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AN ERROR ESTIMATE FOR MMOC-MFEM BASED ON CONVOLUTION FOR POROUS MEDIA FLOW

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Abstract. A modification of the modified method of characteristics (MMOC) is introduced for solving the coupled system of partial differential equations governing miscible displacement in porous media . The pressure-velocity is approximated by a mixed finite element procedure using a Raviart-Thomas space of index k over a uniform grid. The resulting Darcy velocity is post-processed by convolution with Bramble-Schatz kernel and this enhanced velocity is used in the evaluation of the coefficients in MMOC for the concentration equation. If the concentration space is of local degree l, then , the error in the concentration is $O(h_c^{l+1} + h_p^{2k+2})$, which reflects the superconvergence of velocity approximation.

Key Words. Porous medium flow, characteristic methods, Bramble-Schatz kernel, convolution, convergence analysis

1. Introduction

Mathematical models used to describe porous medium flow processes in petroleum reservoir simulation, groundwater contaminant transport, and other applications lead to a coupled system of time-dependent nonlinear partial differential equations (PDEs) [1]. Conventional second-order finite difference or finite element methods (FDMs, FEMs) tend to yield solutions with spurious oscillations. In industrial applications, first-order upwind methods are commonly used to stabilize the numerical approximations, but they tend to generate excessive numerical diffusion and grid-orientation effect [1].

An MMOC-MFEM time-stepping procedure was proposed and successfully applied in the numerical simulation of miscible displacement processes in petroleum reservoir simulation [2], in which the MMOC [3] was used to solve the transport equation while an MFEM scheme [4, 5] was used to solve the pressure equation. The MMOC symmetrizes and stabilizes the transport equation, greatly reduces temporal errors, and so allows for large time steps in a simulation without loss of accuracy. The MFEM schemes generate an accurate approximation to the Darcy velocity, which are required for accurate approximation to the transport because advection and diffusion dispersion in the transport equation are governed by Darcy velocity. The MFEMs minimize the numerical difficulties occurring in finite difference or finite element caused by differentiation of the pressure and then multiplication by rough coefficients [6]. Numerical experiments showed that the MMOC-MFEM type of solution techniques is numerically very competitive [2, 7].

A delicate and rigorous mathematical analysis was conducted in [8], in which an optimal-order error estimate was proved for a family of MMOC-MFEM time stepping procedure for miscible displacement processes in two space dimensions.

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These analysis theoretically confirm the numerical strength and advantage of the MMOC-MFEM time stepping procedure. As noted by the authors [8], however, a primary shortcoming of these results is that they are value only if the Courant number of the numerical discretization tends to zero asymptotically. This constraint is numerically very restrictive and was not observed numerically. In fact, under this assumption, an optimal-order error estimate can be proved for a Galerkin FEM-MFEM time stepping procedure [9], in which a Galerkin FEM is used to solve the transport equation. Furthermore, in the context of a strongly advection-dominated equation, an explicit finite difference method would converge under this assumption [10]. This very restrictive constraint has become a standard assumption in subsequent analysis for the MMOC methods for coupled systems in porous medium flow [11].

The work about superconvergence approximation can be found in [12, 13, 14] for elliptic problems(or pressure equation). A study on superconvergence along Gauss lines for the coupled problem for porous media flow can be found in Ewing [15]. Douglas and Roberts [16] and Douglas and Milner [17] have derived a collection of error estimates for mixed finite element methods for second order elliptic equations. These results include errors in Soblev spaces of negative index and superconvergence approximation, via convolution with Bramble-Schatz kernel, to both the basic dependent variable (in our case, p) and the related gradient field (u). The partition $T_{h_p}$is composed of squares of side length h_p related to a uniform grid over Ω. Based on the idea of [16, 17], Douglas [18] introduced the method of Bramble-Schatz kernel to the miscible displacement problem. The resulting Darcy velocity based on the mixed method is post-processed by convolution with a Bramble-Schatz kernel and this enhanced velocity is used in the evaluation of the coefficient in the Galerkin procedure for the concentration. For a time-continuous scheme, Douglas [18] achieved the superconvergence result $O(h_c^{l+1} + h_p^{2k+2})$, which is obviously higher than the standard optimal error estimate $O(h_c^{l+1} + h_p^{k+1})$ for mixed methods.

The authors of [18] mentioned that it is necessary to discretize the time variable in order to obtain actual numerical information. It seems to be a straightforward task to get the time-stepping procedure and establish the corresponding error estimate, however, the constraint condition between the time step $\triangle t_c$ and the space partition size h_p such as $\triangle t_c = o(h_p)$ had to be required [9]. This condition means that a procedure is guaranteed to converge only if the Courant number tends to zero asymptotically, and it is even more restrictive than the CFL condition for an explicit scheme in the context of a strangely advection-dominated displacement process [10].

Wang [19, 20] proved an optimal-order error estimate for a family of MMOC-MFEM approximation to the coupled system of miscible porous medium flow, which holds even if the Courant number tends to infinity asymptotically. In this way, the estimates justify the numerical advantages and strength of the MMOC-MFEM time-stepping procedure.

The object of this work is to establish and analyze an MFEM-MMOC time stepping procedure for the above model. As in [18], we combine the post-processed Darcy velocity(via convolution with a Bramble-Schatz kernel function) with the evaluation of the concentration variable. The same order of superconvergence rate will be retained in the final error estimates. Here we emphasis what kind of constraint conditions is required for the convergence rate. By introducing a new induction hypothesis, the superconvergence can be derived and the constraint condition between $\triangle t_c$ and h_p will be lightened to be $\triangle t_c = O(h_p^{1/2+3\delta})$ for a small positive constant δ.

The rest of the paper is organized as follows: In §2 we review the mathematical model. In §3 we describe the MMOC-MFEM time-stepping procedure. §4 cites some well established results used in the main analysis. In §5 we prove the main error estimate. In §6 we prove auxiliary lemmas used in §5. §7 contains concluding remarks and future work.

2. Mathematical Model and Notation

We present a mathematical model for porous media flow and introduce the functional spaces used in this paper.

2.1. Mathematical model.

Let $c(\mathbf{x},t)$ be the concentration of an invading fluid or a concerned solute/solvent, and let $p(\mathbf{x},t)$ and $\mathbf{u}(\mathbf{x},t) = (u_1(\mathbf{x},t), u_2(\mathbf{x},t))$ be the pressure and Darcy velocity of the fluid mixture, respectively. The mass conservation for the fluid mixture incorporated with the incompressibility condition, Darcy's law, and the mass conservation for the invading fluid lead to the following system of PDEs [1]:

$$\nabla \cdot \mathbf{u} = q, \qquad \mathbf{u} = -\frac{\mathbf{K}}{\mu(c)}(\nabla p - \rho g \nabla d), \qquad \mathbf{x} \in \Omega,\ t \in [0,T], \tag{1}$$

$$\phi\frac{\partial c}{\partial t} + \mathbf{u}\cdot\nabla c - \nabla\cdot(\mathbf{D}(\mathbf{x},\mathbf{u})\nabla c) = (\bar{c} - c)\bar{q}, \qquad \mathbf{x} \in \Omega,\ t \in [0,T], \tag{2}$$

$$c(\mathbf{x},0) = c_0(\mathbf{x}), \qquad \mathbf{x} \in \Omega, \tag{3}$$

where $\bar{q} = \max\{q, 0\}$ is nonzero at injection wells only. We follow [8] to assume that Ω is a rectangle and that (1)–(3) are Ω-periodic. Throughout the rest of the paper, all functions will be assumed to be spatially Ω-periodic. We assume the medium is homogeneous vertically. $\phi(\mathbf{x})$ and $\mathbf{K}(\mathbf{x})$ are the porosity and the permeability tensor of the medium, respectively, $\mu(c)$ and ρ are the viscosity and the density of the fluid mixture, respectively, g is the gravitational acceleration, $d(\mathbf{x})$ is the reservoir depth, and $q(\mathbf{x},t)$ is the source and sink term. $\mathbf{D}(\mathbf{x},\mathbf{u}) = \phi(\mathbf{x})d_m\,\mathbf{I} + d_t|\mathbf{u}|\mathbf{I} + (d_l - d_t)|\mathbf{u}|\mathbf{E}$ is the diffusion-dispersion tensor, with d_m, d_t, and d_l being the molecular diffusion and the transverse and longitudinal dispersiveness, respectively, $\mathbf{I}$ is the identity tensor, and $\mathbf{E} = (u_i u_j)_{2\times 2}/|\mathbf{u}|^2$. $\bar{c}(\mathbf{x},t)$ is specified at sources and $\bar{c}(\mathbf{x},t) = c(\mathbf{x},t)$ at sinks. $c_0(\mathbf{x})$ is the initial concentration.

Eq. (1) combined with spatial periodicity implies that the pressure $p(\mathbf{x},t)$ can be determined only up to an additive constant for all the time $t \in [0,T]$. But this indeterminacy is of no consequence since $\mathbf{u}$ is uniquely determined by Darcy's law, and only $\mathbf{u}$ (not p) is needed in Eq. (2).

2.2. Notation.

Let $W_q^m(\Omega)$ be the Sobolev spaces consisting of functions whose derivatives up to order-m are q-th integrable on Ω, and $H^m(\Omega) := W_2^m(\Omega)$. Let $L_0^2(\Omega)$ be the subspace of $L^2(\Omega)$ with mean 0, and

$$\begin{aligned}
H^m(\mathrm{div};\Omega) &:= \Big\{\mathbf{f}(\mathbf{x}) = (f_1, f_2)\ :\ f_1, f_2, \nabla\cdot\mathbf{f} \in H^m(\Omega)\Big\},\\
\|\mathbf{f}\|_{H^m(\mathrm{div};\Omega)} &:= \Big(\|f_1\|^2_{H^m(\Omega)} + \|f_2\|^2_{H^m(\Omega)} + \|\nabla\cdot\mathbf{f}\|^2_{H^m(\Omega)}\Big)^{1/2},\\
H_0(\mathrm{div};\Omega) &:= \Big\{\mathbf{f}(\mathbf{x}) \in H^0(\mathrm{div};\Omega)\ :\ \mathbf{f}(\mathbf{x})\cdot\mathbf{n}(\mathbf{x}) = 0,\ \mathbf{x} \in \partial\Omega\ \Big\}.
\end{aligned}$$

For any Banach space X, we introduce Sobolev spaces involving time variable

$$W_q^m(t_1,t_2;X) := \Big\{ f(\mathbf{x},t) \ : \Big\| \frac{\partial^\alpha f}{\partial t^\alpha}(\cdot,t)\Big\|_X \in L^q(t_1,t_2),\ 0\le \alpha \le m,\ 1\le q\le\infty \Big\},$$

$$\|f\|_{W_q^m(t_1,t_2;X)} := \begin{cases} \Big(\sum_{\alpha=0}^{m} \int_{t_1}^{t_2} \Big\| \frac{\partial^\alpha f}{\partial t^\alpha}(\cdot,t)\Big\|_X^q dt\Big)^{1/q}, & 1\le q<\infty, \\ \max_{0\le\alpha\le m} \operatorname{esssup}_{t\in(t_1,t_2)} \Big\| \frac{\partial^\alpha f}{\partial t^\alpha}(\cdot,t)\Big\|_X, & q=\infty. \end{cases}$$

We also define the discrete norms $\|f\|_{\hat{L}_c^\infty(0,T;X)} := \max_{0\le n\le N} \|f(\cdot,t_c^n)\|_X$, $\|f\|_{\hat{L}_p^\infty(0,T;X)} := \max_{0\le m\le M} \|f(\cdot,t_p^m)\|_X$, and $\|f\|_{\hat{L}_c^2(0,T;X)} := \Big(\sum_{n=0}^{N} \|f(\cdot,t_c^n)\|_X^2 \Delta t_c^n\Big)^{1/2}$, with t_c^n and t_p^m being the concentration and pressure time steps defined below (4) and (5), respectively. If $(t_1,t_2)=(0,T)$, we drop it from these notations.

In this paper we use ε to denote an arbitrary small positive number, A_i, K_i, and Q_i to denote fixed positive constants, and Q to denote a generic positive constant that only depend on the constants A_i and K_i and could assume different values at different occurrences.

3. An MMOC-MFEM Time-Stepping Procedure

In this procedure an MFEM scheme is used for the pressure system (1), and an MMOC scheme is used to solve the transport PDE (2).

3.1. An MFEM formulation for the pressure and Darcy velocity. We multiply the second equation in (1) by $\mu(c)\mathbf{K}^{-1}(\mathbf{x})$ and any test functions $\mathbf{v}\in H(\mathrm{div};\Omega)$, and apply the divergence theorem to the ∇p term. We then multiply the first equation in (1) by any test functions $w(\mathbf{x})\in L^2(\Omega)$ and integrate over Ω. The system (1) is expressed as a time-parameterized saddle-point problem of finding a map $(\mathbf{u}(\mathbf{x},t), p(\mathbf{x},t)) \in H(\mathrm{div};\Omega)\times \mathrm{L}^2(\Omega)$ such that

$$\begin{aligned} \int_\Omega \mu(c)\mathbf{K}^{-1}\mathbf{u}\cdot\mathbf{v}d\mathbf{x} - \int_\Omega p\nabla\cdot\mathbf{v}d\mathbf{x} &= \int_\Omega \rho g\nabla d\cdot\mathbf{v}d\mathbf{x}, \\ \int_\Omega w\nabla\cdot\mathbf{u}d\mathbf{x} &= \int_\Omega qwd\mathbf{x}, \\ \forall\ (\mathbf{v}(\mathbf{x}),w(\mathbf{x}))\in H(\mathrm{div};\Omega) &\times\ L^2(\Omega), t\in[0,T]. \end{aligned} \tag{4}$$

We define a temporal partition on the time interval $[0,T]$ for the pressure grid by $0 =: t_p^0 < t_p^1 < \cdots < t_p^m < \cdots < t_p^{M-1} < t_p^M := T$, with $\Delta t_p^m := t_p^m - t_p^{m-1}$ and $\Delta t_p := \max_{1\le m\le M} \Delta t_p^m$. Let $V_h \subset H(\mathrm{div};\Omega)$ and $W_h\subset L^2(\Omega)$ be the MFEM spaces of index $k\ge 0$ on a quasi-uniform partition of $\Omega=\cup\Omega_e^p$ with the diameter h_p [4, 5]. Given a concentration approximation $c_h(\mathbf{x},t_p^m)$ at time t_p^m, the MFEM scheme determines the velocity $\mathbf{u}_h(\mathbf{x},t_p^m)\in V_h$ and the pressure $p_h(\mathbf{x},t_p^m)\in W_h$ such that

$$\begin{aligned} &\int_\Omega \mu(c_h(\mathbf{x},t_p^m))\mathbf{K}^{-1}(\mathbf{x})\mathbf{u}_h(\mathbf{x},t_p^m)\cdot\mathbf{v}_h(\mathbf{x})d\mathbf{x} - \int_\Omega p_h(\mathbf{x},t_p^m)\nabla\cdot\mathbf{v}_h(\mathbf{x})d\mathbf{x} \\ &\qquad = \int_\Omega \rho g\nabla d(\mathbf{x})\cdot\mathbf{v}_h(\mathbf{x})d\mathbf{x}, \qquad \forall\mathbf{v}_h(\mathbf{x})\in V_h, \\ &\int_\Omega w_h(\mathbf{x})\nabla\cdot\mathbf{u}_h(\mathbf{x},t_p^m)d\mathbf{x} = \int_\Omega q(\mathbf{x},t_p^m)w_h(\mathbf{x})d\mathbf{x}, \qquad \forall w_h(\mathbf{x})\in W_h. \end{aligned} \tag{5}$$

3.2. An MMOC-MFEM time-stepping procedure. Note that the velocity field usually changes less rapidly than the concentration. Moreover, at each time step the MFEM system (5) is more expensive to solve than the MMOC scheme for the transport PDE (2). Therefore, a larger time step can be used for the pressure than that for the concentration [9]. It is often computationally convenient to define the time partition for the concentration $0 =: t_c^0 < t_c^1 < \cdots < t_c^n < \cdots < t_c^{N-1} < t_c^N := T$, with $\Delta t_c^n := t_c^n - t_c^{n-1}$ and $\Delta t_c := \max_{1\le n\le N} \Delta t_c^n$, by subdividing the time partition for the pressure. Namely, there exist $0 =: N_0 < N_1 < \cdots < N_m < \cdots < N_{M-1} < N_M := N$ such that $t_c^{N_m} = t_p^m$ for $m = 1, 2, \ldots, M$. For $n = N_{m-1}+1, N_{m-1}+2, \ldots, N_m$, the concentration time step t_c^n relates to the pressure time steps by $t_p^{m-1} < t_c^n \le t_p^m$. In the MMOC scheme we define a velocity approximation $\mathbf{u}_h^{\mathrm{E}}(\mathbf{x}, t_c^n)$ by an extrapolation of $\mathbf{u}_h(\mathbf{x}, t_p^{m-1})$ and earlier values [9]

$$\mathbf{u}_h^{\mathrm{E}}(\mathbf{x}, t_c^n) := \begin{cases} \Big(1 + \dfrac{t_c^n - t_p^{m-1}}{\Delta t_p^{m-1}}\Big)\mathbf{u}_h(\mathbf{x}, t_p^{m-1}) - \dfrac{t_c^n - t_p^{m-1}}{\Delta t_p^{m-1}}\mathbf{u}_h(\mathbf{x}, t_p^{m-2}), \\ \qquad\qquad N_{m-1}+1 \le n \le N_m, \quad 2 \le m \le M, \\ \mathbf{u}_h(\mathbf{x}, 0), \quad 1 \le n \le N_1, \quad m = 1. \end{cases} \tag{6}$$

We often utilize the fact that velocity is smoother than the concentration to use a much larger grid size h_p than h_c and to further reduce computational cost since (5) is more expensive to solve than (11).

We present the modified method of characteristics as a time-stepping procedure for (2). Let τ denote the unit vector in the direction of $(\mathbf{u}^E, \phi)$ in $\Omega \times [0,T]$ and set $\sigma(\mathbf{x}) = (|\mathbf{u}^E(\mathbf{x})|^2 + \phi(\mathbf{x})^2)^{1/2}$. The hyperbolic part of (2), $\phi \partial c/\partial t + \mathbf{u}^E \cdot \nabla c$ can be viewed as a directional or material derivative

$$\phi \frac{\partial c}{\partial t}(\mathbf{x}, t_c^n) + \mathbf{u}^E(\mathbf{x}, t_c^n) \cdot \nabla c(\mathbf{x}, t_c^n) = \sigma \frac{dc(\mathbf{x}, t_c^n)}{d\tau}, \tag{7}$$

which in turn can be approximated by a backward difference along the characteristics

$$\begin{aligned} \sigma \frac{dc}{d\tau}(\mathbf{x}, t_c^n) &= \sigma \frac{c(\mathbf{x}, t_c^n)) - c(\mathbf{x}^*, t_c^{n-1})}{\Delta t_c^n \sqrt{1 + |\mathbf{u}^E(\mathbf{x}, t_c^n)|^2/\phi(\mathbf{x})^2}} + R(\mathbf{x}, t_c^n) \\ &= \phi \frac{c(\mathbf{x}, t_c^n) - c(\mathbf{x}^*, t_c^{n-1})}{\Delta t_c^n} + R(\mathbf{x}, t_c^n). \end{aligned} \tag{8}$$

Here and subsequently, we set

$$\mathbf{x}^* = \mathbf{x} - \frac{\mathbf{u}^E(\mathbf{x}, t_c^n)}{\phi(\mathbf{x})}\Delta t_c^n, \ \mathbf{x} = \tilde{\mathbf{x}} - \frac{\mathbf{u}^E(\tilde{\mathbf{x}}, t_c^n)}{\phi(\tilde{\mathbf{x}})}\Delta t_c^n, \ \mathbf{x}_h^* = \mathbf{x} - \frac{K_h * \mathbf{u}_h^E(\mathbf{x}, t_c^n)}{\phi(\mathbf{x})}\Delta t_c^n, \tag{9}$$

where K_h is the Bramble-Schatz kernel and the symbol $'*'$ is convolution operation(for more details, see next section) and

$$\begin{aligned} R(\mathbf{x}, t_c^n) &= \sigma \frac{dc(\mathbf{x}, t_c^n)}{d\tau} - \phi \frac{c(\mathbf{x}, t_c^n) - c(\mathbf{x}^*, t_c^{n-1})}{\Delta t_c^n} \\ &= \frac{\phi}{\Delta t_c^n} \int_{(\mathbf{x}^*, t_c^{n-1})}^{(\mathbf{x}, t_c^n)} \left[|\mathbf{x} - \mathbf{x}^*|^2 + (\tau - t_c^{n-1})^2\right]^{1/2} \frac{d^2 c}{d\tau^2} d\tau. \end{aligned} \tag{10}$$

The time difference (8) will be combined with a standard Galerkin procedure in the space variable. For $h_c > 0$ and an integer $l \ge 1$, let $M_h \subset W_\infty^1(\Omega)$ be an FEM space, which contains the space of continuous piecewise polynomials of degree at most l on a quasi-uniform partition of diameter h_c. Then we obtain a weak form of (2) by multiplying by a test function in $H^1(\Omega)$ and integrating by parts in the diffusion-dispersion term. Let $c_h(\mathbf{x}, 0)$ be an approximation to $c_0(\mathbf{x})$ (e.g., its L^2 or Ritz

projection, or interpolation). Then an MMOC-MFEM time-stepping procedure is formulated as follows:
For $m = 1, \ldots, M$, solve the MFEM scheme (5) at the pressure time step t_p^{m-1}.
For $n = N_{m-1}+1, N_{m-1}+2, \ldots, N_m$, solve the following MMOC scheme at each concentration time step t_c^n: Find $c_h(\mathbf{x}, t_c^n) \in M_h$ such that for all $z_h(\mathbf{x}) \in M_h$

$$\begin{aligned}&\int_\Omega \phi(\mathbf{x}) c_h(\mathbf{x}, t_c^n) z_h(\mathbf{x}) d\mathbf{x} + \Delta t_c^n \int_\Omega \bar{q}(\mathbf{x}, t_c^n) c_h(\mathbf{x}, t_c^n) z_h(\mathbf{x}) d\mathbf{x} \\ &\quad + \Delta t_c^n \int_\Omega \nabla z_h(\mathbf{x}) \cdot \mathbf{D}(\mathbf{x}, K_h * \mathbf{u}_h^E(\mathbf{x}, t_c^n)) \nabla c_h(\mathbf{x}, t_c^n) d\mathbf{x} \\ &\quad = \int_\Omega \phi(\mathbf{x}) c_h(\mathbf{x}_h^*, t_c^{n-1}) z_h(\mathbf{x}) d\mathbf{x} + \Delta t_c^n \int_\Omega \bar{q}(\mathbf{x}, t_c^n) \bar{c}(\mathbf{x}, t_c^n) z_h(\mathbf{x}) d\mathbf{x}.\end{aligned} \tag{11}$$

4. Preliminaries

4.1. About the spaces and projection. The finite element space M_h has the approximation and inverse properties [21] for $1 \le m \le l+1, 1 \le p, q \le \infty$

$$\inf_{z_h \in M_h} (\|z - z_h\|_{L^q} + h_c \|z - z_h\|_{W_q^1}) \le A_1 h_c^{m + (\frac{2}{q} - \frac{2}{p})} \|z\|_{W_p^m}, \ \forall z \in W_p^m(\Omega), \tag{12}$$

$$\begin{aligned}&\|z_h\|_{H^1} \le K_1 h_c^{-1} \|z_h\|_{L^2}, \qquad \|z_h\|_{L^\infty} \le K_1 \, |\log h_c|^{1/2} \|z_h\|_{H^1}, \\ &\|z_h\|_{W_q^m} \le K_1 h_c^{-(1-\frac{2}{q})} \|z_h\|_{H^m}, \qquad \forall z_h \in M_h, \ m = 0, 1.\end{aligned} \tag{13}$$

The MFEM spaces (V_h,W_h) possess approximation and inverse properties [4, 21, 22]as follows, for $2 \le p, \quad q \le +\infty$ and $1 \le m \le k+1$

$$\begin{aligned}&\inf_{\mathbf{v}_h \in V_h} \|\mathbf{v} - \mathbf{v}_h\|_{L^q} \le A_2 h_p^{m + (\frac{2}{q} - \frac{2}{p})} \|\mathbf{v}\|_{W_p^m}, \qquad \forall \mathbf{v} \in W_p^m, \\ &\inf_{\mathbf{v}_h \in V_h} \|\mathbf{v} - \mathbf{v}_h\|_{H(div)} \le A_2 h_p^m \|\mathbf{v}\|_{H^m(div)}, \qquad \forall \mathbf{v} \in H^m(div), \\ &\inf_{g_h \in W_h} \|g - g_h\|_{L^2} \le A_2 h_p^m \|g\|_{H^m}, \qquad \forall g \in H^m,\end{aligned} \tag{14}$$

$$\|\mathbf{v}_h\|_{L^q} \le K_2 h_p^{\frac{2}{q} - \frac{2}{p}} \|\mathbf{v}_h\|_{L^p}, \quad \|\mathbf{v}_h\|_{W_q^1} \le K_2 h_p^{-1} \|\mathbf{v}_h\|_{L^q}, \quad \forall \mathbf{v}_h \in V_h. \tag{15}$$

In (15) $\|\mathbf{v}_h\|_{W_q^1} := (\sum_{\Omega_e^p \subset \Omega} \|\mathbf{v}_h\|^q_{W_q^1(\Omega_e^p)})^{1/q}$ for $2 \le q < +\infty$, or $\max_{\forall \Omega_e^p \subset \Omega} \|\mathbf{v}_h\|_{W_\infty^1(\Omega_e^p)}$ for $q = +\infty$, where $\Omega_e^p \subset \Omega$ denotes the elements of the pressure mesh.

Let $\tilde{c}(\mathbf{x}, t) \in M_h, \ t \in [0, T]$, be the Ritz projection of $c(\mathbf{x}, t)$ defined by [23]

$$\begin{aligned}&\int_\Omega \nabla \chi(\mathbf{x}) \cdot \mathbf{D}(\mathbf{x}, \mathbf{u}(\mathbf{x}, t)) \nabla \tilde{c}(\mathbf{x}, t) d\mathbf{x} + \int_\Omega \chi(\mathbf{x})(1 + \bar{q}(\mathbf{x}, t)) \tilde{c}(\mathbf{x}, t) d\mathbf{x} \\ &\quad = \int_\Omega \nabla \chi(\mathbf{x}) \cdot \mathbf{D}(\mathbf{x}, \mathbf{u}(\mathbf{x}, t)) \nabla c(\mathbf{x}, t) d\mathbf{x} + \int_\Omega \chi(\mathbf{x})(1 + \bar{q}(\mathbf{x}, t)) c(\mathbf{x}, t) d\mathbf{x} \\ &\quad = - \int_\Omega \chi(\mathbf{x}) \phi \frac{\partial c}{\partial t}(\mathbf{x}, t) d\mathbf{x} - \int_\Omega \chi(\mathbf{x}) \mathbf{u}(\mathbf{x}, t) \cdot \nabla c(\mathbf{x}, t) d\mathbf{x} \\ &\qquad + \int_\Omega \chi(\mathbf{x}) c(\mathbf{x}, t) d\mathbf{x} + \int_\Omega \chi(\mathbf{x}) \bar{q}(\mathbf{x}, t) \bar{c}(\mathbf{x}, t) d\mathbf{x} \quad \forall \chi \in M_h.\end{aligned} \tag{16}$$

The following estimates hold [21, 22, 23] for $2 \le q \le +\infty, \ 1 \le m \le l+1$:

$$\begin{aligned}\|\tilde{c} - c\|_{L^\infty(L^q)} + h_c \|\tilde{c} - c\|_{L^\infty(W_q^1)} &\le A_1 h_c^{m + (\frac{2}{q} - \frac{2}{p})} \|c\|_{L^\infty(W_p^m)}, \\ \|\tilde{c} - c\|_{H^1(L^q)} &\le A_1 h_c^m \|c\|_{H^1(W_q^m)}.\end{aligned} \tag{17}$$

Here the constant A_1 is independent of c and h_c.

Let $\mathrm{I}c(\mathbf{x}, t) \in M_h, \ t \in [0, T]$, be the interpolant of $c(\mathbf{x}, t)$. We use the estimates (12) with $p = q = +\infty$, (13) with $q = +\infty$, and (17) with $q = 2$ to conclude that for $c \in L^\infty(W_\infty^1 \cap H^2)$

$$
\begin{aligned}
\|\tilde{c}\|_{L^\infty(W^1_\infty)} &\le \|\tilde{c} - \mathrm{I}c\|_{L^\infty(W^1_\infty)} + \|\mathrm{I}c - c\|_{L^\infty(W^1_\infty)} + \|c\|_{L^\infty(W^1_\infty)} \\
&\le K_1 h_c^{-1} \|\tilde{c} - \mathrm{I}c\|_{L^\infty(H^1)} + (A_1+1)\|c\|_{L^\infty(W^1_\infty)} \\
&\le K_1 h_c^{-1} \big(\|\tilde{c} - c\|_{L^\infty(H^1)} + \|c - \mathrm{I}c\|_{L^\infty(H^1)} \big) \\
&\quad + (A_1+1)\|c\|_{L^\infty(W^1_\infty)} \\
&\le 2A_1K_1\|c\|_{L^\infty(H^2)} + (A_1+1)\|c\|_{L^\infty(W^1_\infty)} =: K_3.
\end{aligned} \tag{18}
$$

Similarly, we define a mapping: $H(\mathrm{div}) \times \mathrm{L}^2_0 \to \mathrm{V_h} \times \mathrm{W_h}$ by

$$
\begin{aligned}
&\int_\Omega \mu(c(\mathbf{x},t))\mathbf{K}^{-1}(\mathbf{x})(\tilde{\mathbf{u}}(\mathbf{x},t) - \mathbf{u}(\mathbf{x},t)) \cdot \mathbf{v}_h(\mathbf{x}) d\mathbf{x} \\
&\qquad - \int_\Omega (\tilde{p}(\mathbf{x},t) - p(\mathbf{x},t)) \nabla \cdot \mathbf{v}_h(\mathbf{x}) d\mathbf{x} = 0, \quad \forall \mathbf{v}_h \in V_h, \\
&\int_\Omega w_h(\mathbf{x}) \nabla \cdot (\tilde{\mathbf{u}}(\mathbf{x},t) - \mathbf{u}(\mathbf{x},t)) d\mathbf{x} = 0, \qquad \forall w_h \in W_h.
\end{aligned} \tag{19}
$$

The following estimates hold, e.g., for Raviart-Thomas spaces [5, 9, 24]:

$$
\begin{aligned}
&\|\tilde{\mathbf{u}} - \mathbf{u}\|_{L^\infty(H(div))} + \|\tilde{p} - p\|_{L^\infty(L^2)} \\
&\quad \le A \left(\inf_{\mathbf{v}_h \in V_h} \|\mathbf{u} - \mathbf{v}_h\|_{L^\infty(H(\mathrm{div}))} + \inf_{g_h \in W_h} \|p - g_h\|_{L^\infty(L^2)} \right) \\
&\quad \le A_2 h_p^{k+1} (\|\mathbf{u}\|_{L^\infty(H^{k+1}(\mathrm{div}))} + \|p\|_{L^\infty(H^{k+1})}), \\
&\|\tilde{\mathbf{u}} - \mathbf{u}\|_{L^\infty(L^\infty)} \le A_2 h_p |\log h_p|^{\frac{1}{2}} \|\mathbf{u}\|_{L^\infty(W^1_\infty)}.
\end{aligned} \tag{20}
$$

Here A_2 is independent of h_p, $\mathbf{u}$, p, and c.

We let $\mathbf{Iu} \in V_h$ be an interpolant of $\mathbf{u}$. We use the estimates (14) (15) and (20) to conclude that

$$
\begin{aligned}
\|\tilde{\mathbf{u}}\|_{L^\infty(W^1_\infty)} &\le \|\tilde{\mathbf{u}} - \mathbf{Iu}\|_{L^\infty(W^1_\infty)} + \|\mathbf{Iu} - \mathbf{u}\|_{L^\infty(W^1_\infty)} + \|\mathbf{u}\|_{L^\infty(W^1_\infty)} \\
&\le K_2 h_p^{-1} \|\tilde{\mathbf{u}} - \mathbf{Iu}\|_{L^\infty(L^\infty)} + (A_2+1)\|\mathbf{u}\|_{L^\infty(W^1_\infty)} \\
&\le K_2 h_p^{-1} \big(\|\tilde{\mathbf{u}} - \mathbf{u}\|_{L^\infty(L^\infty)} + \|\mathbf{u} - \mathbf{Iu}\|_{L^\infty(L^\infty)} \big) \\
&\quad + (A_2+1)\|\mathbf{u}\|_{L^\infty(W^1_\infty)} \\
&\le \big(A_2K_2 |\log h_p|^{\frac{1}{2}} + 2A_2 + 1 \big) \|\mathbf{u}\|_{L^\infty(W^1_\infty)} \\
&\le K_4 |\log h_p|^{\frac{1}{2}}.
\end{aligned} \tag{21}
$$

For the analysis in §5 we introduce an extrapolation of the exact velocity $\mathbf{u}$

$$
\mathbf{u}^{\mathrm{E}}(\mathbf{x}, t_c^n) := \begin{cases} \Big(1 + \dfrac{t_c^n - t_p^{m-1}}{\Delta t_p^{m-1}}\Big)\mathbf{u}(\mathbf{x}, t_p^{m-1}) - \dfrac{t_c^n - t_p^{m-1}}{\Delta t_p^{m-1}} \mathbf{u}(\mathbf{x}, t_p^{m-2}), \\ \qquad\qquad N_{m-1}+1 \le n \le N_m, \quad 2 \le m \le M, \\ \mathbf{u}(\mathbf{x}, 0), \quad 1 \le n \le N_1, \quad m = 1. \end{cases} \tag{22}
$$

Then we routinely see that for $2 \le q \le +\infty$

$$
\begin{aligned}
&\|\mathbf{u}^E(\cdot,t) - \mathbf{u}(\cdot,t)\|_{L^q} \\
&\quad \le \begin{cases} A_3 (\Delta t_p)^{\frac{3}{2}} \|\mathbf{u}\|_{H^2(t_p^{m-2}, t_p^m; L^q)}, & \forall t \in [t_p^{m-1}, t_p^m], \ m \ge 2, \\ A_3 \Delta t_p^1 \|\mathbf{u}\|_{W^1_\infty(t_p^0, t_p^1; L^q)}, & \forall t \in [t_p^0, t_p^1], \quad m = 1. \end{cases}
\end{aligned} \tag{23}
$$

4.2. About the extension of the velocity. Let K_h be the Bramble-Schatz kernel function defined by [18, 25, 26]

$$K_h(x) = \prod_{m=1}^{2} \big(\sum_{i=-k}^{k} h_p^{-1} k_i' g_{k+2}(h_p^{-1} x_m - i) \big), \tag{24}$$

where

$$g_l(s) = (\chi_{[-1/2,1/2]} * g_{l-1})(s), \quad g_1(s) = \chi_{[-1/2,1/2]}(s), \tag{25}$$

$$k'_{-i} = k'_i = \frac{1}{2} k_i, \quad for \quad i = 1, \cdots, k+2, \quad and \quad k'_0 = k_0, \tag{26}$$

$$\sum_{i=0}^{k} k_i \int_{\mathbb{R}} g_k(y)(y+i)^{2n} dy = \delta_{0n}, \quad n = 0, \cdots, k. \tag{27}$$

Here $\chi_{[-1/2,1/2]}$ is the characteristics function on $[-1/2, 1/2]$. It is known that, in the periodic case considered here,

$$\|K_h * w - w\| \le Q \|w\|_r h_p^r, \quad 0 \le r \le 2k+2, \tag{28}$$

$$\|D^\nu (K_h * w)\|_m \le Q \|\partial^\nu w\|_m, \quad m \in Z, \tag{29}$$

where $D^\nu = \frac{\partial^{|\nu|}}{\partial x_1^{\nu_1} \partial x_2^{\nu_2}}, \nu = (\nu_1, \nu_2)$ and ∂^ν is the corresponding forward, divided difference with step length h_p, and

$$\|w\| \le Q \sum_{\nu \le s} \|D^\nu w\|_{-s}, \quad 0 \le s \in Z. \tag{30}$$

It follows from [17] that

$$\|\partial^\nu (\mathbf{u} - \tilde{\mathbf{u}})\|_{-(k+1)} \le Q(c) \|p\|_{2k+4} h_p^{2k+2}, \tag{31}$$

for $|\nu| \le k+1$. So , by (28)–(31)

$$\begin{aligned} \|\mathbf{u} - K_h * \tilde{\mathbf{u}}\| \le & \ \|\mathbf{u} - K_h * \mathbf{u}\| + \|K_h * (\mathbf{u} - \tilde{\mathbf{u}})\| \\ \le & \ Q\{\|\mathbf{u}\|_{2k+2} h_p^{2k+2} + \sum_{|\nu| \le k+1} \|D^\nu (K_h * (\mathbf{u} - \tilde{\mathbf{u}}))\|_{-(k+1)}\} \\ \le & \ Q\{\|\mathbf{u}\|_{2k+2} h_p^{2k+2} + \sum_{|\nu| \le k+1} \|\partial^\nu (\mathbf{u} - \tilde{\mathbf{u}}\|_{-(k+1)}\} \\ \le & \ Q(c) \|p\|_{2k+4} h_p^{2k+2} \end{aligned} \tag{32}$$

Similarly, it follows from estimates for difference quotients for $\nabla \cdot (\mathbf{u} - \tilde{\mathbf{u}})$ and $(p - \tilde{p})$ that [17]

$$\|\nabla \cdot (\mathbf{u} - K_h * \tilde{\mathbf{u}})\| \le Q(c) \|p\|_{2k+4} h_p^{2k+2}, \tag{33}$$

$$\|p - K_h * \tilde{p}\| \le Q(c) \|p\|_{2k+4} h_p^{2k+2}. \tag{34}$$

It will be useful to note some relations between $\mathbf{u}_h$, $K_h * \mathbf{u}_h$ and $K_h * \tilde{\mathbf{u}}$. First, since

$$(\alpha(c)(\mathbf{u}_h - \tilde{\mathbf{u}}), z) - (\nabla \cdot z, P - \tilde{p}) = ([\alpha(c) - \alpha(c_h)]\tilde{\mathbf{u}}, z), \quad z \in V_h, \tag{35}$$

$$(\nabla \cdot (\mathbf{u}_h - \tilde{\mathbf{u}}), w), \quad w \in W_h, \tag{36}$$

where $\alpha(c) = \mu(c)/K$, the known boundedness of $\tilde{\mathbf{u}}$ in L^∞ leads immediately to the bound

$$\|\mathbf{u}_h - \tilde{\mathbf{u}}\|_{H(div)} + \|P - \tilde{p}\| \le Q \|c - c_h\|. \tag{37}$$

Then (29) implies that

$$\|K_h * (\mathbf{u}_h - \tilde{\mathbf{u}})\|_{H(div)} + \|K_h * (P - \tilde{p})\| \le Q \|c - c_h\|. \tag{38}$$

5. An Optimal-Order Error Estimate

We prove an optimal-order error estimate for the MMOC-MFEM time-stepping procedure with any order of approximating polynomials ($k \geq 0$, $l \geq 1$).

Theorem 5.1. *Suppose that the solution $(c, p, \mathbf{u})$ of problem (1)–(3) satisfies $c \in L^\infty(W^{l+1}_{2+\delta}) \cap L^\infty(W^1_\infty) \cap H^1(H^{l+1})$, $p \in L^\infty(H^{k+1})$, and $\mathbf{u} \in L^\infty(H^{k+1}(\mathrm{div}) \cap \mathrm{W}^1_\infty) \cap \mathrm{W}^1_\infty(\mathrm{L}^\infty) \cap \mathrm{H}^2(\mathrm{L}^2)$. Let $(c_h(\mathbf{x}, t^n_c), p_h(\mathbf{x}, t^m_p), \mathbf{u}_h(\mathbf{x}, t^m_p))$ be the solution of the MMOC-MFEM time-stepping procedure (5) and (11) with $l \geq 1$ and $k \geq 0$. Assume that the discretization parameters obey the relations*

$$
\begin{aligned}
&\Delta t_c = O(h_p^{1-\delta}), \quad \Delta t_c = O(h_c^{1/2+3\delta}), \quad h_c^{l+1} = O(h_p^{3/2}),\\
&\Delta t_p^1 = O(h_p^{2/3}), \quad \Delta t_p = O(h_p^{1/2}),
\end{aligned} \tag{39}
$$

where δ is an arbitrary small positive constant. There exist positive constants h^*_c, h^*_p, Δt^*_c, Δt^*_p, and Q^* such that the following optimal-order error estimate holds for $0 < h_c \leq h^*_c$, $0 < h_p \leq h^*_p$, $0 < \Delta t_c \leq \Delta t^*_c$, and $0 < \Delta t_p \leq \Delta t^*_p$:

$$
\begin{aligned}
&\|c_h - c\|_{\hat{L}^\infty_c(L^2)} + h_c\|c_h - c\|_{\hat{L}^2_c(H^1)}\\
&\quad +\|\mathbf{u}_h - \mathbf{u}\|_{\hat{L}^\infty_p(H(\mathrm{div}))} + \|p_h - p\|_{\hat{L}^\infty_p(L^2)}\\
&\quad \leq Q^*\Delta t^n_c \Big\|\frac{d^2 c}{d\tau^2}\Big\|_{L^2(L^2)} + Q^*((\Delta t^1_p)^{3/2} + (\Delta t_p)^2)\|\mathbf{u}\|_{H^2(L^2)}\\
&\qquad +Q^* h_c^{l+1}(\|c\|_{L^\infty(H^{l+1})} + \|c\|_{H^1(H^{l+1})})\\
&\qquad +Q^* h_p^{2k+2}(\|\mathbf{u}\|_{L^\infty(H^{k+1}(\mathrm{div}))} + \|p\|_{L^\infty(H^{2k+4})}).
\end{aligned} \tag{40}
$$

The constant $Q^* = Q^*(h^*_c, h^*_p, \Delta t^*_c, \Delta t^*_p, T)$, but Q^* is independent of the discretization parameters h_c, h_p, Δt_c, or Δt_p.

To prove the theorem, we use Eqs. (4), (5) and (19) to derive a relation

$$
\begin{aligned}
&\int_\Omega \mu(c_h(\mathbf{x}, t^m_p))\mathbf{K}^{-1}(\mathbf{x})(\mathbf{u}_h(\mathbf{x}, t^m_p) - \tilde{\mathbf{u}}(\mathbf{x}, t^m_p)) \cdot \mathbf{v}_h(\mathbf{x})d\mathbf{x}\\
&\qquad - \int_\Omega (p_h(\mathbf{x}, t^m_p) - \tilde{p}(\mathbf{x}, t^m_p))\nabla \cdot \mathbf{v}_h(\mathbf{x})d\mathbf{x}\\
&\qquad = \int_\Omega (\mu(c(\mathbf{x}, t^m_p)) - \mu(c_h(\mathbf{x}, t^m_p)))\mathbf{K}^{-1}(\mathbf{x})\tilde{\mathbf{u}}(\mathbf{x}, t^m_p) \cdot \mathbf{v}_h(\mathbf{x})d\mathbf{x},\\
&\int_\Omega w_h(\mathbf{x})\nabla \cdot (\mathbf{u}_h(\mathbf{x}, t^m_p) - \tilde{\mathbf{u}}(\mathbf{x}, t^m_p))d\mathbf{x} = 0, \quad \forall (\mathbf{v}_h, w_h) \in V_h \times W_h.
\end{aligned}
$$

Combining this equation with (21) yields an estimate [4]

$$
\begin{aligned}
&\|\mathbf{u}_h(\cdot, t^m_p) - \tilde{\mathbf{u}}(\cdot, t^m_p)\|_{H(\mathrm{div})} + \|p_h(\cdot, t^m_p) - \tilde{p}(\cdot, t^m_p)\|\\
&\quad \leq Q(1 + \|\tilde{\mathbf{u}}(\cdot, t^m_p)\|_{L^\infty})\ \|c_h(\cdot, t^m_p) - c(\cdot, t^m_p)\|\\
&\quad \leq A_4\|c_h(\cdot, t^m_p) - c(\cdot, t^m_p)\|, \qquad 0 \leq m \leq M.
\end{aligned} \tag{41}
$$

For convenience, we have dropped the subscript L^2. Moreover, we use the stability estimate of the saddle-point problem in the L_q norm to get [27, 28]

$$
\|\mathbf{u}_h(\cdot, t^m_p) - \tilde{\mathbf{u}}(\cdot, t^m_p)\|_{L^q} \leq A_5\|c_h(\cdot, t^m_p) - c(\cdot, t^m_p)\|_{L^q},\ \forall\ 2 \leq q < \infty. \tag{42}
$$

The estimates (20) and (41) show that the bound on $\|\mathbf{u}_h - \mathbf{u}\|_{\hat{L}^\infty_p(H(\mathrm{div}))} + \|p_h - p\|_{\hat{L}^\infty_p(L^2)}$ in (40) is a consequence of the bound on $\|c_h - c\|_{\hat{L}^\infty_c(L^2)}$. To analyze $\|c_h - c\|_{\hat{L}^\infty_c(L^2)}$, we set $\xi(\mathbf{x}, t^n_c) := c_h(\mathbf{x}, t^n_c) - \tilde{c}(\mathbf{x}, t^n_c)$ and $\eta(\mathbf{x}, t^n_c) := \tilde{c}(\mathbf{x}, t^n_c) - c(\mathbf{x}, t^n_c)$.

Note that $c_h - c = \xi + \eta$ and that the estimate for η is known from (17). The key to prove the theorem is to derive an estimate of the form (40) for ξ. We use (15), (21), and (42) to get

$$
\begin{aligned}
\|(\mathbf{u}_h(\cdot, t_p^j)\|_{W^1_\infty} &\le \|(\mathbf{u}_h - \tilde{\mathbf{u}})(\cdot, t_p^j)\|_{W^1_\infty} + \|\tilde{\mathbf{u}}(\cdot, t_p^j)\|_{W^1_\infty} \\
&\le K_2 h_p^{-1-2/q} \|(\mathbf{u}_h - \tilde{\mathbf{u}})(\cdot, t_p^j)\|_{L^q} + K_4 |\log h_p|^{\frac{1}{2}} \\
&\le K_2 A_5 h_p^{-1-2/q} \|(c_h - c)(\cdot, t_p^j)\|_{L^q} + K_4 |\log h_p|^{\frac{1}{2}}.
\end{aligned}
\tag{43}
$$

On the other hand, the initial approximation $c_h(\mathbf{x}, 0)$ to $c(\mathbf{x}, 0)$ satisfies

$$
\|c_h(\cdot, 0) - c(\cdot, 0)\|_{L^q} \le K_5 h_c^{l+1}. \tag{44}
$$

We prove the theorem by induction on m. We base on (43) with $j = 0$ and (44) to assume that for a properly chosen $q = q(\delta)$ (to be given above (55))

$$
\|\mathbf{u}_h(\cdot, t_j^p)\|_{W^1_\infty} \le 5K_4 \Delta t_c^{-1} h_p^{\frac{\delta}{2}}, \qquad \forall\ 0 \le j \le m-1. \tag{45}
$$

To derive an error equation, we use Eqs. (7)–(9) to rearrange Eq. (16) at $t = t_c^{N_{m-1}+1}, \cdots, t_c^{N_m}$ for any $z_h \in M_h$ as follows:

$$
\begin{aligned}
&\int_\Omega \nabla z_h(\mathbf{x}) \cdot \mathbf{D}(\mathbf{x}, K_h * \mathbf{u}_h^E(\mathbf{x}, t_c^n)) \nabla \tilde{c}(\mathbf{x}, t_c^n) d\mathbf{x} + \int_\Omega z_h(\mathbf{x})(1 + \bar{q}) \tilde{c}(\mathbf{x}, t_c^n) d\mathbf{x} \\
&\quad = -\int_\Omega z_h(\mathbf{x}) \Big(\phi \frac{\partial c}{\partial t} + \mathbf{u}^E \cdot \nabla c \Big)(\mathbf{x}, t_c^n) d\mathbf{x} + \int_\Omega z_h(\mathbf{x}) c(\mathbf{x}, t_c^n) d\mathbf{x} \\
&\qquad + \int_\Omega z_h(\mathbf{x}) \bar{q}(\mathbf{x}, t_n) \bar{c}(\mathbf{x}, t_c^n) d\mathbf{x} + \int_\Omega z_h(\mathbf{x}) (\mathbf{u}^E - \mathbf{u})(\mathbf{x}, t_c^n) \cdot \nabla c(\mathbf{x}, t_c^n) d\mathbf{x} \\
&\qquad + \int_\Omega \nabla z_h(\mathbf{x}) \cdot \big(\mathbf{D}(\mathbf{x}, K_h * \mathbf{u}_h^E(\mathbf{x}, t_c^n)) - \mathbf{D}(\mathbf{x}, \mathbf{u}(\mathbf{x}, t_c^n)) \big) \nabla \tilde{c}(\mathbf{x}, t_c^n) d\mathbf{x} \\
&\quad = -\int_\Omega z_h(\mathbf{x}) \Big(\phi(\mathbf{x}) \frac{c(\mathbf{x}, t_c^n) - c(\mathbf{x}^*, t_c^{n-1})}{\Delta t_c^n} + R(\mathbf{x}, t_c^n) \Big) d\mathbf{x} \\
&\qquad + \int_\Omega z_h(\mathbf{x}) c(\mathbf{x}, t_c^n) d\mathbf{x} + \int_\Omega z_h(\mathbf{x}) \bar{q}(\mathbf{x}, t_c^n) \bar{c}(\mathbf{x}, t_c^n) d\mathbf{x} \\
&\qquad + \int_\Omega z_h(\mathbf{x}) (\mathbf{u}^E - \mathbf{u})(\mathbf{x}, t_c^n) \cdot \nabla c(\mathbf{x}, t_c^n) d\mathbf{x} \\
&\qquad + \int_\Omega \nabla z_h(\mathbf{x}) \cdot \big(\mathbf{D}(\mathbf{x}, K_h * \mathbf{u}_h^E(\mathbf{x}, t_c^n)) - \mathbf{D}(\mathbf{x}, \mathbf{u}(\mathbf{x}, t_c^n)) \big) \nabla \tilde{c}(\mathbf{x}, t_c^n) d\mathbf{x}.
\end{aligned}
\tag{46}
$$

We subtract Eq. (11) from Eq. (46) multiplied by Δt_c^n and choose $z_h = \xi(\mathbf{x}, t_c^n)$ in the resulting equation to obtain

$$
\begin{aligned}
&\int_\Omega \phi(\mathbf{x})\xi(\mathbf{x}, t_c^n)^2 d\mathbf{x} + \Delta t_c^n \int_\Omega \nabla\xi(\mathbf{x}, t_c^n) \cdot \mathbf{D}(\mathbf{x}, K_h * \mathbf{u}_h^E(\mathbf{x}, t_c^n))\nabla\xi(\mathbf{x}, t_c^n) d\mathbf{x} \\
&= \int_\Omega \phi(\mathbf{x})\xi(\mathbf{x}, t_c^n)\xi(\mathbf{x}, t_c^{n-1}) d\mathbf{x} + \Delta t_c^n \int_\Omega \xi(\mathbf{x}, t_c^n) R(\mathbf{x}, t_c^n) d\mathbf{x} \\
&\quad + \Delta t_c^n \int_\Omega \xi(\mathbf{x}, t_c^n)\big(\mathbf{u} - \mathbf{u}^E\big)(\mathbf{x}, t_c^n) \cdot \nabla c(\mathbf{x}, t_c^n) d\mathbf{x} \\
&\quad - \int_\Omega \xi(\mathbf{x}, t_c^n)\phi(\mathbf{x})(\eta(\mathbf{x}, t_c^n) - \eta(\mathbf{x}, t_c^{n-1})) d\mathbf{x} \\
&\quad + \Delta t_c^n \int_\Omega \xi(\mathbf{x}, t_c^n)\eta(\mathbf{x}, t_c^n) d\mathbf{x} - \Delta t_c^n \int_\Omega \bar{q}(\mathbf{x}, t_c^n)\xi(\mathbf{x}, t_c^n)^2 \\
&\quad + \Delta t_c^n \int_\Omega \nabla z_h(\mathbf{x}) \cdot \big(\mathbf{D}(\mathbf{x}, \mathbf{u}(\mathbf{x}, t_c^n)) - \mathbf{D}(\mathbf{x}, K_h * \mathbf{u}_h^E(\mathbf{x}, t_c^n))\big)\nabla\tilde{c}(\mathbf{x}, t_c^n) d\mathbf{x} \\
&\quad + \int_\Omega \phi(\mathbf{x})(c_h(\mathbf{x}_h^*, t_c^{n-1}) - c(\mathbf{x}^*, t_c^{n-1}))\xi(\mathbf{x}, t_c^n)(\mathbf{x}) d\mathbf{x} \\
&\quad - \int_\Omega \phi(\mathbf{x})(c_h(\mathbf{x}, t_c^{n-1}) - c(\mathbf{x}, t_c^{n-1}))\xi(\mathbf{x}, t_c^n) d\mathbf{x}.
\end{aligned} \tag{47}
$$

By Cauchy-inequality, the first term on the right side is bounded by

$$
\Big|\int_\Omega \phi(\mathbf{x})\xi(\mathbf{x}, t_c^n)\xi(\mathbf{x}, t_c^{n-1}) d\mathbf{x}\Big| \le \frac{1}{2}\int_\Omega \phi(\mathbf{x})\xi^2(\mathbf{x}, t_c^n) d\mathbf{x} + \frac{1}{2}\int_\Omega \phi(\mathbf{x})\xi^2(\mathbf{x}, t_c^{n-1}) d\mathbf{x}.
$$

We use (10) to bound the second term on the right side of (47) as follows:

$$
\begin{aligned}
&\Delta t_c^n \Big|\int_\Omega R(\mathbf{x}, t_c^n)\xi(\mathbf{x}, t_c^n) d\mathbf{x}\Big| \\
&\le (\Delta t_c^n)^{3/2} \Big\|\frac{\rho^3}{\phi}\Big\|_{L^\infty}^{1/2} \|\xi(\cdot, t_c^n)\| \Big(\int_\Omega \int_{(\mathbf{x}^*, t_c^{n-1})}^{(\mathbf{x}, t_c^n)} \Big|\frac{d^2 c}{d\tau^2}\Big|^2 d\tau d\mathbf{x}\Big)^{1/2} \\
&\le (\Delta t_c^n)^{3/2} \Big\|\frac{\rho^2}{\phi}\Big\|_{L^\infty} \|\xi(\cdot, t_c^n)\| \Big(\int_\Omega \int_{t_c^{n-1}}^{t_c^n} \Big|\frac{d^2 c}{d\tau^2}(\bar{\tau}\mathbf{x}^* + (1-\bar{\tau})\mathbf{x}, t)\Big|^2 d\tau d\mathbf{x}\Big)^{1/2} \\
&\le Q(\Delta t_c^n)^2 \Big\|\frac{d^2 c}{d\tau^2}\Big\|_{L^2(t_c^{n-1}, t_c^n; L^2)}^2 + Q\Delta t_c^n \|\xi(\cdot, t_c^n)\|^2.
\end{aligned} \tag{48}
$$

In (48) we have used a change of variable to replace $(\bar{\tau}\mathbf{x}^* + (1-\bar{\tau})\mathbf{x}, t)$ with $(\mathbf{x}, t)$ at the cost of a multiplicative constant.

We use (23) to estimate the third term on the right side of (47) by

$$
\begin{aligned}
&\Delta t_c^n \Big|\int_\Omega \xi(\mathbf{x}, t_c^n)\big(\mathbf{u} - \mathbf{u}^E\big)(\mathbf{x}, t_c^n) \cdot \nabla c(\mathbf{x}, t_c^n) d\mathbf{x}\Big| \\
&\quad \le \Delta t_c^n \|(\mathbf{u} - \mathbf{u}^E)(\cdot, t_c^n)\| \|\nabla c(\cdot, t_c^n)\|_{L^\infty} \|\xi(\cdot, t_c^n)\| \\
&\quad \le Q\Delta t_c^n \|\xi(\cdot, t_c^n)\|^2 + Q\delta_{m,1}\Delta t_c^n (\Delta t_p^1)^2 \|\mathbf{u}\|_{W_\infty^1(0, t_1^p; L^2)} \\
&\qquad + Q(1 - \delta_{m,1})\Delta t_c^n (\Delta t_p)^3 \|\mathbf{u}\|_{H^2(t_{m-2}^p, t_m^p; L^2)},
\end{aligned}
$$

where $\delta_{i,j} = 1$ if $i = j$ or 0 otherwise.

The fourth term on the right side of (47) is bounded by

$$
\begin{aligned}
&\Big|\int_\Omega \xi(\mathbf{x}, t_c^n)\phi(\eta(\mathbf{x}, t_c^n) - \eta(\mathbf{x}, t_c^{n-1})) d\mathbf{x}\Big| \\
&\quad = \Big|\int_\Omega \xi(\mathbf{x}, t_c^n)\phi \int_{t_c^{n-1}}^{t_c^n} \frac{\eta(\mathbf{x}, t)}{\partial t} dt d\mathbf{x}\Big| \\
&\quad \le A_1^2 h_c^{2l+2} \|c\|_{H^1(t_c^{n-1}, t_c^n; H^{m+1})}^2 + Q\Delta t_c^n \|\xi(\cdot, t_c^n)\|^2.
\end{aligned}
$$

We bound the fifth and sixth terms on the right-hand side of Eq. (47) by

$$\begin{aligned}&\Delta t_c^n\Big|\int_\Omega \bar{q}(\mathbf{x},t_c^n)\xi^2(\mathbf{x},t_c^n)d\mathbf{x}-\int_\Omega \eta(\mathbf{x},t_c^n)\xi(\mathbf{x},t_c^n)d\mathbf{x}\Big|\\ &\le Q\Delta t_c^n\|\xi(\cdot,t_c^n)\|^2+\Delta t_c^n\|\eta(\cdot,t_c^n)\|^2\\ &\le Q\Delta t_c^n\|\xi(\cdot,t_c^n)\|^2+A_1^2\Delta t_c^n h_c^{2l+2}\|c\|^2_{L^\infty(H^{l+1})}.\end{aligned}$$

The last three terms on the right-hand side of (47) will be analyzed in Lemmas 6.2 and 6.3, respectively. They are bounded by

$$\begin{aligned}&\Delta t_c^n\Big|\int_\Omega \nabla\xi(\mathbf{x},t_c^n)\cdot(\mathbf{D}(\mathbf{x},K_h*\mathbf{u}_h^E(\mathbf{x},t_c^n))-\mathbf{D}(\mathbf{x},\mathbf{u}(\mathbf{x},t_c^n)))\nabla\tilde{c}(\mathbf{x},t_c^n)d\mathbf{x}\Big|\\ &\le \varepsilon\Delta t_c^n\|\nabla\xi(\cdot,t_c^n)\|^2+Q\Delta t_c^n(\|\xi(\cdot,t_p^{m-1})\|^2+\|\xi(\cdot,t_p^{m-2})\|^2)\\ &\quad+Q\Delta t_c^n\Big(h_c^{2l+2}\|c\|^2_{L^\infty(H^{l+1})}+h_p^{4k+4}(\|\mathbf{u}\|^2_{L^\infty(H^{k+1}(\mathrm{div}))}+\|p\|^2_{L^\infty(H^{2k+4})})\\ &\quad+\delta_{m,1}(\Delta t_p^1)^2\|\mathbf{u}\|^2_{W_1^\infty(0,t_1^p;L^2)}+(1-\delta_{m,1})(\Delta t_p)^3\|\mathbf{u}\|^2_{H^2(t_{m-2}^p,t_p^m;L^2)}\Big),\end{aligned}\tag{49}$$

and

$$\begin{aligned}&\Big|\int_\Omega \phi\big[c_h(\mathbf{x}_h^*,t_c^{n-1})-c(\mathbf{x}^*,t_c^{n-1})\big]\xi(\mathbf{x},t_n)d\mathbf{x}\\ &\quad-\int_\Omega \phi\big[c_h(\mathbf{x},t_c^{n-1})-c(\mathbf{x},t_c^{n-1})\big[\xi(\mathbf{x},t_n)d\mathbf{x}\Big|\\ &\le \varepsilon\Delta t_c^n(\|\nabla\xi(\cdot,t_{n-1}^c)\|^2+\|\xi(\cdot,t_n^c)\|^2_{H^1})\\ &\quad+Q\Delta t_c^n(\|\xi(\cdot,t_p^{m-1})\|^2+\|\xi(\cdot,t_p^{m-2})\|^2+\|\xi(\cdot,t_n^c)\|^2)\\ &\quad+Q\Delta t_c^n\Big(h_c^{2l+2}\|c\|^2_{L^\infty(H^{l+1})}+h_p^{4k+4}(\|\mathbf{u}\|^2_{L^\infty(H^{k+1}(\mathrm{div}))}+\|p\|^2_{L^\infty(H^{2k+4})})\Big).\end{aligned}\tag{50}$$

We incorporate the preceding estimates into Eq. (47) to obtain

$$\begin{aligned}&\int_\Omega \phi(\mathbf{x})\xi^2(\mathbf{x},t_c^n)d\mathbf{x}+\Delta t_c^n\int_\Omega \nabla\xi(\mathbf{x},t_c^n))\cdot\mathbf{D}(\mathbf{x},K_h*\mathbf{u}_h^E(\mathbf{x},t_c^n))\nabla\xi(\mathbf{x},t_c^n)d\mathbf{x}\\ &\le \frac{1}{2}\int_\Omega \phi(\mathbf{x})\xi^2(\mathbf{x},t_c^n)d\mathbf{x}+\frac{1}{2}\int_\Omega \phi(\mathbf{x})\xi^2(\mathbf{x},t_c^{n-1})d\mathbf{x}+\varepsilon\Delta t_c^n(\|\nabla\xi(\cdot,t_c^{n-1})\|^2\\ &\quad+\|\xi(\cdot,t_n^c)\|^2)+Q\Delta t_c^n\Big(\|\xi(\cdot,t_c^n)\|^2+\|\xi(\cdot,t_c^{n-1})\|^2\\ &\quad+\|\xi(\cdot,t_p^{m-1})\|^2+\|\xi(\cdot,t_p^{m-2})\|^2\Big)+Q(\Delta t_c^n)^2\Big\|\frac{d^2c}{d\tau^2}\Big\|^2_{L^2(t_c^{n-1},t_c^n;L^2)}\\ &\quad+Q\Delta t_c^n\Big(\delta_{m,1}(\Delta t_p^1)^2\|\mathbf{u}\|^2_{W^1_\infty(0,t_1^p;L^2)}\\ &\quad+(1-\delta_{m,1})(\Delta t_p)^3\|\mathbf{u}\|^2_{H^2(t_{m-2}^p,t_p^m;L^2)}\Big)\\ &\quad+Qh_c^{2l+2}\Big(\Delta t_c^n\|c\|^2_{L^\infty(H^{l+1})}+\|c\|^2_{H^1(t_c^{n-1},t_c^n;H^{l+1})}\Big)\\ &\quad+Q\Delta t_c^n h_p^{4k+4}\Big(\|\mathbf{u}\|^2_{L^\infty(H^{k+1}(\mathrm{div}))}+\|p\|^2_{L^\infty(H^{2k+4})}\Big).\end{aligned}\tag{51}$$

We choose $\varepsilon = \frac{1}{2}|\mathbf{D}|_{min}$, and sum this estimate for $n = 1, 2, \ldots, n^*$, with $n^* \le N_m$, and cancel the like terms to obtain

$$
\begin{aligned}
&\int_\Omega \phi(\mathbf{x})\xi^2(\mathbf{x}, t_c^{n^*})d\mathbf{x} + |\mathbf{D}|_{min}\sum_{n=1}^{n^*}\Delta t_c^n\|\nabla\xi(\cdot, t_c^n)\|^2 \\
&\le Q\sum_{n=1}^{n^*}\Delta t_c^n\|\xi(\cdot, t_c^n)\|^2 + Q\big((\Delta t_p)^4 + (\Delta t_p^1)^3\big)\|\mathbf{u}\|^2_{H^2(L^2)} \\
&\quad +Q(\Delta t_c^n)^2\Big\|\frac{d^2c}{d\tau^2}\Big\|^2_{L^2(L^2)} + Qh_c^{2l+2}\big(\|c\|^2_{L^\infty(H^{l+1})} + \|c\|^2_{H^1(H^{l+1})}\big) \\
&\quad +Qh_p^{4k+4}\big(\|\mathbf{u}\|^2_{L^\infty(H^{k+1}(\mathrm{div}))} + \|p\|^2_{L^\infty(H^{2k+4})}\big).
\end{aligned}
\tag{52}
$$

We choose Δt_c^n small enough such that $Q\Delta t_c^n < \phi_{min}/2$ and apply Gronwall inequality to (52) to get

$$
\begin{aligned}
&\|\xi\|_{\hat{L}_c^\infty(0,t_c^{n^*};L^2)} + \|\nabla\xi\|_{\hat{L}_c^2(0,t_c^{n^*};L^2)} \\
&\le Q_1\Delta t_c\Big\|\frac{d^2c}{d\tau^2}\Big\|_{L^2(L^2)} + Q_2\big((\Delta t_p)^2 + (\Delta t_p^1)^{3/2}\big)\|\mathbf{u}\|_{H^2(L^2)} \\
&\quad +Q_3h_c^{l+1}\big(\|c\|_{L^\infty(H^{l+1})} + \|c\|_{H^1(H^{l+1})}\big) \\
&\quad +Q_4h_p^{2k+2}\big(\|\mathbf{u}\|_{L^\infty(H^{k+1}(\mathrm{div}))} + \|p\|_{L^\infty(H^{2k+4})}\big) \\
&\le Q_5\Delta t_c + Q_6(h_c^{l+1} + h_p^{2k+2} + (\Delta t_p)^2 + (\Delta t_p^1)^{3/2}).
\end{aligned}
\tag{53}
$$

Combining this estimate with (17), we get (40) at once.

It remains to check the induction hypothesis (45) for $j = m$. We use (17), (21),(43),(53), and the embedding and inverse inequality in time to obtain

$$
\begin{aligned}
&\|\mathbf{u}_h(\cdot, t_p^m)\|_{W^1_\infty} \\
&\le K_2A_5h_p^{-1-\frac{2}{q}}(\|\xi(\cdot, t_p^m)\|_{L^q} + \|\eta(\cdot, t_p^m)\|_{L^q}) + K_4|\log h_p|^{\frac{1}{2}} \\
&\le K_2A_5h_p^{-1-\frac{2}{q}}\big(\|\xi(\cdot, t_p^m)\|_{H^1} + A_1h_c^{l+1+(\frac{2}{q}-\frac{2}{2+\delta})}\|c\|_{L^\infty(W^{l+1}_{2+\delta})}\big) \\
&\quad +K_4|\log h_p|^{\frac{1}{2}} \\
&\le K_2A_5h_p^{-1-\frac{2}{q}}\Big[(\Delta t_c)^{-1/2}\big(Q_5\Delta t_c + Q_6(h_c^{l+1} + h_p^{2k+2} + (\Delta t_p)^2 \\
&\quad +(\Delta t_p^1)^{3/2})\big) + A_1h_c^{l+1+(\frac{2}{q}-\frac{2}{2+\delta})}\|c\|_{L^\infty(W^{l+1}_{2+\delta})}\Big] + K_4|\log h_p|^{\frac{1}{2}} \\
&\le \Delta t_c^{-1}h_p^{\frac{\delta}{2}}\Big[K_2A_5h_p^{-1-\frac{2}{q}-\frac{\delta}{2}}\Big(Q_5(\Delta t_c)^{3/2} \\
&\quad +Q_6(\Delta t_c)^{1/2}\big((\Delta t_p)^2 + (\Delta t_p^1)^{3/2}\big)\Big) \\
&\quad +K_2A_5Q_6(\Delta t_c)^{1/2}h_p^{2k+1-\frac{2}{q}-\frac{\delta}{2}} + K_2A_5Q_6(\Delta t_c)^{1/2}h_p^{-1-\frac{2}{q}-\frac{\delta}{2}}h_c^{l+1} \\
&\quad +A_1K_2A_5\Delta t_ch_p^{-1-\frac{2}{q}-\frac{\delta}{2}}h_c^{l+1+(\frac{2}{q}-\frac{2}{2+\delta})}\|c\|_{L^\infty(W^{l+1}_{2+\delta})} \\
&\quad +K_4\Delta t_c|\log h_p|^{\frac{1}{2}}h_p^{-\frac{\delta}{2}}\Big].
\end{aligned}
\tag{54}
$$

We note that $2/q+\delta/2 = 1/2-2\delta$ if we choose $q = 4/(1-5\delta)$. We use the condition (39) to conclude that there exist positive h_p^*, h_c^*, Δt_p^* and Δt_c^* that are independent of m in (45), such that the following estimates hold for $0 < h_p < h_p^*$, $0 < h_c < h_c^*$,

$0 < \Delta t_p < \Delta t_p^*$, and $0 < \Delta t_c < \Delta t_c^*$:

$$\begin{aligned}
&K_2 A_5 h_p^{-1-\frac{2}{q}-\frac{\delta}{2}}\Big(Q_5(\Delta t_c)^{3/2} + Q_6(\Delta t_c)^{1/2}\big((\Delta t_p)^2 + (\Delta t_p^1)^{3/2}\big)\Big)\\
&\qquad = O(h_p^{\frac{\delta}{2}} + h_p^{\frac{3\delta}{2}}) \le K_4,\\
&K_2 A_5 Q_6 (\Delta t_c)^{1/2} h_p^{2k+1-\frac{2}{q}-\frac{\delta}{2}} = O(h_p^{2k+1+\frac{3\delta}{2}}) \le K_4,\\
&K_2 A_5 Q_6 (\Delta t_c)^{1/2} h_p^{-1-\frac{2}{q}-\frac{\delta}{2}} h_c^{l+1} = O((\Delta t_c)^{1/2} h_p^{-\frac{2}{q}-\frac{\delta}{2}}) = O(h_p^{\frac{3\delta}{2}}) \le K_4, \qquad (55)\\
&A_1 K_2 A_5 \Delta t_c h_p^{-1-\frac{2}{q}-\frac{\delta}{2}} h_c^{l+1+(\frac{2}{q}-\frac{2}{2+\delta})} \|c\|_{L^\infty(W^{l+1}_{2+\delta})}\\
&\qquad = O(\Delta t_c h_p^{2\delta} h_c^{-\frac{1+5\delta}{2}}) = O(h_p^{2\delta}) \le K_4,\\
&K_4 \Delta t_c |\log h_p|^{\frac{1}{2}} h_p^{-\frac{\delta}{2}} = O(h_p^{1-\frac{3\delta}{2}} |\log h_p|^{\frac{1}{2}}) \le K_4.
\end{aligned}$$

We combine (54) and (55) to conclude that (45) holds for $j = m$.

6. Auxiliary Lemmas

We prove several lemmas that were used in the proof of Theorem 5.1.

Lemma 6.1. *Under the conditions of Theorem 5.1, the Jacobian matrix*

$$\frac{\partial \mathbf{y}(\bar z, \mathbf{x})}{\partial \mathbf{x}} = \mathbf{I} - \Big[\frac{\partial}{\partial \mathbf{x}}\Big(\frac{\mathbf{u}^E(\mathbf{x}, t_c^n)}{\phi(\mathbf{x})}\Big) + \bar z \frac{\partial}{\partial \mathbf{x}}\Big(\frac{\big(K_h * \mathbf{u}_h^E - \mathbf{u}^E\big)(\mathbf{x}, t_c^n)}{\phi(\mathbf{x})}\Big)\Big]\Delta t_c^n, \quad 0 \le \bar z \le 1,$$

of the transform

$$\begin{aligned}
\mathbf{y}(\bar z, \mathbf{x}) &:= (1-\bar z)\mathbf{x}^* + \bar z \mathbf{x}_h^*\\
&= \mathbf{x} - \Big[\frac{\mathbf{u}^E(\mathbf{x}, t_c^n)}{\phi(\mathbf{x})} + \bar z \frac{\big(K_h * \mathbf{u}_h^E - \mathbf{u}^E\big)(\mathbf{x}, t_c^n)}{\phi(\mathbf{x})}\Big]\Delta t_c^n, \quad 0 \le \bar z \le 1,
\end{aligned} \tag{56}$$

is bounded in the Frobenius norm $|\cdot|_2$ *by*

$$\Big|\frac{\partial \mathbf{y}(\bar z, \mathbf{x})}{\partial \mathbf{x}} - \mathbf{I}\Big|_2^2 = o(1), \quad 0 \le \bar z \le 1. \tag{57}$$

In addition,

$$\det\Big(\frac{\partial \mathbf{y}(\bar z, \mathbf{x})}{\partial \mathbf{x}}\Big) = 1 + o(1), \qquad 0 \le \bar z \le 1. \tag{58}$$

Finally, the following estimate holds for any $w(\mathbf{x}) \in H^1(\Omega)$*:*

$$\int_0^1 \int_\Omega \Big|\nabla w((1-\bar z)\mathbf{x}^* + \bar z \mathbf{x}_h^*, t_c^{n-1})\Big|^2 d\mathbf{x} ds \le Q \|\nabla w(\cdot, t_c^{n-1})\|^2_{L^2(\Omega)}. \tag{59}$$

Proof: We know $\partial(\mathbf{u}(\mathbf{x}, t_c^n))/\partial \mathbf{x}$ is bounded since $\mathbf{u} \in L^\infty(W^1_\infty)$. On the other hand, By (21), (6), (45) and the definition of convolution,

$$\|K_h * \mathbf{u}_h^{\mathrm{E}}\|_{W^1_\infty(\Omega)} \le Q \|\mathbf{u}_h^{\mathrm{E}}\|_{W^1_\infty(\Omega)} \le Q K_4 (\triangle t_c^n)^{-1} h_p^{\frac{\delta}{2}}$$

Thus, by condition (39), we have

$$\frac{\partial \mathbf{y}(\bar z, \mathbf{x})}{\partial \mathbf{x}} = (1 + Q\Delta t_c^n + Q K_4 h_p^{\frac{\delta}{2}})\mathbf{I} = (1 + o(1))\mathbf{I}. \tag{60}$$

The estimates (57) and (58) are consequences of (60).

The transform (56) is a diffeomorphism for a given smooth velocity $\mathbf{u}(\mathbf{x}, t)$ [3]. This proves (59) in the context of linear advection-diffusion PDEs [3, 29, 30, 31, 32, 33]. In the current context, $\mathbf{x}_h^*$ is determined by a numerical velocity $K_h * \mathbf{u}_h^{\mathrm{E}}$

obtained from an MFEM approximation and convolution operation. Consequently, the transform (56) is not one-to-one anymore [8]. In general, the transform could be infinitely many-to-one asymptotically. To prove (59) we let Ω_e^p run over all the elements in the pressure mesh

$$
\begin{aligned}
&\int_0^1 \int_\Omega \Big|\nabla w((1-\bar z)\mathbf{x}^* + \bar z \mathbf{x}_h^*, t_c^{n-1})\Big|^2 d\mathbf{x} d\bar z \\
&\quad = \int_0^1 \sum_{\Omega_e^p \subset \Omega} \int_{\mathbf{y}(\bar z, \Omega_e^p)} \Big|\nabla w(\mathbf{y}, t_c^{n-1})\Big|^2 \det\Big(\frac{\partial \mathbf{x}}{\partial \mathbf{y}}\Big) d\mathbf{y} d\bar z \\
&\quad \le 2\int_0^1 \sum_{\Omega_e^p \subset \Omega} \int_{\mathbf{y}(\bar z, \Omega_e^p)} \Big|\nabla w(\mathbf{y}, t_c^{n-1})\Big|^2 d\mathbf{y} d\bar z.
\end{aligned}
\tag{61}
$$

(58) shows that $\mathbf{y}(\bar z, \mathbf{x})$ is a one-to-one mapping on each pressure element Ω_e^p and that $\mathbf{y}(\bar z, \mathbf{x})$ maps each Ω_e^p into itself and its immediate-neighboring elements. This implies that the sum in (61) is bounded by finitely many multiples of $\|\nabla w(\mathbf{x}, t_c^n)\|^2_{L^2(\Omega)}$, with a repetition factor of the number of neighbors of an element Ω_e^p that is bounded since the partition is quasi-uniform.

Lemma 6.2. *Under the conditions of Theorem 5.1, estimate (49) holds for* $n = N_{m-1}+1, \ldots, N_m$.

Proof. It is straightforward to see that [9] $|\mathbf{D}(\mathbf{x},\mathbf{u}) - \mathbf{D}(\mathbf{x},\mathbf{v})| \le Q\,|\mathbf{u}-\mathbf{v}|$. With this we bound the left-hand side of (49) by

$$
\begin{aligned}
&\Big|\int_\Omega \nabla\xi(\mathbf{x}, t_c^n)\cdot(\mathbf{D}(\mathbf{x}, K_h * \mathbf{u}_h^E(\mathbf{x}, t_c^n)) - \mathbf{D}(\mathbf{x}, \mathbf{u}(\mathbf{x}, t_c^n)))\nabla\tilde c(\mathbf{x}, t_c^n) d\mathbf{x}\Big| \\
&\quad \le Q\|\nabla\xi(\cdot, t_c^n)\|\ \|K_h * \mathbf{u}_h^E(\cdot, t_c^n) - \mathbf{u}(\cdot, t_c^n)\|\ \|\tilde c(\cdot, t_c^n)\|_{W^1_\infty} \\
&\quad \le Q\|\nabla\xi(\cdot, t_c^n)\|\ \{\|K_h * \mathbf{u}_h^E(\cdot, t_c^n) - K_h * \tilde{\mathbf{u}}^E(\cdot, t_c^n)\| \\
&\qquad + \|K_h * \tilde{\mathbf{u}}^E(\cdot, t_c^n) - \mathbf{u}^E(\cdot, t_c^n)\| + \|\mathbf{u}^E(\cdot, t_c^n) - \mathbf{u}(\cdot, t_c^n)\|\},
\end{aligned}
\tag{62}
$$

where at the last "$\le$" sign we have used (18).

The last term in the bracket is bounded in (23). We use the estimate (32) to bound the second term in the bracket to get

$$
\begin{aligned}
&\|K_h * \tilde{\mathbf{u}}^E(\cdot, t_c^n) - \mathbf{u}^E(\cdot, t_c^n)\| \\
&\quad \le Q(\|K_h * \tilde{\mathbf{u}}(\cdot, t_p^{m-2}) - \mathbf{u}(\cdot, t_p^{m-2})\| + \|K_h * \tilde{\mathbf{u}}(\cdot, t_p^{m-1}) - \mathbf{u}(\cdot, t_p^{m-1})\|) \\
&\quad \le Q h_p^{2k+4}(\|\mathbf{u}\|_{L^\infty(H^{k+1}(div))} + \|p\|_{L^\infty(H^{2k+4})}).
\end{aligned}
$$

We use (17) and (38) and (41) to bound the first term in the bracket of (62) by

$$
\begin{aligned}
&\|K_h * \mathbf{u}_h^E(\cdot, t_c^n) - K_h * \tilde{\mathbf{u}}^E(\cdot, t_c^n)\| \\
&\quad \le Q\ (\|K_h * \mathbf{u}_h(\cdot, t_p^{m-2}) - K_h * \mathbf{u}(\cdot, t_p^{m-2})\| + \|K_h * \mathbf{u}_h(\cdot, t_p^{m-1}) - K_h * \tilde{\mathbf{u}}(\cdot, t_p^{m-1})\|) \\
&\quad \le Q\ (\|c_h(\cdot, t_p^{m-2}) - c(\cdot, t_p^{m-2})\| + \|c_h(\cdot, t_p^{m-1}) - c(\cdot, t_p^{m-1})\|) \\
&\quad \le Q\ (\|\xi(\cdot, t_p^{m-2})\| + \|\xi(\cdot, t_p^{m-1})\|) + Q h_c^{l+1}\|c\|_{L^\infty(H^{l+1})}.
\end{aligned}
$$

We combine these two estimates with (23) to complete the proof.

Lemma 6.3. *Under the conditions of Theorem 5.1, estimate (50) holds for* $n = N_{m-1}+1, \ldots, N_m$.

Proof. First we rewrite the last two terms on the right side of (47) as follows:

$$
\begin{aligned}
&\int_\Omega \phi(\mathbf{x})\big[c_h(\mathbf{x}_h^*, t_c^{n-1}) - c(\mathbf{x}^*, t_c^{n-1})\big]\xi(\mathbf{x}, t_c^n)d\mathbf{x} \\
&\quad - \int_\Omega \phi(\mathbf{x})\big[c_h(\mathbf{x}, t_c^{n-1}) - c(\mathbf{x}, t_c^{n-1})\big]\xi(\mathbf{x}, t_c^n)d\mathbf{x} \\
&= \int_\Omega \phi(\mathbf{x})\big[\tilde{c}(\mathbf{x}_h^*, t_c^{n-1}) - \tilde{c}(\mathbf{x}^*, t_c^{n-1})\big]\xi(\mathbf{x}, t_c^n)d\mathbf{x} \\
&\quad + \int_\Omega \phi\big[\xi(\mathbf{x}_h^*, t_c^{n-1}) - \xi(\mathbf{x}^*, t_c^{n-1})\big]\xi(\mathbf{x}, t_c^n)d\mathbf{x} \\
&\quad + \int_\Omega \phi\big[\xi(\mathbf{x}^*, t_c^{n-1}) - \xi(\mathbf{x}, t_c^{n-1})\big]\xi(\mathbf{x}, t_c^n)d\mathbf{x} \\
&\quad + \int_\Omega \phi\big[\eta(\mathbf{x}^*, t_c^{n-1}) - \eta(\mathbf{x}, t_c^{n-1})\big]\xi(\mathbf{x}, t_c^n)d\mathbf{x}.
\end{aligned} \tag{63}
$$

The first term on the right side of (63) is rewritten as:

$$
\begin{aligned}
&\int_\Omega \phi(\mathbf{x})\big[\tilde{c}(\mathbf{x}_h^*, t_c^{n-1}) - \tilde{c}(\mathbf{x}^*, t_c^{n-1})\big]\xi(\mathbf{x}, t_c^n)d\mathbf{x} \\
&= \int_\Omega \phi(\mathbf{x})\Big[\int_0^1 \nabla\tilde{c}((1-\bar{z})\mathbf{x}^* + \bar{z}\mathbf{x}_h^*, t_c^{n-1})d\bar{z}\Big]\cdot(\mathbf{x}_h^* - \mathbf{x}^*)\xi(\mathbf{x}, t_c^n)d\mathbf{x} \\
&= \Delta t_c^n \int_\Omega \phi(\mathbf{x})\Big[\int_0^1 \nabla\tilde{c}((1-\bar{z})\mathbf{x}^* + \bar{z}\mathbf{x}_h^*, t_c^{n-1})d\bar{z}\Big] \\
&\qquad \cdot(K_h * \mathbf{u}_h^E - \mathbf{u}^E)(\mathbf{x}, t_c^n)\xi(\mathbf{x}, t_c^n)d\mathbf{x}.
\end{aligned} \tag{64}
$$

Note that

$$
\|g_{\tilde{c}}\|_{L^\infty} = \Big\|\int_0^1 \nabla\tilde{c}((1-\bar{z})\mathbf{x}^* + \bar{z}\mathbf{x}_h^*, t_c^{n-1})d\bar{z}\Big\|_{L^\infty} \le \|\tilde{c}(\cdot, t_c^{n-1})\|_{W_\infty^1}, \tag{65}
$$

the first term in (63) leads to

$$
\begin{aligned}
&\Big|\int_\Omega \phi\big[\tilde{c}(\mathbf{x}_h^*, t_c^{n-1}) - \tilde{c}(\mathbf{x}^*, t_c^{n-1})\big]\xi(\mathbf{x}, t_c^n)d\mathbf{x}\Big| \\
&\le \Delta t_c^n \big\|g_{\tilde{c}}\big\|_{L^\infty}\big\|(\mathbf{u}^E - K_h * \mathbf{u}_h^E)(\cdot, t_c^n)\big\|\big\|\xi(\cdot, t_c^n)\big\| \\
&\le Q\Delta t_c^n \big\|(\mathbf{u}^E - K_h * \mathbf{u}_h^E)(\cdot, t_c^n)\big\|\big\|\xi(\cdot, t_c^n)\big\|.
\end{aligned} \tag{66}
$$

In Lemma 6.2 we showed that

$$
\begin{aligned}
&\big\|(\mathbf{u}^E - K_h * \mathbf{u}_h^E)(\cdot, t_c^n)\big\|^2 \le Q(\|\xi(\cdot, t_p^{m-1})\|^2 + \|\xi(\cdot, t_p^{m-2})\|^2) \\
&\quad + Q\Big(h_c^{2l+2}\|c\|^2_{L^\infty(H^{l+1})} + h_p^{4k+4}(\|\mathbf{u}\|^2_{L^\infty(H^{k+1}(div))} + \|p\|^2_{L^\infty(H^{2k+4})})\Big).
\end{aligned} \tag{67}
$$

We combine (66) and (67) to bound the first term in (63) as follows:

$$
\begin{aligned}
&\Big|\int_\Omega \phi(\mathbf{x})\big[\tilde{c}(\mathbf{x}_h^*, t_c^{n-1}) - \tilde{c}(\mathbf{x}^*, t_c^{n-1})\big]\xi(\mathbf{x}, t_c^n)d\mathbf{x}\Big| \\
&\quad \le Q\Delta t_c^n(\|\xi(\cdot, t_p^{m-1})\|^2 + \|\xi(\cdot, t_p^{m-2})\|^2 + \|\xi(\cdot, t_n^c)\|^2) \\
&\qquad + Q\Delta t_c^n\big(h_c^{2l+2}\|c\|^2_{L^\infty(H^{l+1})} + h_p^{4k+4}(\|\mathbf{u}\|^2_{L^\infty(H^{k+1}(div))} + \|p\|^2_{L^\infty(H^{2k+4})})\big).
\end{aligned}
$$

Similar to (64), the second term on the right side of (63) can be rewritten as

$$
\begin{aligned}
&\int_\Omega \phi(\mathbf{x})\big[\xi(\mathbf{x}_h^*, t_c^{n-1}) - \xi(\mathbf{x}^*, t_c^{n-1})\big]\xi(\mathbf{x}, t_c^n)d\mathbf{x} \\
&\quad = \Delta t_c^n \int_\Omega g_\xi(\mathbf{x})\cdot(K_h * \mathbf{u}_h^E - \mathbf{u}^E)(\mathbf{x}, t_c^n)\xi(\mathbf{x}, t_c^n)d\mathbf{x}.
\end{aligned} \tag{68}
$$

However, since $\|g_\xi\|_{W_\infty^1}$ is not uniformly bounded, we cannot treat this term in the same way to (66). From (67), it is clear that $\big\|(\mathbf{u}^E - K_h * \mathbf{u}_h^E)(\cdot, t_c^n)\big\| =$

$o(|\log h_c|^{-1/2})$, since our theorem will prove that $\|\xi(\cdot,t_p^{m-i})\| = O(h_c^{l+1} + h_p^{2k+2} + \Delta t_c + (\Delta t_p^1)^{3/2} + (\Delta t_p)^2)$. Thus we apply Lemma 6.1 and (13) to (68) to obtain

$$\begin{aligned}
&\Big|\int_\Omega \phi(\mathbf{x})\big[\xi(\mathbf{x}_h^*,t_c^{n-1}) - \xi(\mathbf{x}^*,t_c^{n-1})\big]\xi(\mathbf{x},t_c^n)d\mathbf{x}\Big| \\
&\le Q\Delta t_c^n \big\|g_\xi\big\|\big\|(\mathbf{u}^E - K_h * \mathbf{u}_h^E)(\cdot,t_c^n)\big\|\big\|\xi(\cdot,t_c^n)\big\|_{L^\infty} \\
&\le Q\Delta t_c^n \big\|(\mathbf{u}^E - K_h * \mathbf{u}_h^E)(\cdot,t_c^n)\big\||\log h_c|^{1/2})\|\nabla\xi(\cdot,t_c^{n-1})\|\|\xi(\cdot,t_n^c)\|_{H^1} \\
&\le \varepsilon\Delta t_c^n(\|\nabla\xi(\cdot,t_c^{n-1})\|^2 + \|\xi(\cdot,t_n^c)\|_{H^1}^2).
\end{aligned}$$

Following the proof of Lemma 6.1, we obtain the same results if the transform is replaced by $\mathbf{y}(\bar{z},\mathbf{x}) = \mathbf{x} + \bar{z}(\mathbf{x}^* - \mathbf{x})$. Then we bound the third term on the right side of (63) by

$$\begin{aligned}
&\Big|\int_\Omega \phi\big[\xi(\mathbf{x}^*,t_c^{n-1}) - \xi(\mathbf{x},t_c^{n-1})\big]\xi(\mathbf{x},t_n^c)d\mathbf{x}\Big| \\
&= \Big|\int_\Omega \phi \int_0^1 \nabla\xi(\mathbf{x} + s(\mathbf{x}^* - \mathbf{x}),t_c^{n-1})ds \cdot (\mathbf{x}^* - \mathbf{x})\xi(\mathbf{x},t_n^c)d\mathbf{x}\Big| \\
&\le Q\Delta t_c^n\|\nabla\xi(\cdot,t_c^{n-1})\| \ \|\xi(\cdot,t_n^c)\| \\
&\le Q\Delta t_c^n\|\xi(\cdot,t_n^c)\|^2 + \varepsilon\Delta t_c^n\|\nabla\xi(\cdot,t_c^{n-1})\|^2.
\end{aligned}$$

We similarly bound the fourth term on the right side of (63) by (17)

$$\begin{aligned}
&\int_\Omega \phi(\mathbf{x})\big[\eta(\mathbf{x}^*,t_c^{n-1}) - \eta(\mathbf{x},t_c^{n-1})\big]\xi(\mathbf{x},t_n^c)d\mathbf{x} \\
&= \int_\Omega \phi(\mathbf{x})\eta(\mathbf{x},t_c^{n-1})\xi(\tilde{\mathbf{x}},t_n^c)d\tilde{\mathbf{x}} - \int_\Omega \phi(\mathbf{x})\eta(\mathbf{x},t_c^{n-1})\xi(\mathbf{x},t_n^c)d\mathbf{x} \\
&\le \int_\Omega \phi(\mathbf{x})|\eta(\mathbf{x},t_c^{n-1})(\xi(\tilde{\mathbf{x}},t_n^c) - \xi(\mathbf{x},t_n^c))|d\mathbf{x} + Q\Delta t_c^n \int_\Omega \phi(\mathbf{x})|\eta(\mathbf{x},t_c^{n-1})\xi(\tilde{\mathbf{x}},t_n^c)|d\mathbf{x} \\
&\le Q\Delta t_c^n \int_\Omega \phi(\mathbf{x})|\eta(\mathbf{x},t_c^{n-1})| \int_0^1 |\nabla\xi(\mathbf{x} + s(\tilde{\mathbf{x}} - \mathbf{x}),t_c^{n-1})|dsd\mathbf{x} \\
&\quad + Q\Delta t_c^n \int_\Omega \phi(\mathbf{x})|\eta(\mathbf{x},t_c^{n-1})\xi(\tilde{\mathbf{x}},t_n^c)|d\mathbf{x} \\
&\le Q\Delta t_c^n h_c^{2l+2}\|c\|_{L^\infty(H^{l+1})}^2 + \varepsilon\Delta t_c^n\|\nabla\xi(\cdot,t_c^{n-1})\|^2 + Q\Delta t_c^n\|\xi(\cdot,t_n^c)\|^2.
\end{aligned}$$

Combining the preceding estimates finishes the proof.

7. Concluding Remarks and Future Work

In this paper we proposed an MMOC-MFEM time stepping procedure based on the Darcy velocity processed by convolution with Bramble-Schatz kernel functions for miscible displacement processes in porous medium flow. The convergence rate proved above is $O(h_c^{l+1}+h_p^{2k+2})$, which reflects that the superconvergence of velocity approximation is retained to the concentration approximation. This is an extension of the result for time-continuous case in [18] to time-stepping case.

An error estimate similar to Theorem 5.1 was proved in [9] for a Galerkin FEM-MFEM time-stepping procedure for problem (1)–(3) and in [8] for an MMOC-MFEM time-stepping procedure. These estimates require a restrictive condition that

$$\Delta t_c = o(h_p). \tag{69}$$

In other words, these procedures are guaranteed to converge only if the Courant number tends to zero asymptotically, which is more restrictive than the CFL condition for an explicit scheme in the context of a strongly advection-dominated displacement process [10].

In Theorem 5.1 the restriction (69) is relaxed to be

$$\Delta t_c = O(h_p^{1/2+3\delta}). \tag{70}$$

This implies that the MMOC-MFEM time-stepping procedure converges for any size of Courant numbers. This is especially important for the MMOC-MFEM time-stepping procedure, since the strength of the MMOC scheme is really reflected in the large time steps allowed.

Recently, some novel uniform estimates were established for convection-diffusion equations[34, 35, 36, 37, 38, 39, 40, 41], we will follow the ideas there to conduct valuable error analysis for coupled problems.

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