin{aligned} &\widehat{x}_{i+\frac{1}{2}} = \widetilde{x}_{i+\frac{1}{2}}, \;\; 1 \le i \le M-1, \;\; \widehat{x}_{\frac{1}{2}} = \frac{1}{3}\widetilde{x}_1, \;\; \widehat{x}_{M+\frac{1}{2}} = \frac{1}{3}\left(2 + \widetilde{x}_M\right), \\ &\widehat{y}_j = \frac{1}{3}\big(\widetilde{y}_{j-\frac{1}{2}} + \widetilde{y}_j + \widetilde{y}_{j+\frac{1}{2}}\big), \;\; 1 \le j \le N; \end{aligned} \tag{5.15}$$

hence, using Lemma 3.1, we find that

$$\frac{1}{\widetilde{h}_{i+\frac{1}{2}}} \int_{\widetilde{K}_{i+\frac{1}{2},j}} \theta dxdy = \frac{\widetilde{k}_j}{2} \theta\big(\widehat{x}_{i+\frac{1}{2}}, \widehat{y}_j\big) + O'(\overline{\rho}^3), \tag{5.16}$$

where, due to (3.5), $O'(\overline{\rho}^3)$ is bounded by $c|\theta|_{\mathcal{C}^2}\overline{\rho}^3$ for a constant c independent of the mesh sizes.

We then simplify I_1 by using (4.7) and (5.16):

$$\begin{aligned} I_1 &= \sum_{\substack{0\le i\le M\\1\le j\le N}} \int_{\widetilde{K}_{i+\frac{1}{2},j}} \widetilde{h}_{i+\frac{1}{2}}^{-1} \left(\widetilde{u}_{i+1,j} - \widetilde{u}_{i,j}\right) \theta dxdy \\ &= (\text{by changing the indices and using } u_{0,j} = u_{M+1,j} = 0) \\ &= \sum_{\substack{0\le i\le M\\1\le j\le N}} \widetilde{u}_{i,j} \Big\{ \widetilde{h}_{i-\frac{1}{2}}^{-1} \int_{\widetilde{K}_{i-\frac{1}{2},j}} \theta dxdy - \widetilde{h}_{i+\frac{1}{2}}^{-1} \int_{\widetilde{K}_{i+\frac{1}{2},j}} \theta dxdy \Big\} \\ &= -\frac{1}{2} \sum_{\substack{0\le i\le M\\1\le j\le N}} \widetilde{u}_{i,j} \widetilde{k}_j \Big\{ \theta\big(\widehat{x}_{i+\frac{1}{2}},\, \widehat{y}_j\big) - \theta\big(\widehat{x}_{i-\frac{1}{2}},\, \widehat{y}_j\big) \Big\} + O'(\overline{\rho}) |\widetilde{u}_h|_{\widetilde{V}_h}. \end{aligned} \tag{5.17}$$

Using (3.7) for θ along $(\widehat{x}_{i-\frac{1}{2}}, \widehat{x}_{i+\frac{1}{2}})$ at x_i, we write I_1 in (5.17) in the form:

$$I_1 = -\frac{1}{2} \sum_{\substack{1\le i\le M\\1\le j\le N}} \widetilde{u}_{i,j} D_x \theta\left(\widetilde{x}_i, \widehat{y}_j\right) \widetilde{h}_i \widetilde{k}_j + O'(\overline{\rho}) |\widetilde{u}_h|_{\widetilde{V}_h}. \tag{5.18}$$

Now, to obtain the expression of I_2 similar to (5.18), we consider the quadrilaterals $\widetilde{K}_{i,j+\frac{1}{2}}$, $1 \le i \le M$, $0 \le j \le N$ with areas $2^{-1}\widetilde{h}_i\widetilde{k}_{j+\frac{1}{2}}$ and centers $(\widehat{x}_i, \widehat{y}_{j+\frac{1}{2}})$:

$$\begin{aligned} &\widehat{x}_i = \frac{1}{3}\big(\widetilde{x}_{i-\frac{1}{2}} + \widetilde{x}_i + \widetilde{x}_{i+\frac{1}{2}}\big), \;\; 1 \le i \le M, \\ &\widehat{y}_{j+\frac{1}{2}} = \widetilde{y}_{j+\frac{1}{2}}, \;\; 1 \le j \le N-1, \;\; \widehat{y}_{\frac{1}{2}} = \frac{1}{3}\widetilde{y}_1, \;\; \widehat{y}_{N+\frac{1}{2}} = \frac{1}{3}\left(2 + \widetilde{y}_N\right). \end{aligned} \tag{5.19}$$

Using Lemma 3.1 on $\widetilde{K}_{i,j+\frac{1}{2}}$, we find that, for $1 \leq i \leq M$, $0 \leq j \leq N$,

$$\frac{1}{\widetilde{h}_i}\int_{\widetilde{K}_{i,j+\frac{1}{2}}} \theta dxdy = \frac{\widetilde{k}_{j+\frac{1}{2}}}{2}\theta\big(\widehat{x}_i, \widehat{y}_{j+\frac{1}{2}}\big) + O'(\overline{\rho}^3), \tag{5.20}$$

where $O'(\overline{\rho}^3)$ is bounded by $c|\theta|_{\mathcal{C}^2}\overline{\rho}^3$ as in (5.16).
Due to the definition of $\widetilde{\nabla}_h$ in (4.8), (4.9), using (3.7), we write I_2 in (5.14) in the form:
(5.21)

$$\begin{aligned}
I_2 \;&= \sum_{\substack{1\leq i\leq M\\ 1\leq j\leq N}} \int_{\widetilde{K}_{i,j+\frac{1}{2}}} \frac{1}{4\widetilde{h}_i}\left(\widetilde{u}_{i+1,j+1} + \widetilde{u}_{i+1,j} - \widetilde{u}_{i-1,j+1} - \widetilde{u}_{i-1,j}\right)\theta dxdy\\
&= \text{(changing indices for } i \text{ and using (5.19) and (5.20))}\\
&= -\frac{1}{8}\sum_{\substack{1\leq i\leq M\\ 1\leq j\leq N}} \left(\widetilde{u}_{i,j+1} + \widetilde{u}_{i,j}\right)\widetilde{k}_{j+\frac{1}{2}}\Big\{\theta\big(\widehat{x}_{i+1}, \widehat{y}_{j+\frac{1}{2}}\big) - \theta\big(\widehat{x}_{i-1}, \widehat{y}_{j+\frac{1}{2}}\big)\Big\}\\
&\quad + O'(\overline{\rho})|\widetilde{u}_h|_{\widetilde{V}_h}\\
&= -\frac{1}{8}\sum_{\substack{1\leq i\leq M\\ 1\leq j\leq N}} \left(\widetilde{u}_{i,j+1} + \widetilde{u}_{i,j}\right)\widetilde{k}_{j+\frac{1}{2}} D_x\theta\big(\widetilde{x}_i, \widehat{y}_{j+\frac{1}{2}}\big)\big(\widehat{x}_{i+1} - \widehat{x}_{i-1}\big) + O'(\overline{\rho})|\widetilde{u}_h|_{\widetilde{V}_h},
\end{aligned}$$

where, using (5.20) and treating boundary terms for $i = 0, M$, $O'(\overline{\rho})$ is bounded by $c|\theta|_{\mathcal{C}^2}\overline{\rho}$ for a constant c independent of the mesh sizes.
For I_2 in (5.21), we change the indices for j, and use (3.9) and the analog of (4.6) in the y direction. As a result, we find

$$I_2 \;= -\frac{1}{4}\sum_{\substack{1\leq i\leq M\\ 1\leq j\leq N}} \widetilde{u}_{i,j} D_x\theta(\widetilde{x}_i, \overline{y}_j)(\widehat{x}_{i+1} - \widehat{x}_{i-1})\widetilde{k}_j + O'(\overline{\rho})|\widetilde{u}_h|_{\widetilde{V}_h}, \tag{5.22}$$

where

$$\overline{y}_j = \frac{\widetilde{k}_{j+\frac{1}{2}}\widehat{y}_{j+\frac{1}{2}} + \widetilde{k}_{j-\frac{1}{2}}\widehat{y}_{j-\frac{1}{2}}}{\widetilde{k}_{j+\frac{1}{2}} + \widetilde{k}_{j-\frac{1}{2}}}. \tag{5.23}$$

We also notice that, for $2 \leq i \leq M-1$,

$$\begin{aligned}
\widehat{x}_{i-1} - \widehat{x}_{i+1} \;&= \text{(using (5.15) for } \widehat{x}_i \text{, and (2.6), (2.8), (4.2) and (4.5))}\\
&= -2\widetilde{h}_i + \frac{1}{12}\left\{(h_i - h_{i+2}) + (h_i - h_{i-2})\right\},
\end{aligned} \tag{5.24}$$

and hence, using the assumption (5.9), (5.22) yields

$$I_2 \;= -\frac{1}{2}\sum_{\substack{1\leq i\leq M\\ 1\leq j\leq N}} \widetilde{u}_{i,j} D_x\theta(\widetilde{x}_i, \overline{y}_j)\widetilde{h}_i\widetilde{k}_j + O'(\eta_1)|\widetilde{u}_h|_{\widetilde{V}_h}, \tag{5.25}$$

where $O'(\eta_1)$ is bounded by $c\eta_1$ for a constant c independent of the mesh sizes.

Now, since the area of $\widetilde{K}_{i,j}$ is equal to $\widetilde{h}_i\widetilde{k}_j$, from (5.14), (5.18) and (5.25), we can rewrite I_h^x in the form:

$$I_h^x = -\int_\Omega \widetilde{u}_h D_h^x\theta dxdy + O'(\eta_1)|\widetilde{u}_h|_{\widetilde{V}_h}, \tag{5.26}$$

with

$$D_h^x\theta = \sum_{\substack{1\le i\le M\\ 1\le j\le N}} \frac{1}{2}\big\{D_x\theta(\widetilde{x}_i,\widehat{y}_j)+D_x\theta(\widetilde{x}_i,\overline{y}_j)\big\}\chi_{\widetilde{K}_{i,j}}. \tag{5.27}$$

Then, since θ is in $\mathcal{C}_0^\infty(\mathbb{R}^2)$, it is easy to see that, as $\overline{\rho}\to 0$, $D_h^x\theta$ converges to $D_x\theta$ strongly in $L^2(\Omega)$ and we now conclude that

$$I_h^x \to -\int_\Omega \phi_0 D_x\theta dxdy \text{ as } \overline{\rho}\to 0; \tag{5.28}$$

the proof of $(C2)$ for FD is complete.

6. A map between the FD and FV spaces.

To prove the $(C2)$ property for finite volumes, we introduce the following map $\Lambda_h : \widetilde{V}_h \to V_h$ between the FD and FV spaces:

$$\Lambda_h\Big(\sum_{\substack{1\le i\le M\\ 1\le j\le N}} \widetilde{u}_{i,j}\chi_{\widetilde{K}_{i,j}}\Big) = \sum_{\substack{1\le i\le M\\ 1\le j\le N}} \widetilde{u}_{i,j}\chi_{K_{i,j}}. \tag{6.1}$$

Its inverse Λ_h^{-1} mapping V_h into $\widetilde{V}_h$ is defined by

$$\Lambda_h^{-1}\Big(\sum_{\substack{1\le i\le M\\ 1\le j\le N}} u_{i,j}\chi_{K_{i,j}}\Big) = \sum_{\substack{1\le i\le M\\ 1\le j\le N}} u_{i,j}\chi_{\widetilde{K}_{i,j}}. \tag{6.2}$$

We now state and prove a lemma estimating the L^2 norms of $u_h - \Lambda_h^{-1}u_h$ and of $\widetilde{u}_h - \Lambda_h\widetilde{u}_h$.

Lemma 6.1. *For any $u_h \in V_h$ and $\widetilde{u}_h \in \widetilde{V}_h$, we have*

$$\begin{aligned} &|u_h - \Lambda_h^{-1}u_h|_{L^2(\Omega)} \le \sqrt{30}\underline{\alpha}^{-\frac{1}{2}}\overline{\rho}\|u_h\|_{V_h},\\ &|\widetilde{u}_h - \Lambda_h\widetilde{u}_h|_{L^2(\Omega)} \le \sqrt{30}\underline{\alpha}^{-\frac{1}{2}}\overline{\rho}\|\widetilde{u}_h\|_{\widetilde{V}_h}. \end{aligned} \tag{6.3}$$

Proof. We only prove $(6.3)_1$. By the points ordering relations (2.8) and (4.2) (see also Fig. 6), we see that $K_{i,j}$ can only intersect its neighbors $\widetilde{K}_{i,j\pm1}$, $\widetilde{K}_{i\pm1,j}$, $\widetilde{K}_{i\pm1,j\pm1}$ and that

$$\big|u_h - \Lambda_h^{-1}u_h\big| = \begin{cases} |u_{i,j}-u_{i,j\pm1}|, \text{ on } K_{i,j}\cap\widetilde{K}_{i,j\pm1},\\ |u_{i,j}-u_{i\pm1,j}|, \text{ on } K_{i,j}\cap\widetilde{K}_{i\pm1,j},\\ |u_{i,j}-u_{i-1,j\pm1}|, \text{ on } K_{i,j}\cap\widetilde{K}_{i-1,j\pm1},\\ |u_{i,j}-u_{i+1,j\pm1}|, \text{ on } K_{i,j}\cap\widetilde{K}_{i+1,j\pm1},\\ 0, \text{ on } K_{i,j}\setminus\Big(\widetilde{K}_{i,j\pm1}\cup\widetilde{K}_{i\pm1,j}\cup\widetilde{K}_{i\pm1,j\pm1}\Big). \end{cases} \tag{6.4}$$

Thus

$$\begin{aligned}\int_{K_{i,j}}\big|u_h-\Lambda_h^{-1}u_h\big|^2 \le \quad & h_ik_j\left\{|u_{i,j}-u_{i,j\pm1}|^2+|u_{i,j}-u_{i\pm1,j}|^2\right.\\ &\left.+|u_{i,j}-u_{i-1,j\pm1}|^2+|u_{i,j}-u_{i+1,j\pm1}|^2\right\}.\end{aligned} \tag{6.5}$$

For the last two terms in the right-hand side of (6.5), we write

$$\begin{aligned} &|u_{i,j}-u_{i-1,j\pm1}|^2 \le 2\,|u_{i,j}-u_{i-1,j}|^2 + 2\,|u_{i-1,j}-u_{i-1,j\pm1}|^2,\\ &|u_{i,j}-u_{i+1,j\pm1}|^2 \le 2\,|u_{i,j}-u_{i+1,j}|^2 + 2\,|u_{i+1,j}-u_{i+1,j\pm1}|^2, \end{aligned} \tag{6.6}$$

and hence

$$\int_{K_{i,j}} \left|u_h - \Lambda_h^{-1} u_h\right|^2 \leq 5h_i k_j \left\{|u_{i,j} - u_{i,j\pm 1}|^2 + |u_{i,j} - u_{i\pm 1,j}|^2 + |u_{i-1,j} - u_{i-1,j\pm 1}|^2 + |u_{i+1,j} - u_{i+1,j\pm 1}|^2\right\}. \tag{6.7}$$

By summing in i and j, we find

(6.8)

$$\begin{aligned}
&\int_{\Omega} \left|u_h - \Lambda_h^{-1} u_h\right|^2 dxdy \\
&\leq 5 \sum_{\substack{1\leq i\leq M\\1\leq j\leq N}} h_i k_j \left\{|u_{i,j} - u_{i,j\pm 1}|^2 + |u_{i,j} - u_{i\pm 1,j}|^2 \right.\\
&\quad \left. + |u_{i-1,j} - u_{i-1,j\pm 1}|^2 + |u_{i+1,j} - u_{i+1,j\pm 1}|^2\right\} \\
&\leq \text{(by changing indices)} \\
&\leq 15 \sum_{1\leq i\leq M} \overline{h} \Big\{ \sum_{1\leq j\leq N} k_j \left|u_{i,j} - u_{i,j\pm 1}\right|^2 \Big\} + 5 \sum_{1\leq j\leq N} \overline{k} \Big\{ \sum_{1\leq i\leq M} h_i \left|u_{i,j} - u_{i\pm 1,j}\right|^2 \Big\} \\
&\leq \text{(with (2.13) and (2.19))} \\
&\leq 30 \underline{h}^{-1} \overline{hk}^2 \sum_{\substack{1\leq i\leq M\\0\leq j\leq N}} \frac{\left|u_{i,j+1} - u_{i,j}\right|^2}{k_{j+\frac{1}{2}}} h_i + 10 \underline{k}^{-1} \overline{kh}^2 \sum_{\substack{0\leq i\leq M\\1\leq j\leq N}} \frac{\left|u_{i+1,j} - u_{i,j}\right|^2}{h_{i+\frac{1}{2}}} k_j \\
&\leq 30 \underline{\alpha}^{-1} \overline{\rho}^2 \|u_h\|_{V_h}^2;
\end{aligned}$$

$(6.3)_1$ follows. □

We pursue in our task of proving the property $(C2)$ for finite volumes, and now we want to compare $\nabla_h u_h$ and $\widetilde{\nabla}_h \Lambda_h^{-1} u_h$. We recall that the domains of constancy of the FV derivatives $\nabla_h^x u_h$, $\nabla_h^y u_h$ are the quadrilaterals $K_{i+\frac{1}{2},j}$ and $K_{i,j+\frac{1}{2}}$; for the finite differences, those are the quadrilaterals $\widetilde{K}_{i+\frac{1}{2},j}$ and $\widetilde{K}_{i,j+\frac{1}{2}}$. Considering for instance $K_{i+\frac{1}{2},j}$, we notice that this quadrilateral may only intersect the quadrilaterals $\widetilde{K}_{i+\frac{1}{2},j+s}$, $s = 0, \pm 1$ and $K_{i+r,j\pm\frac{1}{2}}$, $r = 0, 1$; see Fig. 7. To obtain the property $(C2)$ for FV, we impose an additional technical assumption on the mesh, namely:

$$\begin{aligned}
&\sup_{\substack{2\leq i\leq M-1\\2\leq j\leq N-1}} \frac{1}{\underline{hk}} \left|K_{i,j+\frac{1}{2}} \setminus \left(K_{i,j+\frac{1}{2}} \cap \widetilde{K}_{i,j+\frac{1}{2}}\right)\right| = \eta_3 \to 0 \text{ as } \overline{\rho} \to 0, \\
&\sup_{\substack{2\leq i\leq M-1\\2\leq j\leq N-1}} \frac{1}{\underline{hk}} \left|K_{i+\frac{1}{2},j} \setminus \left(K_{i+\frac{1}{2},j} \cap \widetilde{K}_{i+\frac{1}{2},j}\right)\right| = \eta_4 \to 0 \text{ as } \overline{\rho} \to 0.
\end{aligned} \tag{6.9}$$

We notice that the areas of $K_{i,j+\frac{1}{2}}$ and $\widetilde{K}_{i,j+\frac{1}{2}}$ are respectively $2^{-1} h_i k_{j+\frac{1}{2}}$ and $2^{-1} \widetilde{h}_i k_{j+\frac{1}{2}}$, and therefore, there exits $0 < \widehat{\widehat{h}}_i \leq \min(h_i, \widetilde{h}_i)$ such that

$$\left|K_{i,j+\frac{1}{2}} \cap \widetilde{K}_{i,j+\frac{1}{2}}\right| = \frac{1}{2} \widehat{\widehat{h}}_i k_{j+\frac{1}{2}}. \tag{6.10}$$

Thanks to (6.10), we can rewrite the assumptions (6.9) in the form:

$$\sup_{2\leq i\leq M-1} \frac{h_i - \widehat{\widehat{h}}_i}{\underline{h}} = \eta_3 \to 0, \quad \sup_{2\leq j\leq N-1} \frac{k_j - \widehat{\widehat{k}}_j}{\underline{k}} = \eta_4 \to 0 \text{ as } \overline{\rho} \to 0, \tag{6.11}$$

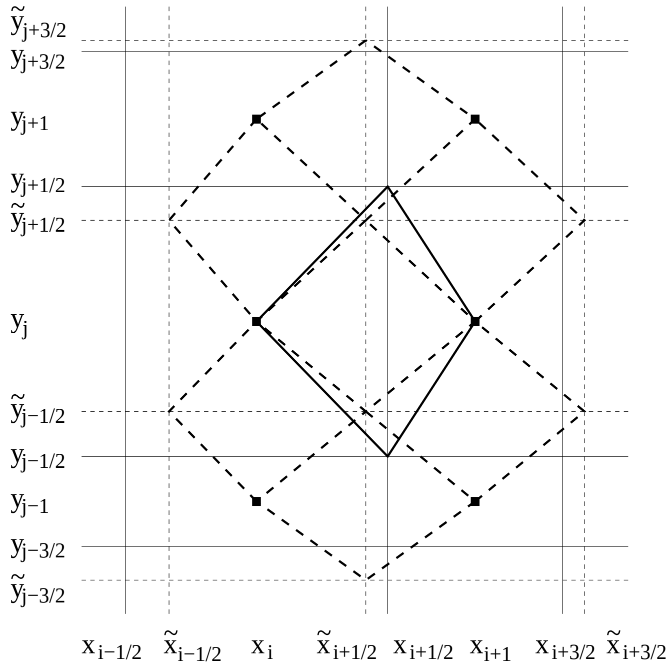

FIGURE 7. $K_{i+\frac{1}{2},j}$ (thick solid lines) and nearby $\widetilde{K}_{i,j+\frac{1}{2}}$, $\widetilde{K}_{i+\frac{1}{2},j}$ (thick dashed lines) which may intersect.

and, due to the assumptions (5.9) and (6.11), we find that

(6.12)

$$\begin{aligned}
\sup_{3\le i\le M-2}\frac{\widehat{\widehat{h}}_{i+1}-\widehat{\widehat{h}}_{i-1}}{\underline{h}} &= \sup_{3\le i\le M-2}\Big(\frac{\widehat{\widehat{h}}_{i+1}-h_{i+1}}{\underline{h}}+\frac{h_{i+1}-h_{i-1}}{\underline{h}}+\frac{h_{i-1}-\widehat{\widehat{h}}_{i-1}}{\underline{h}}\Big)\\
&\le \eta_1+\eta_3,\\
\sup_{3\le j\le N-2}\frac{\widehat{\widehat{k}}_{j+1}-\widehat{\widehat{k}}_{j-1}}{\underline{k}} &= \sup_{3\le j\le N-2}\Big(\frac{\widehat{\widehat{k}}_{j+1}-k_{j+1}}{\underline{k}}+\frac{k_{j+1}-k_{j-1}}{\underline{k}}+\frac{k_{j-1}-\widehat{\widehat{k}}_{j-1}}{\underline{k}}\Big)\\
&\le \eta_2+\eta_4.
\end{aligned}$$

We now state the following lemma which is the last ingredient needed to show the property $(C2)$ for FV.

Lemma 6.2. *Under the assumptions (2.18), (2.19), (5.9) and (6.11), we have that, for any $\varphi\in C_0^\infty(\mathbb{R}^2)$, and $u_h\in V_h$,*

$$\begin{aligned}
\left|\int_\Omega\left(\nabla_h^x u_h-\widetilde{\nabla}_h^x\Lambda_h^{-1}u_h\right)\varphi\,dxdy\right| &\le c(\eta_1+\eta_3+\eta_4)\|u_h\|_{V_h},\\
\left|\int_\Omega\left(\nabla_h^y u_h-\widetilde{\nabla}_h^y\Lambda_h^{-1}u_h\right)\varphi\,dxdy\right| &\le c(\eta_2+\eta_3+\eta_4)\|u_h\|_{V_h},
\end{aligned}\tag{6.13}$$

for a constant c depending on φ, but independent of the mesh sizes.

Proof. We only prove the first inequality in (6.13).
We observe from (2.13) and (4.8) that $\nabla_h^x u_h-\widetilde{\nabla}_h^x\ \Lambda_h^{-1}u_h$ vanishes on the sets

$K_{i+\frac{1}{2},j} \cap \widetilde{K}_{i+\frac{1}{2},j}$, but it may not on the sets $K_{i,j+\frac{1}{2}} \cap \widetilde{K}_{i,j+\frac{1}{2}}$.
On each $K_{i,j+\frac{1}{2}} \cap \widetilde{K}_{i,j+\frac{1}{2}}$, by using (2.13), (2.14), (4.5), (4.8) and (4.9), we find

$$\begin{aligned}
&\left(\nabla_h^x u_h - \widetilde{\nabla}_h^x \, \Lambda_h^{-1} u_h\right)_{K_{i,j+\frac{1}{2}} \cap \widetilde{K}_{i,j+\frac{1}{2}}} \\
&\quad = h_i^{-1}\big(u_{i+\frac{1}{2},j+\frac{1}{2}} - u_{i-\frac{1}{2},j+\frac{1}{2}}\big) - \widetilde{h}_i^{-1}\big(\widetilde{u}_{i+\frac{1}{2},j+\frac{1}{2}} - \widetilde{u}_{i-\frac{1}{2},j+\frac{1}{2}}\big) \\
&\quad = J_1 + J_2,
\end{aligned} \tag{6.14}$$

where

$$\begin{aligned}
J_1 &= \frac{h_{i+1}k_j u_{i,j+1} + h_{i+1}k_{j+1}u_{i,j}}{h_i(h_i + h_{i+1})(k_j + k_{j+1})} - \frac{h_{i-1}k_j u_{i,j+1} + h_{i-1}k_{j+1}u_{i,j}}{h_i(h_{i-1} + h_i)(k_j + k_{j+1})}, \\
J_2 &= \frac{k_j u_{i+1,j+1} + k_{j+1}u_{i+1,j}}{(h_i + h_{i+1})(k_j + k_{j+1})} - \frac{k_j u_{i-1,j+1} + k_{j+1}u_{i-1,j}}{(h_{i-1} + h_i)(k_j + k_{j+1})} \\
&\quad - \frac{u_{i+1,j+1} + u_{i+1,j} - u_{i-1,j+1} - u_{i-1,j}}{h_{i-1} + 2h_i + h_{i+1}}.
\end{aligned} \tag{6.15}$$

We now rewrite the terms J_1 and J_2 in the form:

$$\begin{aligned}
J_1 &= \frac{(h_{i+1} - h_{i-1})}{(h_{i-1} + h_i)(h_i + h_{i+1})(k_j + k_{j+1})}\left[k_j u_{i,j+1} + k_{j+1}u_{i,j}\right] \\
&= \frac{(h_{i+1} - h_{i-1})(k_j - k_{j+1})}{2(h_{i-1} + h_i)(h_i + h_{i+1})(k_j + k_{j+1})}\left(u_{i,j+1} - u_{i,j}\right) + J_1', \\
J_1' &= \frac{h_{i+1} - h_{i-1}}{2(h_{i-1} + h_i)(h_i + h_{i+1})}\left(u_{i,j+1} + u_{i,j}\right),
\end{aligned} \tag{6.16}$$

and

$$\begin{aligned}
J_2 &= \left[\frac{1}{2(h_i + h_{i+1})} - \frac{1}{h_{i-1} + 2h_i + h_{i+1}}\right]\left(u_{i+1,j+1} + u_{i+1,j}\right) \\
&\quad + \left[\frac{-1}{2(h_{i-1} + h_i)} + \frac{1}{h_{i-1} + 2h_i + h_{i+1}}\right]\left(u_{i-1,j+1} + u_{i-1,j}\right) \\
&\quad + \frac{k_j - k_{j+1}}{2(k_j + k_{j+1})}\left[\frac{u_{i+1,j+1} - u_{i+1,j}}{h_i + h_{i+1}} - \frac{u_{i-1,j+1} - u_{i-1,j}}{h_{i-1} + h_i}\right] \\
&= \frac{k_j - k_{j+1}}{2(k_j + k_{j+1})}\left[\frac{u_{i+1,j+1} - u_{i+1,j}}{h_i + h_{i+1}} - \frac{u_{i-1,j+1} - u_{i-1,j}}{h_{i-1} + h_i}\right] + J_2' \\
J_2' &= \frac{h_{i-1} - h_{i+1}}{2(h_i + h_{i+1})(h_{i-1} + 2h_i + h_{i+1})}\left(u_{i+1,j+1} + u_{i+1,j}\right) \\
&\quad + \frac{h_{i-1} - h_{i+1}}{2(h_{i-1} + h_i)(h_{i-1} + 2h_i + h_{i+1})}\left(u_{i-1,j+1} + u_{i-1,j}\right).
\end{aligned} \tag{6.17}$$

We can combine as follows the terms J_1' and J_2' in (6.16) and (6.17):

$$\begin{aligned}
&\frac{h_{i-1} - h_{i+1}}{2(h_i + h_{i+1})(h_{i-1} + 2h_i + h_{i+1})}\left[(u_{i+1,j+1} - u_{i,j+1}) + (u_{i+1,j} - u_{i,j})\right] \\
&+ \frac{h_{i-1} - h_{i+1}}{2(h_{i-1} + h_i)(h_{i-1} + 2h_i + h_{i+1})}\left[(u_{i-1,j+1} - u_{i,j+1}) + (u_{i-1,j} - u_{i,j})\right].
\end{aligned} \tag{6.18}$$

Then, using (6.16), (6.17) and (6.18), we can rewrite (6.14) in the form:
(6.19)

$$\begin{aligned}
&J_1+J_2=K_1+K_2+K_3,\\
&K_1=\frac{(h_{i+1}-h_{i-1})(k_j-k_{j+1})}{2(h_{i-1}+h_i)(h_i+h_{i+1})(k_j+k_{j+1})}\left(u_{i,j+1}-u_{i,j}\right),\\
&K_2=\frac{h_{i-1}-h_{i+1}}{2(h_i+h_{i+1})(h_{i-1}+2h_i+h_{i+1})}\left[(u_{i+1,j+1}-u_{i,j+1})+(u_{i+1,j}-u_{i,j})\right]\\
&\quad+\frac{h_{i-1}-h_{i+1}}{2(h_{i-1}+h_i)(h_{i-1}+2h_i+h_{i+1})}\left[(u_{i-1,j+1}-u_{i,j+1})+(u_{i-1,j}-u_{i,j})\right],\\
&K_3=\frac{k_j-k_{j+1}}{2(k_j+k_{j+1})}\left\{\frac{u_{i+1,j+1}-u_{i+1,j}}{h_i+h_{i+1}}-\frac{u_{i-1,j+1}-u_{i-1,j}}{h_{i-1}+h_i}\right\}.
\end{aligned}$$

Using (2.16) and the assumption (5.9), we then treat the term K_1:

$$\begin{aligned}
&\Big|\sum_{\substack{1\le i\le M\\0\le j\le N}}\int_{K_{i,j+\frac12}\cap\widetilde{K}_{i,j+\frac12}}K_1\varphi dxdy\Big|\\
&\qquad\le c\sum_{\substack{1\le i\le M\\0\le j\le N}}\frac{|h_{i+1}-h_{i-1}|}{\underline{h}^2}|u_{i,j+1}-u_{i,j}|\Big|\int_{K_{i,j+\frac12}\cap\widetilde{K}_{i,j+\frac12}}\varphi dxdy\Big|\\
&\qquad\le c\sum_{\substack{1\le i\le M\\0\le j\le N}}|h_{i+1}-h_{i-1}||u_{i,j+1}-u_{i,j}|\\
&\qquad\le c\big(\eta_1+O'(\overline{\rho})\big)\|u_h\|_{V_h},
\end{aligned}\tag{6.20}$$

where c is a positive constant depending on φ, but independent of the mesh sizes. Using the estimate similar to (6.20), we also obtain

$$\Big|\sum_{\substack{1\le i\le M\\0\le j\le N}}\int_{K_{i,j+\frac12}\cap\widetilde{K}_{i,j+\frac12}}K_2\varphi dxdy\Big|\le c\big(\eta_1+O'(\overline{\rho})\big)\|u_h\|_{V_h}.\tag{6.21}$$

For the term K_3, we use (3.5) on $K_{i,j+\frac12}\cap\widetilde{K}_{i,j+\frac12}$ with area $2^{-1}\widehat{\widehat{h}}_i k_{j+\frac12}$ and barycenter $(x_{i,j+\frac12},y_{i,j+\frac12})$:
(6.22)

$$\int_{K_{i,j+\frac12}\cap\widetilde{K}_{i,j+\frac12}}\varphi dxdy=\frac12\widehat{\widehat{h}}_i k_{j+\frac12}\varphi_{i,j+\frac12}+O'(\overline{\rho}^4),\quad \varphi_{i,j+\frac12}=\varphi\big(x_{i,j+\frac12},y_{i,j+\frac12}\big),$$

where $O'(\overline{\rho}^4)$ is bounded by $c|\varphi|_{\mathcal{C}^2}\overline{\rho}^4$. Then, by (2.6) and (6.22), we write
(6.23)

$$\Big|\sum_{\substack{1\le i\le M\\0\le j\le N}}\int_{K_{i,j+\frac12}\cap\widetilde{K}_{i,j+\frac12}}K_3\varphi dxdy\Big|=\Big|\sum_{\substack{1\le i\le M\\0\le j\le N}}K_3\frac12\widehat{\widehat{h}}_i k_{j+\frac12}\varphi_{i,j+\frac12}\Big|+O'(\overline{\rho})\|u_h\|_{V_h},$$

and, by changing the indices in (6.23), we find

(6.24)

$$\Big|\sum_{\substack{1\le i\le M\\0\le j\le N}}\int_{K_{i,j+\frac12}\cap\widetilde{K}_{i,j+\frac12}}K_3\varphi dxdy\Big|$$
$$=\Big|\sum_{\substack{1\le i\le M\\0\le j\le N}}\frac{k_j-k_{j+1}}{8}(u_{i,j+1}-u_{i,j})\Big\{\frac{\widehat{\widehat{h}}_{i-1}}{h_{i-1}+h_i}\varphi_{i-1,j+\frac12}-\frac{\widehat{\widehat{h}}_{i+1}}{h_i+h_{i+1}}\varphi_{i+1,j+\frac12}\Big\}\Big|$$
$$+O'(\overline{\rho})\|u_h\|_{V_h}.$$

We also find that

$$\frac{\widehat{\widehat{h}}_{i-1}}{h_{i-1}+h_i}\varphi_{i-1,j+\frac12}-\frac{\widehat{\widehat{h}}_{i+1}}{h_i+h_{i+1}}\varphi_{i+1,j+\frac12}$$
$$=\frac12\Big\{\frac{\widehat{\widehat{h}}_{i-1}}{h_{i-1}+h_i}+\frac{\widehat{\widehat{h}}_{i+1}}{h_i+h_{i+1}}\Big\}(\varphi_{i-1,j+\frac12}-\varphi_{i+1,j+\frac12})\tag{6.25}$$
$$+\frac12\Big\{\frac{\widehat{\widehat{h}}_{i-1}}{h_{i-1}+h_i}-\frac{\widehat{\widehat{h}}_{i+1}}{h_i+h_{i+1}}\Big\}(\varphi_{i-1,j+\frac12}+\varphi_{i+1,j+\frac12}),$$

and

$$\frac{\widehat{\widehat{h}}_{i-1}}{h_{i-1}+h_i}-\frac{\widehat{\widehat{h}}_{i+1}}{h_i+h_{i+1}}$$
$$=\frac{h_i\big(\widehat{\widehat{h}}_{i-1}-\widehat{\widehat{h}}_{i+1}\big)+\widehat{\widehat{h}}_{i-1}\,(h_{i+1}-h_{i-1})+\big(\widehat{\widehat{h}}_{i-1}-\widehat{\widehat{h}}_{i+1}\big)h_{i-1}}{(h_{i-1}+h_i)\,(h_i+h_{i+1})}\tag{6.26}$$
$$=\frac{\widehat{\widehat{h}}_{i-1}-\widehat{\widehat{h}}_{i+1}}{h_i+h_{i+1}}+\frac{\widehat{\widehat{h}}_{i-1}\,(h_{i+1}-h_{i-1})}{(h_{i-1}+h_i)\,(h_i+h_{i+1})}.$$

Therefore, using (6.25) and (6.26), we can bound (6.24) by $L_1+L_2+L_3$ where
(6.27)

$$L_1=\Big|\sum_{\substack{1\le i\le M\\0\le j\le N}}\Big\{\frac{k_j-k_{j+1}}{16}(u_{i,j+1}-u_{i,j})\Big[\frac{\widehat{\widehat{h}}_{i-1}}{h_{i-1}+h_i}+\frac{\widehat{\widehat{h}}_{i+1}}{h_i+h_{i+1}}\Big]\big(\varphi_{i-1,j+\frac12}-\varphi_{i+1,j+\frac12}\big)\Big\}\Big|,$$
$$L_2=\Big|\sum_{\substack{1\le i\le M\\0\le j\le N}}\frac{k_j-k_{j+1}}{16}(u_{i,j+1}-u_{i,j})\frac{\widehat{\widehat{h}}_{i-1}-\widehat{\widehat{h}}_{i+1}}{h_i+h_{i+1}}\big(\varphi_{i-1,j+\frac12}+\varphi_{i+1,j+\frac12}\big)\Big|,$$
$$L_3=\Big|\sum_{\substack{1\le i\le M\\0\le j\le N}}\Big\{\frac{k_j-k_{j+1}}{16}(u_{i,j+1}-u_{i,j})\frac{\widehat{\widehat{h}}_{i-1}\,(h_{i+1}-h_{i-1})}{(h_{i-1}+h_i)\,(h_i+h_{i+1})}\big(\varphi_{i-1,j+\frac12}+\varphi_{i+1,j+\frac12}\big)\Big\}\Big|.$$

We control the L_i, $1\le i\le 3$, terms of (6.27): for a positive constant c independent of the mesh sizes,

$$L_1\le c\sum_{\substack{1\le i\le M\\0\le j\le N}}|u_{i,j+1}-u_{i,j}|\,\Big|\varphi_{i-1,j+\frac12}-\varphi_{i+1,j+\frac12}\Big|\,\overline{\rho}\le c|D\varphi|_{L^2}\|u_h\|_{V_h}\overline{\rho}.\tag{6.28}$$

Under the assumption (6.12), we use the analog of (6.20) and find

$$L_2 \le c \sum_{\substack{1\le i\le M\\ 0\le j\le N}} |u_{i,j+1} - u_{i,j}| \left|\widehat{\widehat{h}}_{i+1} - \widehat{\widehat{h}}_{i-1}\right| \le c\big(\eta_1 + \eta_3 + O'(\overline{\rho})\big) \|u_h\|_{V_h} . \tag{6.29}$$

Under the assumption (5.9), the term L_3 can be easily treated as (6.20) and we finally obtain, from (6.19)-(6.21) and (6.27)-(6.29), that

$$\Big| \sum_{\substack{1\le i\le M\\ 0\le j\le N}} \int_{K_{i,j+\frac12}\cap \widetilde{K}_{i,j+\frac12}} \left(\nabla_h^x u_h - \widetilde{\nabla}_h^x \Lambda_h^{-1} u_h\right) \varphi dxdy\Big| \le c(\eta_1+\eta_3)\|u_h\|_{V_h}. \tag{6.30}$$

Now, to treat $\nabla_h^x u_h - \widetilde{\nabla}_h^x\ \Lambda_h^{-1} u_h$ on $K_{i,j+\frac12} \setminus \big(K_{i,j+\frac12} \cap \widetilde{K}_{i,j+\frac12}\big)$ or $K_{i+\frac12,j} \setminus \big(K_{i+\frac12,j} \cap \widetilde{K}_{i+\frac12,j}\big)$, we recall that the quadrilateral $K_{i+\frac12,j}$ may only intersect the quadrilaterals $\widetilde{K}_{i+\frac12,j+s}$, $s = 0, \pm1$ and $K_{i+r,j\pm\frac12}$, $r = 0, 1$, and similarly the $K_{i,j+\frac12}$ may only intersect the quadrilaterals $\widetilde{K}_{i+s,j+\frac12}$, $s = 0, \pm1$ and $\widetilde{K}_{i\pm\frac12,j+r}$, $r = 0, 1$; see Fig. 7. We then, observe that

(6.31)

$$\begin{aligned}
&\left|\nabla_h^x u_{h|K_{i+\frac12,j}}\right| \Big(\text{ or } \left|\widetilde{\nabla}_h^x \Lambda_h^{-1} u_{h|\widetilde{K}_{i+\frac12,j}}\right|\Big) \le \frac{1}{\underline{h}} |u_{i+1,j} - u_{i,j}| ,\\
&\left|\widetilde{\nabla}_h^x \Lambda_h^{-1} u_{h|\widetilde{K}_{i,j+\frac12}}\right| \le \frac{1}{4\underline{h}} \left\{|u_{i+1,j+1} - u_{i-1,j+1}| + |u_{i+1,j} - u_{i-1,j}|\right\},\\
&\left|\nabla_h^x u_{h|K_{i,j+\frac12}}\right| \le \frac{1}{8\underline{h}} \{|u_{i+1,j+1} - u_{i+1,j}| + |u_{i+1,j+1} - u_{i,j+1}| + |u_{i+1,j} - u_{i,j}|\\
&\qquad + |u_{i,j+1} - u_{i,j}| + |u_{i,j+1} - u_{i-1,j+1}| + |u_{i,j} - u_{i-1,j}| + |u_{i-1,j+1} - u_{i-1,j}|\} .
\end{aligned}$$

If we set

$$K^*_{i,j} = \left\{K_{i,j+\frac12} \setminus \left(K_{i,j+\frac12} \cap \widetilde{K}_{i,j+\frac12}\right)\right\} \cup \left\{K_{i+\frac12,j} \setminus \left(K_{i+\frac12,j} \cap \widetilde{K}_{i+\frac12,j}\right)\right\}, \tag{6.32}$$

where the indices are $1 \le i \le M$, $0 \le j \le N$ for $K_{i,j+\frac12}$, and $0 \le i \le M$, $1 \le j \le N$ for $K_{i+\frac12,j}$, then, thanks to (6.9) and (6.31), we find that, for a constant c independent of the mesh sizes,

$$\begin{aligned}
&\left|\sum_{i,j} \int_{K^*_{i,j}} \left(\nabla_h^x u_h - \widetilde{\nabla}_h^x\ \Lambda_h^{-1} u_h\right) \varphi dxdy\right|\\
&\qquad \le \sum_{i,j} \left\{|\nabla_h^x u_h| + \left|\widetilde{\nabla}_h^x \Lambda_h^{-1} u_h\right|\right\} \left|\int_{K^*} \varphi dxdy\right|\\
&\qquad \le \sum_{i,j} \left\{|\nabla_h^x u_h| + \left|\widetilde{\nabla}_h^x \Lambda_h^{-1} u_h\right|\right\} \overline{hk}\, (\eta_3 + \eta_4)\\
&\qquad \le \text{(by changing indices in (6.31) and treating the boundary terms)}\\
&\qquad \le c\, (\eta_3 + \eta_4) \sum_{i,j} \{|u_{i+1,j} - u_{i,j}| + |u_{i,j+1} - u_{i,j}|\}\, \overline{h}\\
&\qquad \le c\, (\eta_3 + \eta_4)\, \|u_h\|_{V_h} .
\end{aligned} \tag{6.33}$$

Consequently, from (6.30) and (6.33), we obtain

$$
\begin{aligned}
&\Big|\int_\Omega \Big(\nabla_h^x u_h - \widetilde{\nabla}_h^x \Lambda_h^{-1} u_h\Big)\varphi dxdy\Big| \\
&\quad = \Big| \sum_{\substack{1\le i\le M\\ 0\le j\le N}} \int_{K_{i,j+\frac12}} \Big(\nabla_h^x u_h - \widetilde{\nabla}_h^x \Lambda_h^{-1} u_h\Big)\varphi dxdy \\
&\qquad + \sum_{\substack{0\le i\le M\\ 1\le j\le N}} \int_{K_{i+\frac12,j}} \Big(\nabla_h^x u_h - \widetilde{\nabla}_h^x \Lambda_h^{-1} u_h\Big)\varphi dxdy\Big| \\
&\quad \le |(6.30)| + |(6.33)| \le c(\eta_1+\eta_3+\eta_4)\|u_h\|_{V_h},
\end{aligned} \tag{6.34}
$$

and this is $(6.13)_1$ as desired. □

Remark 6.1. *It is noteworthy that, since η_3 in (6.11) is not equal to zero for the case where $h_{2i} = h$ and $h_{2i+1} = 2h$, the assumption (6.11) is somewhat more restrictive than the assumption (6.12); in (6.12), $\widehat{\widehat{h}}_{i+1} - \widehat{\widehat{h}}_{i-1}$, $3 \le i \le M-2$, are equal to zero when $h_{2i} = h$ and $h_{2i+1} = 2h$.*

Remark 6.2. *For the case where $h_{2i} = h$, $h_{2i+1} = 2h$, it is enough to impose the condition (6.12) and, for such a special case, we can prove Lemma 6.2 by using the discrete integration by parts. But here we assume (6.11) to handle more complicated meshes, for which, the geometric complexity of the mesh prevents us from using the discrete integration by parts.*

Now, thanks to Lemma 6.2, we deduce the convergence result of the FV in the following theorem:

Theorem 6.1. *Under the assumptions (2.18), (2.19), (5.9) and (6.11), the $(C2)$ property for the external approximation of $H_0^1(\Omega)$ by the FV spaces V_h holds true. Hence, with (3.3) and $(3.4)_1$, we conclude that the FV approximation is stable and convergent.*

Proof. Consider $\{u_h\} \in V_h$ such that $p_h u_h \rightharpoonup \phi = (\phi_0, \phi_1, \phi_2)$ weakly in F:

$$
\begin{cases}
u_h \rightharpoonup \phi_0 \text{ weakly in } L^2(\Omega),\\
\nabla_h^x u_h \rightharpoonup \phi_1 \text{ weakly in } L^2(\Omega),\\
\nabla_h^y u_h \rightharpoonup \phi_2 \text{ weakly in } L^2(\Omega).
\end{cases} \tag{6.35}
$$

Then, to prove the $(C2)$ property for the FV method, we have to verify that $\phi_0 \in H_0^1(\Omega)$ and $(\phi_1, \phi_2) = D\phi_0$. But, since the property $(C2)$ for the FD method holds, it is sufficient to show that

$$
\begin{cases}
\Lambda_h^{-1} u_h \rightharpoonup \phi_0 \text{ weakly in } L^2(\Omega),\\
\widetilde{\nabla}_h^x \Lambda_h^{-1} u_h \rightharpoonup \phi_1 \text{ weakly in } L^2(\Omega),\\
\widetilde{\nabla}_h^y \Lambda_h^{-1} u_h \rightharpoonup \phi_2 \text{ weakly in } L^2(\Omega).
\end{cases} \tag{6.36}
$$

Property $(6.36)_1$ is true by Lemma 6.1 and the boundedness of $\|u_h\|_{V_h}$; $(6.36)_2$ and $(6.36)_3$ are valid because of Lemma 6.2 and the boundedness of $\|u_h\|_{V_h}$ again. Hence we obtain the property $(C2)$ for the FV method. □

7. An application.

Now, as an application of the convergence result for the FV method, we briefly show how one can implement the FV method to approximate the solutions of some typical elliptic equations with Dirichlet boundary condition, and prove the convergence results.

We consider a general two dimensional Dirichlet problem in the form:

$$\begin{cases} -\partial_\alpha a_{\alpha\beta}(x,y)\partial_\beta u + \partial_\alpha\big(b_\alpha(x,y)u\big) + g(x,y)u = f(x,y) \text{ in } \Omega=(0,1)^2 \subset \mathbb{R}^2, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{7.1}$$

where the Einstein summation convention is understood for the Greek indices $\alpha, \beta = 1, 2$. For simplicity, we assume that, for each $\alpha, \beta = 1, 2$,

$$a_{\alpha,\beta}, g, f \in \mathcal{C}^0(\overline{\Omega}), \quad b_\alpha \in \mathcal{C}^2(\overline{\Omega}), \tag{7.2}$$

and, for the coercivity of the problem (7.1), we also impose the following properties on $a_{\alpha\beta}(x,y)$, $b_\alpha(x,y)$ and $g(x,y)$:

$$\begin{cases} a_{\alpha\beta}(x,y)\xi_\alpha\xi_\beta \geq \kappa_1|\boldsymbol{\xi}|^2, \ \forall\boldsymbol{\xi}=(\xi_1,\xi_2)\in\mathbb{R}^2, \\ \dfrac{1}{2}\partial_\alpha b_\alpha(x,y) + g(x,y) \geq \kappa_2 > 0, \end{cases} \tag{7.3}$$

for suitable strictly positive constants κ_1 and κ_2.

The variational form of (7.1) is classical:

$$\textit{To find } u \in V = H_0^1(\Omega) \textit{ such that } \mathrm{a}(u,v) = < l, v >, \ \forall v \in V, \tag{7.4}$$

where

$$\begin{aligned} \mathrm{a}(u,v) &= \int_\Omega \big(a_{\alpha\beta}\partial_\beta u\partial_\alpha v - b_\alpha u\partial_\alpha v + guv\big)dxdy, \\ < l, v > &= \int_\Omega fv dxdy. \end{aligned} \tag{7.5}$$

Note that, thanks to the Lax-Milgram theorem, we obtain the existence and uniqueness of the solution u of (7.1) in $V = H_0^1(\Omega)$.

To construct the FV approximation of (7.5), we use the spaces V_h introduced in Section 2 as well as all the notations of the previous sections as needed. We first consider the convection term in (7.1) which is the most problematic: we start by integrating this term over a rectangle $K_{i,j}$ for fixed i, j, and use the Stokes formula:

$$\begin{aligned} \int_{K_{i,j}} \partial_\alpha\big(b_\alpha u\big)dxdy &= \int_{\partial K_{i,j}} \big(b_1 u, b_2 u\big)\cdot \mathbf{n}_{i,j} dS \\ &= F_{i+\frac{1}{2},j} - F_{i-\frac{1}{2},j} + F_{i,j+\frac{1}{2}} - F_{i,j-\frac{1}{2}j}, \end{aligned} \tag{7.6}$$

where $\mathbf{n}_{i,j}$ is the unit outer normal on $K_{i,j}$ and where $F_{i+\frac{1}{2},j}$ and $F_{i,j+\frac{1}{2}}$ are the fluxes along the parts of boundary $\{x_{i+\frac{1}{2}}\}\times(y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}})$ and $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})\times\{y_{j+\frac{1}{2}}\}$ respectively. Since the unit outer normal $\mathbf{n}_{i,j}$ of $K_{i,j}$ is $(\pm1,0)$ or $(0,\pm1)$, we can approximate those fluxes in the following way: for $0 \leq i \leq M$, $1 \leq j \leq N$,

$$F_{i+\frac{1}{2},j} = \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} b_1(x_{i+\frac{1}{2}},y)u(x_{i+\frac{1}{2}},y)dy \cong b_1(x_{i+\frac{1}{2}},y_j)u^*_{i+\frac{1}{2},j}k_j, \tag{7.7}$$

and, for $1 \leq i \leq M$, $0 \leq j \leq N$,

$$F_{i,j+\frac{1}{2}} = \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} b_2(x, y_{j+\frac{1}{2}}) u(x, y_{j+\frac{1}{2}}) dx \cong b_2(x_i, y_{j+\frac{1}{2}}) u^*_{i,j+\frac{1}{2}} h_i, \tag{7.8}$$

where

$$u^*_{i+\frac{1}{2},j} = \frac{h_i u_{i,j} + h_{i+1} u_{i+1,j}}{h_i + h_{i+1}}, \quad u^*_{i,j+\frac{1}{2}} = \frac{k_j u_{i,j} + k_{j+1} u_{i,j+1}}{k_j + k_{j+1}}; \tag{7.9}$$

moreover, for any $u_h \in V_h$, we write

$$u^*_h = \sum_{\substack{0 \leq i \leq M \\ 1 \leq j \leq N}} u^*_{i+\frac{1}{2},j} \chi_{K_{i+\frac{1}{2},j}} + \sum_{\substack{1 \leq i \leq M \\ 0 \leq j \leq N}} u^*_{i,j+\frac{1}{2}} \chi_{K_{i,j+\frac{1}{2}}}. \tag{7.10}$$

Thanks to (7.7)-(7.9), for $u_h, v_h \in V_h$, multiplying (7.6) by $v_{i,j}$ and summing over $1 \leq i \leq M$, $1 \leq j \leq N$, we find that

$$\mathrm{a}^*_h(u_h, v_h) = \mathrm{a}^{1*}_h(u_h, v_h) + \mathrm{a}^{2*}_h(u_h, v_h), \tag{7.11}$$

where, remembering that $v_{0,j} = v_{M+1,j} = v_{i,0} = v_{i,N+1} = 0$,

$$\begin{aligned}
\mathrm{a}^{1*}_h(u_h, v_h) &= \sum_{\substack{1 \leq i \leq M \\ 1 \leq j \leq N}} \big\{ b_1(x_{i+\frac{1}{2}}, y_j) u^*_{i+\frac{1}{2},j} - b_1(x_{i-\frac{1}{2}}, y_j) u^*_{i-\frac{1}{2},j} \big\} v_{i,j} k_j \\
&= - \sum_{\substack{1 \leq i \leq M \\ 1 \leq j \leq N}} b_1(x_{i+\frac{1}{2}}, y_j) u^*_{i+\frac{1}{2},j} \big\{ v_{i+1,j} - v_{i,j} \big\} k_j, \\
\mathrm{a}^{2*}_h(u_h, v_h) &= \sum_{\substack{1 \leq i \leq M \\ 1 \leq j \leq N}} \big\{ b_2(x_i, y_{j+\frac{1}{2}}) u^*_{i,j+\frac{1}{2}} - b_2(x_i, y_{j-\frac{1}{2}}) u^*_{i,j-\frac{1}{2}} \big\} v_{i,j} h_i \\
&= - \sum_{\substack{1 \leq i \leq M \\ 1 \leq j \leq N}} b_2(x_i, y_{j+\frac{1}{2}}) u^*_{i,j+\frac{1}{2}} \big\{ v_{i,j+1} - v_{i,j} \big\} h_i.
\end{aligned} \tag{7.12}$$

Now, using (7.9), (7.11) and (7.12), we introduce the FV discrete variational problem of (7.1) in the form:

$$\textit{To find } u_h \in V_h \textit{ such that } \mathrm{a}_h(u_h, v_h) = < l_h, v_h >, \ \forall v_h \in V_h, \tag{7.13}$$

where

$$\begin{aligned}
&\mathrm{a}_h(u_h, v_h) = \big(a_{\alpha\beta} \nabla^\alpha_h u_h, \nabla^\beta_h v_h \big) + \mathrm{a}^*_h(u_h, v_h) + \big(g u_h, v_h \big), \\
&< l_h, v_h > = \big(f, v_h \big);
\end{aligned} \tag{7.14}$$

here $(\cdot, \cdot)$ is the usual L^2 inner product over Ω, and $\nabla^1_h = \nabla^x_h$ and $\nabla^2_h = \nabla^y_h$.

Remark 7.1. *For the sake of simplicity, in the FV discrete variational form (7.13) and (7.14), we only use the fluxes from the convection term of the original equation (7.1). One can also use the fluxes from the diffusion term in (7.1) and construct the corresponding FV discrete variational form; see e.g.* [8] *or* [10]. *However, in such a case, more difficulties may occur in the analysis, e.g. in the computation for the uniform coercivity of the bilinear forms* a_h.

Before we start the analysis of the problem (7.13), we first state and prove a simple, but useful lemma:

Lemma 7.1. *For any u_h in V_h,*

$$|u_h^* - u_h|_{L^2(\Omega)} \leq \underline{\alpha}^{-1}\overline{\rho}\, \|u_h\|_{V_h}\,. \tag{7.15}$$

Proof. We use the points' ordering in (2.8) and notice that each $K_{i,j}$ may only intersect $K_{i,j\pm\frac{1}{2}}$ and $K_{i\pm\frac{1}{2},j}$; see also Fig. 3. Then, using (2.10), (2.17)-(2.19), (7.9) and (7.10), we observe that, on each $K_{i,j}$,

$$\begin{aligned} &\big|(u_h^* - u_h)_{|K_{i,j}}\big| \\ &\quad \leq 2^{-1}\underline{\alpha}^{-1}\big\{|u_{i+1,j} - u_{i,j}| + |u_{i,j} - u_{i-1,j}| + |u_{i,j+1} - u_{i,j}| + |u_{i,j} - u_{i,j-1}|\big\}, \end{aligned} \tag{7.16}$$

and write

$$\begin{aligned} |u_h^* - u_h|^2_{L^2(\Omega)} &= \sum_{\substack{1\leq i\leq M\\ 1\leq j\leq N}} (u_h^* - u_h)^2_{|K_{i,j}} h_i k_j \\ &\leq \underline{\alpha}^{-1} \sum_{\substack{0\leq i\leq M\\ 1\leq j\leq N}} (u_{i+1,j} - u_{i,j})^2 \overline{h} k_j + \underline{\alpha}^{-1} \sum_{\substack{1\leq i\leq M\\ 0\leq j\leq N}} (u_{i,j+1} - u_{i,j})^2 h_i \overline{k} \\ &\leq \underline{\alpha}^{-2}\overline{\rho}^2 \|u_h\|^2_{V_h}; \end{aligned} \tag{7.17}$$

hence (7.15) follows. □

We now set

$$\overline{a} = \max_{\alpha,\beta=1,2}\Big(\sup_\Omega |a_{\alpha\beta}|\Big), \quad \overline{b} = \max_{\alpha=1,2}\Big(\sup_\Omega |b_\alpha|\Big), \quad \overline{g} = \sup_\Omega |g|, \tag{7.18}$$

and we promptly see that the families $\{\mathrm{a}_h\}_h$ and $\{l_h\}_h$ are uniformly continuous with respect to h (the mesh sizes): using (7.14) and (7.15), we find that, for any u_h, v_h in V_h,

$$\begin{aligned} |\mathrm{a}_h(u_h, v_h)| &\leq \overline{a}|\nabla_h^\alpha u_h \nabla_h^\beta v_h|_{L^2(\Omega)} + 2\overline{b}|u_h^*|_{L^2(\Omega)}|\nabla_h v_h|_{L^2(\Omega)} + \overline{g}|u_h v_h|_{L^2(\Omega)} \\ &\leq 2\overline{a}\|u_h\|_{V_h}\|v_h\|_{V_h} + 2\overline{b}|u_h^*|_{L^2(\Omega)}\|u_h\|_{V_h} + \overline{g}|u_h|_{V_h}|u_h|_{V_h} \\ &\leq \big(2\overline{a} + 2\overline{b}c_0 + \overline{g}c_0^2 + \underline{\alpha}^{-1}\overline{\rho}\big)\|u_h\|_{V_h}\|v_h\|_{V_h}, \\ &\leq \big(2\overline{a} + 2\overline{b}c_0 + \overline{g}c_0^2 + \underline{\alpha}^{-1}\big)\|u_h\|_{V_h}\|v_h\|_{V_h}, \end{aligned} \tag{7.19}$$

where $c_0 = \sqrt{2}\underline{\alpha}^{-1}$ is the discrete Poincaré constant in (2.22); hence the family $\{\mathrm{a}_h\}_h$ is uniformly continuous.
We also notice that

$$< l_h, v_h >= \big(f, v_h\big)_{L^2(\Omega)} \leq |f|_{L^2(\Omega)}|v_h|_{V_h}; \tag{7.20}$$

due to the independence of $|f|_{L^2(\Omega)}$ on the mesh sizes, and we have the uniform continuity of the family $\{l_h\}_h$.

Now, to obtain the uniform coercivity of a_h on V_h, we first establish a lemma for the term a_h^*:

Lemma 7.2. *For any u_h in V_h,*

$$\Big|\mathrm{a}_h^*(u_h, u_h) - \int_\Omega \frac{1}{2}\partial_\alpha b_\alpha u_h^2 dxdy\Big| \leq \kappa_3\overline{\rho}\|u_h\|^2_{V_h}, \tag{7.21}$$

for a positive constant κ_3 depending on b_1 and b_2, but independent of the mesh sizes.

Proof. From (7.9), (7.11) and (7.12), using the Taylor expansion of $b_1(x,\cdot)$ at $x = x_i$, we first write the bilinear form a_h^{1*} in the form: for $u_h \in V_h$,

$$\begin{aligned}\mathrm{a}_h^{1*}(u_h,u_h) &= \sum_{\substack{1\le i\le M\\1\le j\le N}} \Big\{ b_1(x_{i+\frac12},y_j)u^*_{i+\frac12,j} - b_1(x_{i-\frac12},y_j)u^*_{i-\frac12,j}\Big\} u_{i,j}k_j \\ &= M_1+M_2+M_3+M_4,\end{aligned}\tag{7.22}$$

where

$$\begin{aligned}M_1 &= \sum_{\substack{1\le i\le M\\1\le j\le N}} \Big\{\frac{h_{i+1}(u_{i+1,j}-u_{i,j})}{h_i+h_{i+1}} - \frac{h_{i-1}(u_{i-1,j}-u_{i,j})}{h_{i-1}+h_i}\Big\} b_1(x_i,y_j)u_{i,j}k_j,\\ M_2 &= \sum_{\substack{1\le i\le M\\1\le j\le N}} \partial_1 b_1(x_i,y_j)u_{i,j}^2 h_i k_j,\\ M_3 &= \frac12 \sum_{\substack{1\le i\le M\\1\le j\le N}} \Big\{\frac{h_{i+1}(u_{i+1,j}-u_{i,j})}{h_i+h_{i+1}} + \frac{h_{i-1}(u_{i-1,j}-u_{i,j})}{h_{i-1}+h_i}\Big\}\partial_1 b_1(x_i,y_j)u_{i,j}h_ik_j.\\ M_4 &= \sum_{\substack{1\le i\le M\\1\le j\le N}} \Big\{\big[b_1(x_{i+\frac12},y_j)-b_1(x_i,y_j)-\frac12\partial_1 b_1(x_i,y_j)h_i\big]u^*_{i+\frac12,j}\\ &\qquad - \big[b_1(x_{i-\frac12},y_j)-b_1(x_i,y_j)+\frac12\partial_1 b_1(x_i,y_j)h_i\big]u^*_{i-\frac12,j}\Big\}u_{i,j}k_j.\end{aligned}\tag{7.23}$$

Since $u_{0,j} = u_{M+1,j} = 0$, after changing the indices i in M_1, we find

$$\begin{aligned}M_1 &= \sum_{\substack{0\le i\le M\\1\le j\le N}} \Big\{(u_{i+1,j}u_{i,j}-u_{i,j}^2)h_{i+1}b_1(x_i,y_j)\\ &\qquad - (u_{i,j}u_{i+1,j}-u_{i+1,j}^2)h_ib_1(x_{i+1},y_j)\Big\}(h_i+h_{i+1})^{-1}k_j\\ &= M_1'+M_1'',\end{aligned}\tag{7.24}$$

with

$$\begin{aligned}M_1' &= -\frac12 \sum_{\substack{0\le i\le M\\1\le j\le N}} \Big\{(u_{i+1,j}-u_{i,j})^2\frac{h_{i+1}b_1(x_i,y_j)-h_ib_1(x_{i+1},y_j)}{h_i+h_{i+1}}k_j\Big\},\\ M_1'' &= \frac12 \sum_{\substack{0\le i\le M\\1\le j\le N}} \Big\{(u_{i+1,j}^2-u_{i,j}^2)\frac{h_{i+1}b_1(x_i,y_j)+h_ib_1(x_{i+1},y_j)}{h_i+h_{i+1}}k_j\Big\}.\end{aligned}\tag{7.25}$$

We can bound the term M_1':

$$|M_1'| \le (2\underline{\alpha})^{-1}\overline{b}\sum_{\substack{0\le i\le M\\1\le j\le N}} \big|u_{i+1,j}-u_{i,j}\big|^2 k_j \le (2\underline{\alpha})^{-1}\overline{b}\overline{\rho}\|u_h\|_{V_h}^2,\tag{7.26}$$

and, thanks to Lemma 3.3, we write the term M_1'' in the form:

$$\begin{aligned}M_1'' &= \frac12 \sum_{\substack{0\le i\le M\\1\le j\le N}} (u_{i+1,j}^2-u_{i,j}^2)b_1(x_{i+\frac12},y_j)k_j\\ &= -\frac12 \sum_{\substack{1\le i\le M\\1\le j\le N}} u_{i,j}^2\big\{b_1(x_{i+\frac12},y_j)-b_1(x_{i-\frac12},y_j)\big\}k_j.\end{aligned}\tag{7.27}$$

Then, using the Taylor expansion of b_1 at x_i again, we find

$$\Big|M_1'' + \frac{1}{2}\sum_{\substack{1\le i\le M\\1\le j\le N}} u_{i,j}^2\partial_1 b_1(x_i,y_j)h_ik_j\Big| \le c\overline{\rho}|u_h|_{V_h}^2. \tag{7.28}$$

We also observe that

$$\begin{aligned}|M_3| &\le (4\underline{\alpha})^{-1}\sup_{\Omega}|\partial_1 b|\sum_{\substack{1\le i\le M\\1\le j\le N}}\Big\{|u_{i+1,j}-u_{i,j}|+|u_{i-1,j}-u_{i,j}|\Big\}u_{i,j}h_ik_j\\ &\le (2\underline{\alpha})^{-2}\sup_{\Omega}|\partial_1 b||u_h|_{V_h}\|u_h\|_{V_h}\overline{\rho} \le c\overline{\rho}\|u_h\|_{V_h}^2,\end{aligned} \tag{7.29}$$

and, using the Taylor expansion and Lemma 7.1, we find

$$|M_4| \le c\sum_{\substack{1\le i\le M\\1\le j\le N}}\big|u_{i+\frac{1}{2},j}^*-u_{i-\frac{1}{2},j}^*\big|u_{i,j}k_j\overline{\rho}^2 \le c\overline{\rho}|u_h^*|_{L^2(\Omega)}|u_h|_{V_h} \le c\overline{\rho}|u_h|_{V_h}^2, \tag{7.30}$$

for a positive constant c depending on b_1, but independent of the mesh sizes. Therefore, combining (7.22)-(7.30) and using the Poincaré inequality, we find

$$\big|\mathrm{a}_h^{1*}(u_h,u_h) - \int_\Omega \frac{1}{2}\partial_1 b_1 u_h^2\,dxdy\big| \le c\overline{\rho}\|u_h\|_{V_h}^2. \tag{7.31}$$

Similarly, one can easily verify that

$$\big|\mathrm{a}_h^{2*}(u_h,u_h) - \int_\Omega \frac{1}{2}\partial_2 b_2 u_h^2\,dxdy\big| \le c\overline{\rho}\|u_h\|_{V_h}^2; \tag{7.32}$$

hence (7.21) follows by (7.31) and (7.32). □

Thanks to (7.3), (7.14) and (7.21), we finally obtain

$$\mathrm{a}_h(u_h,u_h) \ge \kappa_1\|u_h\|_{V_h}^2 + \kappa_2|u_h|_{V_h}^2 - \kappa_3\overline{\rho}\|u_h\|_{V_h}^2, \tag{7.33}$$

and, for sufficiently small $\overline{\rho} < \kappa_1/\kappa_3$, the uniform coercivity of the bilinear continuous forms a_h on V_h follows. Due to (7.19), (7.20) and (7.33), the Lax-Milgram theorem asserts that, for $\overline{\rho} < \kappa_1/\kappa_3$, the equation (7.13)-(7.14) has a unique solution u_h in V_h; we say that u_h is the FV approximate solution of (7.4)-(7.5). To prove that the FV approximate solution u_h converges to the exact solution u as the mesh size decreases, we now introduce the following consistency lemma; then, the convergence result will follow by the general convergence theorem in [3] (see also [15]):

Lemma 7.3. *If the family v_h converges to v strongly in F as $\overline{\rho}\to 0$, and if the family w_h converges to w weakly in F as $\overline{\rho}\to 0$, then*

$$\begin{aligned}&\lim_{\overline{\rho}\to 0}\mathrm{a}_h(v_h,w_h) = \mathrm{a}(v,w),\\ &\lim_{\overline{\rho}\to 0}\mathrm{a}_h(w_h,v_h) = \mathrm{a}(w,v),\\ &\lim_{\overline{\rho}\to 0} < l_h, w_h > = < l, w > .\end{aligned} \tag{7.34}$$

Proof. From the hypotheses of Lemma 7.3, we have

$$\big(v_h, \nabla_h^x v_h, \nabla_h^y v_h\big) \to \big(v, \partial_x v, \partial_y v\big) \text{ strongly in } F = L^2(\Omega)^3, \tag{7.35}$$

and

$$\big(w_h, \nabla_h^x w_h, \nabla_h^y w_h\big) \to \big(w, \partial_x w, \partial_y w\big) \text{ weakly in } F = L^2(\Omega)^3, \tag{7.36}$$

and, by the property $(C2)$ of FV, we notice that v, w are in $V = H_0^1(\Omega)$.
To verify $(7.34)_1$, using (7.9), (7.10) and (7.12), we first write $\mathrm{a}_h^{1*}(v_h, w_h)$ in the form:

$$\mathrm{a}_h^{1*}(v_h, w_h) = -2\Big(b_1^* v_h^*, \textstyle\sum_{\substack{0\le i\le M\\1\le j\le N}} \nabla_h^x w_{h|K_{i+\frac12,j}} \chi_{K_{i+\frac12,j}}\Big), \tag{7.37}$$

where

$$b_1^* = \sum_{\substack{0\le i\le M\\1\le j\le N}} b_1(x_{i+\frac12}, y_j)\chi_{K_{i+\frac12,j}} + \sum_{\substack{1\le i\le M\\0\le j\le N}} b_1(x_i, y_{j+\frac12})\chi_{K_{i,j+\frac12}}. \tag{7.38}$$

Using the Taylor expansion of b_1, one can easily verify that

$$b_1^* \to b_1 \text{ strongly in } L^2(\Omega) \text{ as } \overline{\rho} \to 0, \tag{7.39}$$

and, using (7.15) and (7.35), we also find

$$v_h^* \text{ converge to } v \text{ strongly in } L^2(\Omega) \text{ as } \overline{\rho} \to 0. \tag{7.40}$$

Moreover, from (5.18), (5.28), (6.34) and (7.36), we infer that

$$\sum_{\substack{0\le i\le M\\1\le j\le N}} \nabla_h^x w_{h|K_{i+\frac12,j}} \chi_{K_{i+\frac12,j}} \to \frac{1}{2}\partial_1 w \text{ weakly in } L^2(\Omega) \text{ as } \overline{\rho} \to 0, \tag{7.41}$$

and hence, using (7.39)-(7.41), (7.37) yields

$$\mathrm{a}_h^{1*}(v_h, w_h) \to -(b_1 v, \partial_1 w) \text{ as } \overline{\rho} \to 0. \tag{7.42}$$

With the same result in the y variable, we also find

$$\mathrm{a}_h^{2*}(v_h, w_h) \to -(b_2 v, \partial_2 w) \text{ as } \overline{\rho} \to 0, \tag{7.43}$$

and then, thanks to (7.11), (7.14), (7.35), (7.36), (7.42) and (7.43), we finally obtain $(7.34)_1$.

For $(7.34)_2$, from (7.9), (7.10) and (7.12), we write

$$\mathrm{a}_h^{1*}(w_h, v_h) = -2\Big(\textstyle\sum_{\substack{0\le i\le M\\1\le j\le N}} w_{i+\frac12,j}^* \chi_{K_{i+\frac12,j}}, b^* \nabla_h^x v_h\Big), \tag{7.44}$$

and, using (7.36), one can easily verify that

$$\sum_{\substack{0\le i\le M\\1\le j\le N}} w_{i+\frac12,j}^* \chi_{K_{i+\frac12,j}} \to \frac{1}{2} w \text{ weakly in } L^2(\Omega) \text{ as } \overline{\rho} \to 0. \tag{7.45}$$

Hence, from (7.35), (7.39) and (7.45), we find

$$\mathrm{a}_h^{1*}(w_h, v_h) \to -(w, b_1 \partial_1 v) \text{ as } \overline{\rho} \to 0, \tag{7.46}$$

and similarly,

$$\mathrm{a}_h^{2*}(w_h, v_h) \to -(w, b_2 \partial_2 v) \text{ as } \overline{\rho} \to 0; \tag{7.47}$$

then, by (7.11), (7.14), (7.35), (7.36), (7.46) and (7.47), $(7.34)_2$ follows.

Finally, using (7.36), we promptly notice

$$< l_h, w_h >= (f, w_h) \to (f, w) =< l, w > \text{ as } \overline{\rho} \to 0; \tag{7.48}$$

and the proof of Lemma 7.3 is complete. □

Now, with the general convergence theorem in [3] and [15], we obtain the convergence of the FV approximate solution u_h to the exact solution u:

Theorem 7.1. *Under the hypotheses (7.19), (7.20), (7.33) and (7.34), the FV approximate solution* u_h *of (7.13)-(7.14) converges strongly to the solution* u *of (7.4)-(7.5) in* F *as* $\overline{\rho} \to 0$.

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