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A GALERKIN/LEAST-SQUARE FINITE ELEMENT APPROXIMATION OF BRANCHES OF NONSINGULAR SOLUTIONS OF THE STATIONARY NAVIER-STOKES EQUATIONS *1)

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Abstract

In the author's previous paper [13], a Galerkin/Least-Square type finite element method was proposed and analyzed for the stationary N-S equations. The method is consistent and stable for any combination of discrete velocity and pressure spaces(without requiring the Babuska-Brezzi stability condition). Under the condition that the solution of N-S equations is unique (i.e. in the case of sufficient viscosity or small data), the existence, uniqueness and convergence (at optimal rate) of discrete solution were proved. In this paper, we further investigate the established Galerkin/Least-Square finite element method for the stationary N-S equations. By applying and extending the results of Lopez-Marcos & Sanz-Serna [15], an existence theorem and error estimates are proved in the case of branches of nonsingular solutions.

1. Introduction

For mixed finite element methods solving the stationary (Navier-) Stokes equations in the primitive variables, it is an important convergence stability condition that the Babuska-Brezzi inequality (or inf-sup, or LBB condition) holds for the combination of finite element subspaces[1]. Employment of combinations which fail to satisfy the compatibility condition may yield undesirable pathologies in the approximation of pressure and velocity. Recently, to deal with this potential shortcoming, the so-called CBB[6] or stabilized finite element methods[4], which circumvents the need to satisfy the LBB condition by modifying the variational equations carefully, have been developed under the motivation of SD (or SUPG) methods[7,8]. In addition to works [2-6] on the Stokes problems, a penalty SD type method and a Galerkin/Least-Square method have already been proposed for the stationary Navier-Stokes equations in [11] and [13], respectively. The two methods are stable and different from the method in [12]. So far they have only been analyzed under the condition of unique solution (i.e. in the case of sufficient

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viscosity or small data). Although G. Lube[19] extended the analysis of SD method to quasilinear elliptic problems of second order in the case of branches of nonsingualr solutions following the abstract approach in [1,10], it seems difficult to apply his method to analyze the two stabilized methods[11,13] for the stationary Naver-Stokes equations in the case of high Reynolds number.

In this paper, we further investigate the Galerkin/Least-Square method established in [13] for the stationary N-S equations. By applying and extending the abstract results of Lopez-Marcos & Sanz-Serna[15], the existence, uniqueness and error estimates are proved in the case of branches of nonsingular solutions. It is worth mentioning that the penalty SD type method[11] can be similarly analyzed in the case of branches of nonsingular solutions by our discussions.

For SD method applied to the nonstationary Navier-Stokes equations we referred to papers [8,9,18].

An outline of the paper follows. In Section 2, we introduce some notations which are important for the following presentation. In Section 3, we present the Galerkin/ Least-Square method established in [13]. The main result of existence and convergence of branches of discrete solutions contained in Section 4.

2. Notations and Preliminaries

Throughout this paper, Ω is supposed to be a bounded domain in $R^n, n = 2$ or 3, with a Lipschitz continuous boundary Γ. For a scalar function w on a measurable subset $G \subseteq \Omega$, let $\| w \|_{k,p,G}$ and $|w|_{k,p,G}$ be the usual norm and seminorm on the Sobolev space $W^{k,p}(G)$, respectively. For vectorvalued functions $u = (u_1, \cdots, u_n) \in W^{k,p}(G)^n$ and $v = (v_1, \cdots, v_n) \in L^\infty(G)$ we use the following norms and seminorms, respectively.

$$\|u\|^p_{k,p,G} = \sum_{i=1}^n \|u_i\|^p_{k,p,G}, |u|^p_{k,p,G} = \sum_{i=1}^n |u_i|^p_{k,p,G},$$
$$\|v\|_{0,\infty,G} = \max_i \|v_i\|_{0,\infty,G}.$$

$(\cdot,\cdot)_G$ denotes the inner product in $L^2(G)$ and $L^2(G)^n, G \subseteq \Omega$ respectively. In the case of $G = \Omega$ and $p = 2$ we omit the index G and p. Henceforth, we denote by C a generic constant independent of h. Other notations without being specially explained are used in the usual meaning.

In this paper, we consider the following stationary Navier-Stokes equations with boundary conditions.

$$\begin{cases} -\nu\Delta u + u \cdot \nabla u + \nabla p &= f \text{ in } \Omega., \\ \mathrm{div} u &= 0 \text{ in } \Omega, \\ u|_{\partial\Omega} &= 0, \end{cases} \tag{2.1}$$

where $u = (u_1, \cdots, u_n)$ is velocity vector, p the pressure, $f = (f_1, \cdots, f_n)$ the body force, ν the constant inverse Reynolds number. Problem (2.1) is equivalent to the following variational problem:

($\mathcal{N}$) Find $\hat{u} = (u,p) \in X$ such that for all $\hat{w} = (w,r) \in X$,

$$A(\hat{u},\hat{w}) \equiv \nu a(u,w) + b(u;u,m) - (p,\text{div}w) + (r,\text{div}u) - (f,w) = 0,$$

where

$$a(u,w) = (\nabla u, \nabla w).$$

$$b(u;v,w) = \frac{1}{2}\{(u\cdot\nabla v, w) - (u\cdot\nabla w, v)\} \qquad \forall u,v,w \in V,$$

$$V = H_0^1(\Omega)^n, Q = L_0^2(\Omega); X = V \times Q.$$

Let $\lambda \equiv \frac{1}{\nu}$, the operator $\Phi_\lambda \in (X \to X')$ associated with ($\mathcal{N}$) by

$$< \Phi_\lambda(\hat{u}), \hat{w} >\equiv A\ (\hat{u},\hat{w})$$

is differentiable in sense of Frechet:

$$< \Phi'_\lambda(\hat{u})\hat{v}, \hat{w} >\equiv \nu a(v,w) + b(v;u,w) + b(u;v,w) - (q,\text{div}w) + (r,\text{div}v) \qquad (2.2),$$

where $< \cdot,\cdot >$ denotes the scalar product between X and the corresponding dual space.

Let $\Lambda \subset (0,\infty)$ be a compact interval, $\lambda \equiv \frac{1}{\nu} \in \Lambda$. In this paper, we concern with the approximation of a continuous branch $\lambda \to \hat{u}_\lambda$, of ($\mathcal{N}$), which are nonsingular solutions in the sense that if there exists $\hat{v} = (v,q) \in X$ such that for any $\hat{w} = (w,r) \in X$

$$< \Phi'_\lambda(\hat{u}_\lambda)\hat{v}, \hat{w} >= 0 \qquad (2.3)$$

then $\hat{v} \equiv 0$.

The main assumptions on ($\mathcal{N}$) are:

(A_1): There exists a branch $\{(\lambda,\hat{u}_\lambda) : \lambda \in \Lambda, \hat{u}_\lambda = (u_\lambda,p_\lambda) \in X\}$ of nonsingular solutions of ($\mathcal{N}$).

(A_2): $\forall \lambda \in \Lambda : |u_\lambda|_1 + \|u_\lambda\|_{0,\infty} + \|p_\lambda\|_0 \le C$.

3. Galerkin/Least-Square Finite Element Formulation

Let $\mathcal{T}_h = \{K\}$ be a finite element partition of Ω with $\Omega = \bigcup_{K\in\mathcal{T}_h} K$, which is assumed to be regular in the usual sense; and let $h_K = \text{diam}K$. We also assume that

$$h/h_K < C, \quad \forall K \in \mathcal{T}_h, \quad h = \max_K h_K,$$

so that we can use inverse inequalities.

We introduce the following finite element spaces of velocity and pressure.

$$V_h(\Omega) = \{v \in H_0^1(\Omega)^n : v|_K \in R_m(K), \forall K \in \mathcal{T}_h\},$$

$$Q_h(\Omega) = \{q \in L_0^2(\Omega) \cap H^1(\Omega) : q|_K \in R_t(K), \forall K \in \mathcal{T}_h\},$$

where

$$R_m(K) = \begin{cases} P_m(K), & \text{if } K \text{ is a triangle or tetrahedron} \\ Q_m(K), & \text{if } K \text{ is a quadrilateral or hexahedron} \end{cases}$$

for each integer $m \geq 0, P_m$ and Q_m have the usual meaning.

Let $X_h \equiv V_h \times Q_h$, the Galerkin/Least-Square finite element approximation established in [13] for the stationary N-S equations (2.1) can be formulated as follows:

$(\mathcal{N}_h)$ Find $\hat{u}_h = (u_h, p_h) \in X_h$ such that for all $\hat{w}_h = (w_h, r_h) \in X_h$,

$$B_\delta(u_h, u_h; \hat{u}_h, \hat{w}_h) = 0,$$

where

$$\begin{aligned} B_\delta(u, u_h; \hat{v}, \hat{w}) = \ & \nu a(v, w) + b(u; v, w) - (q, \mathrm{div} w) + (r, \mathrm{div} v) \\ & + \textstyle\sum_{K \in \mathcal{T}_h} \delta_K(-\nu\Delta v + u\nabla v + \nabla q, -\nu\Delta w + u_h\nabla w + \nabla r)_K \\ & -[(f, w) + \textstyle\sum_{K \in \mathcal{T}_h} \delta_K(f, -\nu\Delta w + u_h\nabla w + \nabla r)_K], \end{aligned}$$

for $u \in V, u_h \in V_h, \hat{v} \equiv (v, q), \hat{w} \equiv (w, r) \in V \times (Q \cap H^1(\Omega))$. $\delta_K = ah_K^2, a > 0$ is arbitrary, δ is the piecewise constant function defined by $\delta|_K = \delta_K$.

Remark 1. Assume f belongs to $L^2(\Omega)^n$ and the solution $\hat{u} \equiv (u, p)$ of (2.1) belongs to $(V \cap H^2(\Omega)^n) \times (Q \cap H^1(\Omega))$, i.e.

$$-\nu\Delta u + u\nabla u + \nabla p = f, \quad \text{in} \quad L^2(\Omega)^n$$

holds, then $\hat{u}$ satisfies

$$B_\delta(u, v_h; \hat{u}, \hat{w}_h) = 0, \quad \forall v_h \in V_h, \hat{w}_h \in X_h. \tag{3.1}$$

For a given $\hat{v}_h \equiv (v_h, q_h)$, it is obvious that $B_\delta(v_h, v_h; \hat{v}_h, \cdot)$ is a linear continuous function on X_h, so there exists a nonlinear operator $\Phi_{\lambda,h} : X_h \to X_h'$ with

$$< \Phi_{\lambda,h}(\hat{v}_h), \hat{w}_h > \equiv B_\delta(v_h, v_h; \hat{v}_h, \hat{w}_h), \quad \forall \hat{v}_h, \hat{w}_h \in X_h. \tag{3.2}$$

Therefore, the problem $(\mathcal{N}_h)$ can be rewritten as follows:

$(\mathcal{N}_h)'$ Find $\hat{u}_h \in X_h$ such that for all $\hat{w}_h \in X_h$,

$$< \Phi_{\lambda,h}(\hat{u}_h), \hat{w}_h >= 0.$$

It means that

$$\Phi_{\lambda,h}(\hat{u}_h) = 0 \quad \text{in} \quad X_h. \tag{3.3}$$

4. Existence and Convergence of Finite Element Solutions

The main result of this paper is the following existence and convergence Theorem 4.0 for the G/L-S finite element method $(\mathcal{N}_h)$(or $(\mathcal{N}_h)'$).

Theorem 4.0. *Let*

$$\{(\lambda, \hat{u}_\lambda) : \lambda \in \Lambda, \hat{u}_\lambda \equiv (u_\lambda, p_\lambda) \in (V \cap H^2(\Omega)^n) \times (Q \cap H^1(\Omega))\}$$

be a branch of nonsingular solutions of $(\mathcal{N})$*, then there exist two constant* $h_0 > 0$ *and* $\rho_0 > 0$ *such that for all* $h \in (0, h_0]$ *and* $\lambda \in \Lambda$*, problem* $(\mathcal{N}_h)$ *have a unique solution* $\hat{u}_{\lambda,h} \equiv (u_{\lambda,h}, p_{\lambda,h}) \in X_h$ *in the ball*

$$B(\rho_0) = \{\hat{v} \equiv (v, q) \in X : |v - u_\lambda|_1 + \|q - p_\lambda\|_0 < \rho_0\},$$

which converges to $\hat{u}_\lambda$ *as* $h \to 0$. *Moreover, if*

$$\hat{u}_\lambda \in (V \cap H^{m+1}(\Omega)^n) \times (Q \cap H^l(\Omega))$$

holds for all $\lambda \in \Lambda$, *then we have*

$$|u_\lambda - u_{\lambda,h}|_1 + \|p_\lambda - p_{\lambda,h}\|_0 \le C(\Lambda)(h^m + h^l), \tag{4.1}$$

where the constant $C(\Lambda)$ *depends on the seminorm* $|u_\lambda|_{m+1}, |p_\lambda|_t$, *and the parameter* a.

For the proof of Theorem 4.0, we shall apply and extend the abstract results of Lopez-Marcos & Sanz-Serna[15].

Firstly, let us define norms on X and X'_h:

$$\|\hat{v}\|_X = (|v|_1^2 + \|q\|_0^2)^{1/2}, \quad \forall \hat{v} \equiv (v, q) \in X, \tag{4.2a}$$

$$\|g\|_{X'_h} = \sup_{\hat{v}_h \in X_h} \frac{< g, \hat{v}_h >}{\|\hat{v}_h\|_X}, \quad \forall g \in X'_h. \tag{4.2b}$$

Secondly, let $I_h = (I_h^1, I_h^2) : V \times Q \to V_h \times Q_h$ be the usual Lagrangian interpolation operator[17]. For a discretization of Navier-Stokes equations (2.1) (or ($\mathcal{N}$)):

Find $\hat{u}_h \in X$ such that

$$\Phi_{\lambda,h}(\hat{u}_h) = 0, \tag{4.3}$$

we introduce the concepts of convergence, consistent, and nonlinear stability in the sense of Lopez-Marcos & Sanz-Serna[15].

If $\hat{u} = (u, p)$ and $\hat{u}_h = (u_h, q_h)$ are solutions of (2.1) and (4.3) respectively, then the element $\hat{e}_h \equiv I_h\hat{u} - \hat{u}_h \in X_h$ is, by definition, the global error in $\hat{u}_h$. We say that the discretization (4.3) is convergent if there exists $h_1 > 0$ such that, for each h with $0 < h \le h_1$, (4.3) possesses a solution $\hat{u}_h$ and

$$\lim_{h\to 0} \|I_h\hat{u} - \hat{u}_h\|_X = \lim_{h\to 0} \|\hat{e}_h\|_X = 0 \ .$$

If, furthermore, $\|\hat{e}_h\|_X = 0(h^p)$ as $h \to 0$, then the convergence is said to be of order p.

The local (discretization) error in $I_h\hat{u}$ is defined to be element

$$l_h = \Phi_{\lambda,h}(I_h\hat{u}) \in X'_h.$$

The discretization (4.3) is said to be consistent (resp. consistent of order p) if, as $h \to 0$, we have $\|l_h\|_{X'_h} \to 0$ (resp. $\|l_h\|_{X'_h} = 0(h^p)$).

We also need the concept of nonlinear stability: the one we shall use is that given by Lopez-Marcos & Sanz-Serna[15]. Which is an extension of earlier definition due to Keller[16].

Definition 4.1. *Suppose that, for each* h *in a set* H *of positive number with* $\inf H = 0$, *we define* $R_h \in (0, \infty]$. *Then the discretization* (4.3) *is said to be stable, restricted to the thresholds* R_h, *if there exist positive constants* h_1 *and* S *(the stability constant) such that, for* h *in* H *with* $h \le h_1$, *the open ball* $B(I_h\hat{u}, R_h)$ *is contained in the domain* X_h, *and such that, for any* $\hat{v}_h$ *and* $\hat{w}_h$ *in the ball*

$$\|\hat{v}_h - \hat{w}_h\|_X \le S\|\Phi_{\lambda,h}(\hat{v}_h) - \Phi_{\lambda,h}(\hat{w}_h)\|_{X'_h} \ . \tag{4.4}$$

With these definitions the following convergence result is derived by Lopez-Marcos & Sanz-Serna[15] abstract approach.

Theorem 4.1. *Assume that the discretization* (4.3) *is consistent and stable with threshold R_h. If $\Phi_{\lambda,h}$ is continuous in $B(I_h\hat{u}, R_h)$ and $\|l_h\|_{X_h'} = o(R_h)$ as $h \to 0$, then* (i) *for h small enough the discrete equations* (4.3) *possess a (convergent) solution in $B(I_h\hat{u}, R_h)$.* (ii) *That solution is unique in the ball.* (iii) *As $h \to 0$ the solution converges to $\hat{u}$. The order of convergence is no smaller than the order of consistency.*

Proof. See Lopez-Marcos & Sanz-Serna[15].

We shall use the above definitions to prove the convergence of problem $(\mathcal{N}_h)$ (or $(\mathcal{N}_h)'$). We have the following result.

Theorem 4.2. *For a given $\lambda \in \Lambda$, let*

$$\hat{u}_\lambda = (u_\lambda, p_\lambda) \in (V \cap H^2(\Omega)) \times (Q \cap H^1(\Omega))$$

be a nonsingular solution of problem $(\mathcal{N})$, then there exist two constants $h_0(\lambda) > 0$ and $\rho(\lambda) > 0$ such that for all $h \in (0, h_0(\lambda)]$ problem $(\mathcal{N}_h)$ (or $(\mathcal{N}_h)'$) has a unique solution $\hat{u}_{\lambda,h} \equiv (u_{\lambda,h}, p_{\lambda.h}) \in X_h$ in the ball $B(I_h\hat{u}_\lambda, \rho(\lambda))$ which converges to $\hat{u}_\lambda$ as $h \to 0$. Moreover, if

$$\hat{u}_\lambda = (u_\lambda, p_\lambda) \in (V \cap H^{m+1}(\Omega)^n) \times (Q \cap H^l(\Omega)) \ ,$$

then we have the error estimates:

$$|u_\lambda - u_{\lambda,h}|_1 + \|p_\lambda - p_{\lambda,h}\|_0 \le C(\lambda)(h^m + h^l) \ , \tag{4.5}$$

where the constant $C(\lambda)$ depends on the seminorm $|u_\lambda|_{m+1}, |p_\lambda|_l$ and the parameters λ, a.

We shall prove this theorem by a sequence of Lemmas which prove the consistency and stability of the Galerkin/Least-Square finite element approximation $(\mathcal{N}_h)$ (or $(\mathcal{N}_h)'$).

Since Λ is compact, Theorem 4.0 can be easily proved by Theorem 4.2. We need only to prove Theorem 4.2.

Lemma 4.1. *With $\Phi_{\lambda,h}$ defined by* (3.2) *the inequality*

$$\begin{aligned}\|\Phi_{\lambda,h}(I_h u_\lambda)\|_{X_h'} \le \ & C(\lambda)(|I_h^1 u_\lambda - u_\lambda|_1^2 + \|I_h^2 p_\lambda - p_\lambda\|_0^2 \\ & + \|\delta^{1/2}[-\nu\Delta(I_h^1 u_\lambda - u_\lambda) + \nabla(I_h^2 p_\lambda - p_\lambda)]\|_{0,h}^2)^{1/2}\end{aligned}$$

holds. Hereafter $\|\cdot\|_{0,h} = (\sum_{K\in\mathcal{T}_h} \|\cdot\|_{0,K}^2)^{1/2}, C(\lambda)$ is a constant dependent on λ, but independent of h.

Proof. From (3.1) and (3.2) we have that

$$\begin{aligned}< \Phi_{\lambda,h}(I_h\hat{u}_\lambda), \hat{w}_h > =& B_\delta(I_h^1 u_\lambda, I_h^1 u_\lambda; I_h\hat{u}_\lambda, \hat{w}_h) \\ =& B_\delta(I_h^1 u_\lambda, I_h^1 u_\lambda; I_h\hat{u}_\lambda, \hat{w}_h) - B_\delta(u_\lambda, I_h^1 u_\lambda; \hat{u}_\lambda, \hat{w}_h) \\ =& [\nu a(I_h^1 u_\lambda - u_\lambda, w_h) - (I_h^2 p_\lambda - p_\lambda, \mathrm{div} w_h)] \\ & + [b(u_\lambda; I_h^1 u_\lambda - u_\lambda, w_h) + b(I_h^1 u_\lambda - u_\lambda; I_h^1 u_\lambda, w_h)\end{aligned}$$

$$\begin{aligned}
&+ (r_h, \operatorname{div}(I_h^1 u_\lambda - u_\lambda))] + \sum_{K\in\mathcal{T}_h} \delta_K(-\nu\Delta(I_h^1 u_\lambda - u_\lambda) \\
&+ (I_h^1 u_\lambda - u_\lambda)\nabla I_h^1 u_\lambda + u_\lambda \nabla (I_h^1 u_\lambda - u_\lambda) \\
&+ \nabla(I_h^2 p_\lambda - p_\lambda), -\nu\Delta w_h + I_h^1 u_\lambda \nabla w_h + \nabla r_h)_K \\
=&S_1 + S_2 + S_3 , \qquad \forall \hat{w}_h = (w_h, r_h) \in X_h
\end{aligned} \tag{4.6}$$

By Cauchy-Schwartz inequality we have

$$|S_1| \le C(\nu+1)(|I_h^1 u_\lambda - u_\lambda)_1 + \|I_h^2 p_\lambda - p_\lambda\|_0)|w_h|_1 , \tag{4.7}$$

$$|S_2| \le C(1 + |u_\lambda|_1 + |I_h^1 u_\lambda|_1)|I_h^1 u_\lambda - u_\lambda|_1(|w_h|_1 + \|r_h\|_0) , \tag{4.8}$$

$$\begin{aligned}
|S_3| \le\ & \delta_M^{1/2}\|\delta^{1/2}[-\nu\Delta(I_h^1 u_\lambda - u_\lambda) + (I_h^1 u_\lambda - u_\lambda)\nabla I_h^1 u_\lambda \\
& + u_\lambda\nabla(I_h^1 u_\lambda - u_\lambda) + \nabla(I_h^2 p_\lambda - p_\lambda)]\|_{0,h} \\
& \cdot\| - \nu\Delta w_h + I_h^1 u_\lambda \nabla w_h + \nabla r_h\|_{0,h} ,
\end{aligned} \tag{4.9}$$

where $\delta_M = \max_{x\in\Omega}\delta = ah^2$. By $L^4(\Omega) \to H^1(\Omega)$ for $n \le 3$ and using an inverse inequality we have

$$\begin{aligned}
& \|\delta^{1/2}[(I_h^1 u_\lambda - u_\lambda)\nabla I_h^1 u_\lambda + u_\lambda \nabla(I_h^1 u_\lambda - u_\lambda)]\|_0 \\
\le\ & \delta_M^{1/2}(\|I_h^1 u_\lambda - u_\lambda\|_{L^4}\|\nabla I_h^1 u_\lambda\|_{L^4} + \|u_\lambda\|_{0,\infty}|I_h^1 u_\lambda - u_\lambda|_1) \\
\le\ & Cah(h^{-1}|I_h^1 u_\lambda - u_\lambda\|_1\|I_h^1 u_\lambda\|_1 + \|u_\lambda\|_{0,\infty}|I_h^1 u_\lambda - u_\lambda|_1) \\
\le\ & C(|I_h^1 u_\lambda|_1 + \|u_\lambda\|_{0,\infty})|I_h^1 u_\lambda - u_\lambda|_1
\end{aligned} \tag{4.10}$$

$$\| - \nu\Delta w_h + I_h^1 u_\lambda \nabla w_h + \nabla r_h\|_{0,h} \le Ch^{-1}[(\nu + |I_h^1 u_\lambda|_1)|w_h|_1 + \|r_h\|_0] \tag{4.11}$$

Combining (4.9) to (4.11) we have

$$\begin{aligned}
|S_3| \le\ & C[(I_h^1 u_\lambda|_1 + \|u_\lambda\|_{0,\infty})|I_h^1 u_\lambda - u_\lambda|_1 \\
& + \|\delta^{1/2}(-\nu\Delta(I_h^1 u_\lambda - u_\lambda) + \nabla(I_h^2 p_\lambda - p_\lambda))\|_{0,h}] \\
& \cdot[(\nu + |I_h^1 u_\lambda|_1)|w_h|_1 + \|r_h\|_0]
\end{aligned} \tag{4.12}$$

By (4.6)∼(4.8) and (4.12), we obtain for any $\hat{w}_h \in X_h$,

$$\begin{aligned}
& | < \Phi_{\lambda,h}(I_h\hat{u}_\lambda), \hat{w}_h > | \\
\le\ & C(1 + \nu + |I_h^1 u_\lambda|_1)[(1 + \nu + \|u_\lambda\|_{0,\infty} + |u_\lambda|_1 \\
& + |I_h^1 u_\lambda|_1)|I_h^1 u_\lambda - u_\lambda|_1 + \|I_h^2 p_\lambda - p_\lambda\|_0 \\
& + \|\delta^{1/2}(-\nu\Delta(I_h^1 u_\lambda - u_\lambda) + \nabla(I_h^2 p_\lambda - p_\lambda)\|_{0,h}](|w_h|_1 + \|r_h\|_0)
\end{aligned}$$

By the definition of $\|\cdot\|_{X_h'}$, we have

$$\begin{aligned}
\|\Phi_{\lambda,h}(I_h\hat{u}_\lambda)\|_{X_h'} \le\ & C(1 + \nu + |I_h^1 u_\lambda|_1)[(1 + \nu + \|u_\lambda\|_{0,\infty} \\
& + |u_\lambda|_1 + |I_h^1 u_\lambda|_1) \cdot |I_h^1 u_\lambda - u_\lambda|_1 + \|I_h^2 p_\lambda - p_\lambda\|_0 \\
& + \|\delta^{1/2}(-\nu\Delta(I_h^1 u_\lambda - u_\lambda) + \nabla(I_h^2 p_\lambda - p_\lambda)\|_{0,h}]
\end{aligned}$$

Since $\|u_\lambda\|_{0,\infty}$ and $|u_\lambda|_1$ are bounded, we get

$$\begin{aligned}
\|\Phi_{\lambda,h}(I_h\hat{u}_\lambda)\|_{X_h'} \le\ & C(\lambda)|I_h u_\lambda - u_\lambda|_1 + \|I_h^2 p_\lambda - p_\lambda\|_0 \\
& + \|\delta^{1/2}[-\nu\Delta(I_h^1 u_\lambda - u_\lambda) + \nabla(I_h^2 p_\lambda - p_\lambda)]\|_{0,h}
\end{aligned}$$

from $\lambda = \frac{1}{\nu}$.

We now have the following consistency result for the G/L-S finite element discretization $(\mathcal{N}_h)$ (or $(\mathcal{N}_h)'$).

Lemma 4.2. *If $\hat{u}_\lambda = (u_\lambda, p_\lambda) \in (V \cap H^2(\Omega)^n) \times (Q \cap H^1(\Omega))$ is the solution of problem $(\mathcal{N})$ (or (2.1)), then the G/L-S finite element discretization $\mathcal{N}_h$ (or $(\mathcal{N}_h)'$) is consistent.*

Proof. From Lemma 4.1 and the well-known interpolation error estimates[17], we have

$$\|\Phi_{\lambda,h}(I_h\hat{u}_\lambda)\|_{X_h'} \le C(\lambda)h \ , \tag{4.13}$$

(where $C(\lambda)$ is independent of h) and consequently

$$\|\Phi_{\lambda,h}(I_h\hat{u}_\lambda)\|_{X_h'} = O(h) = o(1) \ .$$

This proves the result.

Consistency of the discretization $(\mathcal{N}_h)$ (or $(\mathcal{N}_h)'$) has been established hence convergence will follow if nonlinear stability can be proved. As establishing (4.4) directly is difficult we follow Lopez-Marcos & Sanz-Serna[15] by considering the stablity of linearized problem and applying the following theorem.

Theorem 4.3. (Lopez-Marcos & Sanz-Serna) *Assume that for each h in H, h sufficiently small, the mapping $\Phi_{\lambda,h}$ is differentiable at each point $\hat{v}_h \in X_h$ in an open ball $B(I_h\hat{u}_\lambda, R_h)$ with Frechet derivative denoted by $\Phi'_{\lambda,h}$. Suppose that the inverse exists with*

$$\|(\Phi'_{\lambda,h}(I_h\hat{u}_\lambda))^{-1}\|_{X_h'} \le L$$

and furthermore assume that there exists a constant $M \in [0,1)$ such that for each $\hat{v}_h \in X_h$ in $B(I_h\hat{u}_\lambda, R_h)$,

$$\|\Phi'_{\lambda,h}(\hat{v}_h) - \Phi'_{\lambda,h}(I_h\hat{u}_\lambda)\|_{L(X_h, X_h')} \le M/L \ . \tag{4.14}$$

Then the discretization (4.3) is nonlinearly stable with threshold R_h and stability constant $S \equiv L/(1-M)$.

We shall prove that the conditions of Theorem 4.3 are fulfilled by the G/L-S finite element discretization $(\mathcal{N}_h)$ (or $(\mathcal{N}_h)'$) and hence it is nonlinearly stable. To this end, it is immediate that the Frechet derivative of the mapping $\Phi_{\lambda,h}$ defined by (3.2) satisfies

$$\begin{aligned} < \Phi'_{\lambda,h}(\hat{u}_h)\hat{v}_h, \hat{w}_h >=& \nu a(v_h, w_h) + b(v_h; u_h, w_h) + b(u_h; v_h, w_h) \\ & - (q_h, \mathrm{div} w_h) + (r_h, \mathrm{div} v_h) \\ & + \sum_{K\in\mathcal{T}_h} \delta_K(-\nu\Delta v_h + u_h\nabla v_h + v_h\nabla u_h + \nabla q_h, \\ & \qquad - \nu\Delta w_h + u_h\nabla w_h + \nabla r_h)_K \\ & + \sum_{K\in\mathcal{T}_h} \delta_K(-\nu\Delta u_h + u_h\nabla u_h + \nabla p_h, v_h\nabla w_h)_K \\ & - \sum_{K\in\mathcal{T}_h} \delta_K(f, v_k\nabla w_h)_K \ . \end{aligned} \tag{4.15}$$

From which the following Lemma can be established.

Lemma 4.3. *For every $\hat{v}_h \in X_h$ there exists $h_0(\lambda) > 0$ such that for each $h, 0 < h \le h_0(\lambda)$,*

$$\|\hat{v}_h\|_X \le L\|\Phi'_{\lambda,h}(I_h\hat{u}_\lambda)\hat{v}_h\|_{X'_h} \ . \tag{4.16}$$

Proof. We prove this by contradiction. Suppose the statement of the Lemma is false, then a sequence $\{\hat{v}_h\}$ with respect to h in X_h can be chosen so that

$$\lim_{h\to 0}\|\Phi'_{\lambda,h}(I_h\hat{u}_\lambda)\hat{v}_h\|_{X'_h} = 0 \tag{4.17}$$

with

$$\|\hat{v}_h\|_X = 1. \tag{4.18}$$

Since any bounded sequence in $H_0^1(\Omega)$ has an L^2 convergent subsequence, by (4.18) there is a weakly converging subsequence in $V \times Q$ which for simplicity we denote again by $\{\hat{v}_h\}$, we may presume that $\hat{v}_h \rightharpoonup \hat{v}$ as $h \to 0$. It follows that for any $\hat{w} \in (V \cap H^2(\Omega)^n) \times (Q \cap H^1(\Omega))$,

$$\lim_{h\to 0} < \Phi'_{\lambda,h}(I_h\hat{u}_\lambda)\hat{v}_h, I_h\hat{w} >= 0 \ . \tag{4.19}$$

Expanding the term $< \Phi'_{\lambda,h}(I_h\hat{u}_\lambda)\hat{v}_h, I_h\hat{w} >$ and taking the limit implies that

$$\nu a(v,w) + b(v;u_\lambda,w) + b(u_\lambda;v,w) - (\mathrm{div} w, q) + (\mathrm{div} v, r) + \lim_{h\to 0} F_h^1 \equiv 0 \ , \tag{4.20}$$

where

$$\begin{aligned}F_h^1 = \ & \textstyle\sum_{K\in\mathcal{T}_h}\delta_K(-\nu\Delta v_h + I_h^1u_\lambda\nabla v_h + v_h\nabla I_h^1u_\lambda + \nabla q_h,\\ & -\nu\Delta I_h^1 w + I_h^1u_\lambda\nabla I_h^1 w + \nabla I_h^2 r)_K\\ & + \textstyle\sum_{K\in\mathcal{T}_h}\delta_K(-\nu\Delta I_h^1u_\lambda + I_h^1u_\lambda\nabla I_h^1u_\lambda + \nabla I_h^2p_\lambda, v_h\nabla I_h^1 w)_K\\ & - \textstyle\sum_{K\in\mathcal{T}_h}\delta_K(f, v_h\nabla I_h^1 w)_K \ .\end{aligned}$$

By Cauchy's inequality, $L^4(\Omega) \to H^1(\Omega)$ for $n \le 3$, an inverse inequality, and (4.18), we have

$$\begin{aligned}|F_h^1| \le & \sum_{K\in\mathcal{T}_h}\delta_K\| - \nu\Delta v_h + I_h^1u_\lambda\nabla v_h + v_h\nabla I_h^1u_\lambda + \nabla q_h\|_{0,K}\\ & \cdot\| - \nu\Delta I_h^1 w + I_h^1u_\lambda\nabla I_h^1 w + \nabla I_h^2 r\|_{0,K}\\ & + \delta_M(\| - \nu\Delta I_h^1u_\lambda + I_h^1u_\lambda\nabla I_h^1u_\lambda + \nabla I_h^2p_\lambda\|_{0,h} + \|f\|_0)\|v_h\nabla I_h^1 w\|_0\\ \le & C\delta_M(\nu h^{-1}|v_h|_1 + h^{-1}\|q_h\|_0 + \|I_h^1u_\lambda\|_{0,4}\\ & \cdot\|\nabla v_h\|_{0,4} + \|v_h\|_{0,4}\|\nabla I_h^1u_\lambda\|_{0,4})\\ & \cdot(\nu\|w\|_2 + \|r\|_1 + \|I_h^1 u\|_{0,4}\|\nabla I_h^1 w\|_{0,4})\\ & + C\delta_M(\|f\|_0 + \nu h^{-1}|I_h^1u_\lambda|_1 + h^{-1}\|I_h^2p_\lambda\|_0\\ & + \|I_h^1u_\lambda\|_{0,4}\|\nabla I_h^1u_\lambda\|_{0,4})\cdot\|v_h\|_{0,4}\|\nabla I_h^1 w\|_{0,4}\\ \le & C\delta_M(\nu h^{-1}|v_h|_1 + h^{-1}\|q_h\|_0 + h^{-3/4}|I_h^1u_\lambda|_1|v_h|_1)\\ & \cdot(\nu\|w\|_2 + \|r\|_1 + |I_h^1u_\lambda|_1\|w\|_2)\end{aligned}$$

$$
\begin{aligned}
&+ C\delta_M(\|f\|_0 + \nu h^{-1}|I_h^1 u_\lambda|_1 + h^{-1}\|p_\lambda\|_0 + h^{-3/4}|I_h^1 u_\lambda|_1^2)|v_h|_1\|w\|_2 \\
\le & C\delta_M h^{-1}(\nu + |u_\lambda|_1)[(\nu + |u_\lambda|_1)\|w\|_2 + \|r\|_1] \\
&+ C\delta_M h^{-1}(\|f\|_0 + \nu|u_\lambda|_1 + \|p_\lambda\|_0 + |u_\lambda|_1^2)\|w\|_2 \\
\le & C(\lambda, f, \hat{u}_\lambda, \hat{w})h \ , \qquad (4.21)
\end{aligned}
$$

where $C(\lambda, f, \hat{u}_\lambda, \hat{w})$ denotes a constant dependent on $\lambda = \frac{1}{\nu}, f, \hat{u}_\lambda$, and $\hat{w}$, but independent of h. By (4.21), we have $\lim_{h\to 0} F_h^1 = 0$, i.e. for

$$
\hat{w} = (w, r) \in (V \cap H^2(\Omega)^n) \times (Q \cap H^1(\Omega)) \ ,
$$

(4.20) becomes

$$
\nu a(v, w) + b(v; u_\lambda, w) + b(u_\lambda; v, w) - (\mathrm{div} w, q) + (\mathrm{div} v, r) = 0 \ , \qquad (4.22)
$$

since $(V \cap H^2(\Omega)^n) \times (Q \cap H^1(\Omega))$ is dense in $V \times Q$, we obtain that

$$
\begin{aligned}
< \Phi'_\lambda(\hat{u}_\lambda)\hat{v}, \hat{w} > \equiv \ & \nu a(v, w) + b(v; u_\lambda, w) + b(u_\lambda; v, w) \\
& -(\mathrm{div} w, q) + (\mathrm{div} v, r) \qquad (4.23) \\
= \ & 0
\end{aligned}
$$

for each $\hat{w} = (w, r) \in V \times Q$. Since $\hat{u}_\lambda = (u_\lambda, p_\lambda)$ is a nonsingular solution of problem $(\mathcal{N})$, we conclude that $\hat{v} \equiv 0$. Thus we have proved that $\|v_h\|_0 \to 0$ as $h \to 0$.

Now we prove that

$$
\lim_{h\to 0}(|v_h|_1 + \|q_h\|_0) = 0 \ .
$$

By (4.17) we get

$$
\lim_{h\to 0} < \Phi'_{\lambda,h}(I_h\hat{u}_\lambda)\hat{v}_h, \hat{v}_h >= 0 \qquad (4.24)
$$

and

$$
< \Phi'_{\lambda,h}(I_h\hat{u}_\lambda)\hat{v}_h, \hat{v}_h > \equiv S_1 + S_2 + S_3 \ , \qquad (4.25)
$$

where

$$
\begin{aligned}
S_1 = \ & \nu a(v_h, v_h) + \textstyle\sum_{K\in\mathcal{T}_h} \delta_K(-\nu\Delta v_h + I_h^1 u_\lambda \nabla v_h + \nabla q_h, \\
& -\nu\nabla v_h + I_h^1 u_\lambda \nabla v_h + \nabla q_h)_K \ , \\
S_2 = \ & b(v_h; I_h^1 u_\lambda, v_h) + b(I_h^1 u_\lambda; v_h, v_h) = b(v_h; I_h^1 u_\lambda, v_h) \ , \\
S_3 = \ & \textstyle\sum_{K\in\mathcal{T}_h} \delta_K(v_h \nabla I_h^1 u_\lambda, -\nu\Delta v_h + I_h^1 u_\lambda \nabla v_h + \nabla q_h)_K \\
& + \textstyle\sum_{K\in\mathcal{T}_h} \delta_K(-\nu\Delta I_h^1 u_\lambda + I_h^1 u_\lambda \nabla I_h^1 u_\lambda + \nabla I_h^2 p_\lambda, v_h\nabla v_h)_K \\
& - \textstyle\sum_{K\in\mathcal{T}_h} \delta_K(f, v_h\nabla v_h)_K \ .
\end{aligned}
$$

It is easy to see that

$$
S_1 = \nu|v_h|_1^2 + \|\delta^{1/2}(-\nu\Delta v_h + I_h^1 u_\lambda \nabla v_h + \nabla q_h)\|_{0,h}^2 \ . \qquad (4.26)
$$

Using Cauchy's inequality, $L^4(\Omega) \to H^1(\Omega)$ and $L^\infty(\Omega) \to H^2(\Omega)$ for $n \le 3$, an inverse estimates, and (4.18), we have

$$
\begin{aligned}
|S_2| \le \ & C(\|v_h\|_0\|v_h\|_{0,4}\|\nabla I_h^1 u_\lambda\|_{0,4} + \|I_h^1 u\|_{0,\infty}|v_h|_1\|v_h\|_0) \\
\le \ & C(\|v_h\|_0|v_h|_1\|\nabla u_\lambda\|_{0,4} + \|u_\lambda\|_{0,\infty}|v_h|_1\|v_h\|_0) \qquad (4.27) \\
\le \ & C\|u_\lambda\|_2\|v_h\|_0 \ ,
\end{aligned}
$$

$$\begin{aligned}
|S_3| \leq\ & \delta_M \|v_h \nabla I_h^1 u_\lambda\|_0 \| -\nu\Delta v_h + I_h^1 u_\lambda \nabla v_h + \nabla q_h\|_{0,h} \\
& +\delta_M(\| -\nu\Delta I_h^1 u_\lambda + I_h^1 u_\lambda \nabla I_h^1 u_\lambda + \nabla I_h^2 p_\lambda\|_{0,h} + \|f\|_0)\|v_h \nabla v_h\|_0 \\
\leq\ & C\delta_M \|v_h\|_{0,4} \|\nabla I_h^1 u_\lambda\|_{0,4} (\nu h^{-1}|v_h|_1 + h^{-1}\|q_h\|_0 + \|I_h^1 u_\lambda\|_{0,\infty}|v_h|_1) \\
& +C\delta_M(\|f\|_0 + \nu\|u_\lambda\|_2 + \|p_\lambda\|_1 + \|u_\lambda\|_2 |u_\lambda|_1)\|v_h\|_{0,4}\|\nabla v_h\|_0 \\
\leq\ & C\delta_M |v_h|_1 \|u_\lambda\|_2 (\nu h^{-1}|v_h|_1 + h^{-1}\|q_h\|_0 + \|u_\lambda\|_{0,\infty}|v_h|_1) \\
& +C\delta_M h^{-3/4}(\|f\|_0 + \nu\|u_\lambda\|_2 + \|p_\lambda\|_1 + \|u_\lambda\|_2 |u_\lambda|_1)|v_h|_1^2 \\
\leq\ & Ch[\|u\|_2(1+\nu+\|u_\lambda\|_2) + \|f\|_0 + \|p_\lambda\|_1] \ .
\end{aligned} \tag{4.28}$$

Combining (4.24) — (4.28) and noting that $\|v_h\|_0 \to 0$ as $h \to 0$, we have

$$\lim_{h\to 0}(\nu|v_h|_1^2 + \|\delta^{1/2}(-\nu\Delta v_h + I_h^1 u_\lambda \nabla v_h + \nabla q_h)\|_{0,h}^2) = 0 \ . \tag{4.29}$$

This implies that

$$\lim_{h\to 0}|v_h|_1 = \lim_{h\to 0}\|q_h\|_0 = 0 \ , \tag{4.30}$$

which is a contradiction. Hence the statement of the Lemma is true.

The proof of nonlinear stability now follows by verifying the second condition of Theorem 4.3. We first prove the following Lemma.

Lemma 4.4. *If $\hat{v}_h$ in X_h, then*

$$\begin{aligned}
& \|\Phi'_{\lambda,h}(\hat{v}_h) - \Phi'_{\lambda,h}(I_h\hat{u}_\lambda)\|_{L(X_h,X_h')} \\
\leq\ & C[(1+\nu+|v_h|_1 + |I_h^1 u_\lambda|_1)|v_h - I_h^1 u_\lambda|_1 + \|q_h - I_h^2 p_\lambda\|_0] \ .
\end{aligned} \tag{4.31}$$

Proof. By definition

$$\|\Phi'_{\lambda,h}(\hat{v}_h) - \Phi'_{\lambda,h}(I_h\hat{u}_\lambda)\|_{L(X_h,X_h')} = \sup_{\|\hat{z}_h\|_X = 1}\|\Phi'_{\lambda,h}(\hat{v}_h) - \Phi'_{\lambda,h}(I_h\hat{u}_\lambda))\hat{z}_h\|_{X_h'} \ , \tag{4.32}$$

while

$$\|\Phi'_{\lambda,h}(\hat{v}_h) - \Phi'_{\lambda,h}(I_h\hat{u}_\lambda))\hat{z}_h\|_{X_h'} = \sup_{\|\hat{z}_h\|_X = 1} < (\Phi'_{\lambda,h}(\hat{v}_h) - \Phi'_{\lambda,h}(I_h\hat{u}_\lambda))\hat{z}_h, \hat{w}_h > \ , \tag{4.33}$$

where $\hat{z}_h \equiv (z_h, \eta_h)$ and $\hat{w}_h \equiv (w_h, r_h)$ are in X_h. Expanding

$$< (\Phi'_{\lambda,h}(\hat{v}_h) - \Phi'_{\lambda,h}(I_h\hat{u}_\lambda))\hat{z}_h, \hat{w}_h > \equiv S_1 + S_2 + S_3 + S_4 \ , \tag{4.34}$$

where

$$\begin{aligned}
S_1 =\ & b(z_h; v_h - I_h^1 u_\lambda, w_h) + b(v_h - I_h^1 u_\lambda; z_h, w_h) \ , \\
S_2 =\ & \textstyle\sum_{K\in\mathcal{T}_h} \delta_K(-\nu\Delta z_h + v_h\nabla z_h + z_h\nabla v_h + \nabla\eta_h, (v_h - I_h^1 u_\lambda)\nabla w_h)_K \ , \\
S_3 =\ & \textstyle\sum_{K\in\mathcal{T}_h} \delta_K((v_h - I_h^1 u_\lambda)\nabla z_h + z_h\nabla(v_h - I_h^1 u_\lambda), -\nu\Delta w_h + I_h^1 u_\lambda w_h + \nabla\gamma_h)_K \ , \\
S_4 =\ & \textstyle\sum_{K\in\mathcal{T}_h} \delta_K(-\nu\Delta(v_h - I_h^1 u_\lambda) + v_h\nabla v_h - I_h^1 u_\lambda \nabla I_h^1 u_\lambda + \nabla(q_h - I_h^2 p_\Lambda), z_h\nabla w_h)_K \ .
\end{aligned}$$

For S_1, it is easy to get

$$|S_1| \leq C(|v_h - I_h^1 u_\lambda|_1 |z_h|_1 |w_h|_1 \ . \tag{4.35}$$

By means of Sobolev's imbedding theorem and an inverse inequality we can prove that

$$\|v\|_{0,\infty} \leq Ch^{-1/2}|v|_1, \qquad \forall v \in V_h \ . \tag{4.36}$$

Using (4.36) and an inverse inequality we have

$$
\begin{aligned}
|S_2| \le\ & C\delta_M(\nu h^{-1}|z_h|_1 + \|v_h\|_{0,\infty}|z_h|_1 + \|z_h\|_{0,\infty}|v_h| + h^{-1}\|\eta_h\|_0) \\
& \cdot\|v_h - I_h^1 u_\lambda\|_{0,\infty}|w_h|_1 \\
\le\ & C\delta_M(\nu h^{-1}|z_h|_1 + h^{-1/2}|v_h|_1|z_h|_1 + h^{-1}\|\eta_h\|_0) \\
& \cdot h^{-1/2}|v_h - I_h^1 u_\lambda|_1|w_h|_1 \\
\le\ & C|v_h - I_h^1 u_\lambda|_1((\nu + |v_h|_1)|z_h|_1 + \|\eta_h\|_0)|w_h|_1\ ,
\end{aligned}
\tag{4.37}
$$

$$
\begin{aligned}
|S_3| \le\ & C\delta_M(\|v_h - I_h^1 u_\lambda\|_{0,\infty}|z_h|_1 + \|z_h\|_{0,\infty}|v_h - I_h^1 u_\lambda|_1) \\
& \cdot(\nu h^{-1}|w_h|_1 + \|I_h^1 u_\lambda\|_{0,\infty}|w_h|_1 + h^{-1}\|r_h\|_0) \\
\le\ & C\delta_M h^{-1/2}|v_h - I_h^1 u_\lambda|_1|z_h|_1 \\
& \cdot(\nu h^{-1}|w_h|_1 + h^{-1/2}|I_h^1 u_\lambda|_1|w_h|_1 + h^{-1}\|r_h\|_0) \\
\le\ & C|v_h - I_h^1 u_\lambda|_1|z_h|_1((\nu + |I_h^1 u|_1)|w_h|_1 + \|r_h\|_0)\ ,
\end{aligned}
\tag{4.38}
$$

$$
\begin{aligned}
|S_4| \le\ & \delta_M\| - \nu\nabla(v_h - I_h^1 u_\lambda) + v_h\nabla v_h - I_h^1 u_\lambda\nabla I_h^1 u_\lambda + \nabla(q_h - I_h^2 p_\lambda)\|_{0,h}\|z_h\nabla w_h\|_0 \\
\le\ & C\delta_M(\nu h^{-1}|v_h - I_h^1 u_\lambda|_1 + \|v_h\|_{0,\infty}|v_h - I_h^1 u_\lambda|_1 \\
& + \|v_h - I_h^1 u_\lambda\|_{0,\infty}\cdot|I_h^1 u_\lambda|_1 \\
& + h^{-1}\|q_h - I_h^2 p_\lambda\|_0)\|z_h\|_{0,\infty}|w_h|_1 \\
\le\ & C\delta_M h^{-3/2}((\nu + |v_h|_1 + |I_h^1 u_\lambda|_1)|v_h - I_h^1 u_\lambda|_1 + \|q_h - I_h^2 p_\lambda\|_0)|z_h|_1|w_h|_1 \\
\le\ & C[(\nu + |v_h|_1 + |I_h^1 u|_1)|v_h - I_h^1 u_\lambda|_1 + \|q_h - I_h^2 p_\lambda\|_0]|z_h|_1|w_h|_1\ .
\end{aligned}
\tag{4.39}
$$

By (4.32) — (4.39), we have

$$
\begin{aligned}
& \|\Phi'_{\lambda,h}(\hat v_h) - \Phi'_{\lambda,h}(I_h\hat u_\lambda)\|_{L(X_h,X'_h)} \\
\le\ & C[(1 + \nu + |v_h|_1 + |I_h^1 u_\lambda|_1)|v_h - I_h^1 u_\lambda|_1 + \|q_h - I_h^2 p_\lambda\|_0]\ .
\end{aligned}
$$

To prove stability we now show that for a given $\epsilon > 0$ and suitable ρ.

$$\|\Phi'_{\lambda,h}(\hat v_h) - \Phi'_{\lambda,h}(I_h\hat u_\lambda)\|_{L(X_h,X'_h)} < \epsilon, \qquad \text{if} \quad \hat v_h \in B(I_h\hat u_\lambda, \rho).$$

We firstly note that

$$(|v_h|_1 + |I_h^1 u_\lambda|_1) \le (2|I_h^1 u_\lambda|_1 + |v_h - I_h^1 u_\lambda|_1)\ ,$$

thus, as $|I_h^1 u_\lambda|_1$ is bounded as $h \to 0$ it follows from Lemma 4.4 that if $\hat v_h \in B(I_h\hat u_\lambda, \rho)$ then there is a constant $\beta(\lambda)$ such that

$$\|\Phi'_{\lambda,h}(\hat v_h) - \Phi'_{\lambda,h}(I_h\hat u_\lambda)\|_{L(X_h,X'_h)} < (\beta + c\rho)\rho\ , \tag{4.40}$$

which can be made smaller than ϵ provided that $\rho(\lambda) < \epsilon D$ for a constant $D(\lambda)$.

Lemma 4.5. (The stability result) *There exist positive constants $\rho(\lambda)$ and S for all $\hat v_h$ and $\hat w_h$ in $B(I_h\hat u_\lambda, \rho)$*

$$\|\hat v_h - \hat w_h\|_X \le S\|\Phi_{\lambda,h}(\hat v_h) - \Phi_{\lambda,h}(\hat w_h)\|_{X'_h}$$

holds, where $\Phi_{\lambda,h}(\cdot)$ is defined as in (3.2).

Proof. This follows from Theorem 4.3, Lemma 4.3, 4.4 and the above discussion.

The proof of Theorem 4.2 follows by combining these Lemmas. Using the consistency and stability results of Lemma 4.2 and 4.5, it follows from Theorem 4.1 that there is a function $\hat{u}_{\lambda,h} = (u_{\lambda,h}, p_{\lambda,h}) \in X_h$ satisfying $(\mathcal{N}_h)$ (or $(\mathcal{N}_h)'$) such that

$$\begin{aligned} \|I_h\hat{u}_\lambda - \hat{u}_{\lambda,h}\|_X \leq \ & C(\lambda)(|I_h^1 u_\lambda - u_\lambda|_1| + \|I_h^2 p_\lambda - p_\lambda\|_0 \\ & + \|\delta^{1/2}[-\nu\Delta(I_h^1 u_\lambda - u_\lambda) + \nabla(I_h^2 p_\lambda - p_\lambda)]\|_{0,h}) \ . \end{aligned} \tag{4.41}$$

Hence, from the triangle inequality and (4.41) it is immediate that

$$\begin{aligned} \|\hat{u}_\lambda - \hat{u}_{\lambda,h}\|_X \leq \ & C(\lambda)(|I_h^1 u_\lambda - u_\lambda|_1 + \|I_h^2 p_\lambda - p_\lambda\|_0 \\ & + \|\delta^{1/2}[-\nu\Delta(I_h^1 u_\lambda - u_\lambda) + \nabla(I_h^2 p_\lambda - p_\lambda)]\|_{0,h}) \end{aligned} \tag{4.41}$$

By using $\hat{u}_\lambda = (u_\lambda, p_\lambda) \in (V \cap H^{m+1}(\Omega)^n) \times (Q \cap H^l(\Omega))$ and the well-known interpolation properties[17], we finally get

$$\|\hat{u}_\lambda - \hat{u}_{\lambda,h}\|_X \leq C(\lambda)(h^m + h^l)$$

for h sufficiently small. This concludes the proof of Theorem 4.2.

Remark 2. If the finite element pressure subspace Q_h belongs only to $L_0^2(\Omega)$, we need to add the boundary integral term $\sum_K \beta h_K \int_{\partial K}[q][r]ds$ to $B_\delta(u, u_h; \hat{v}, \hat{w})$. (Where $\beta > 0, [q] = q_+ - q_-$) in order to obtain corresponding convergences.

Remark 3. Sufficient conditions for convergence were also establihed in [20] for some common iterative methods (such as successive approximation, Newton's method) applied to the G/L-S finite element formulation $(\mathcal{N}_h)$ of the N-S equations.

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