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MODIFIED UPWIND TAYLOR FINITE ELEMENT SCHEMES FOR 1-D CONSERVATION LAWS I. A BASIC IDEA*1)

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Abstract

In this paper, first, modified upwind finite element schemes are presented for two-point value problem, and then a class of modified upwind Taylor finite element schemes are derived for one dimensional linear hyperbolic equation. The main point of the paper is how to consider the upwind property to construct base functions to make the schemes obtained be MmB (or TVD). Numerical experiments are given to show that the method is efficient to solve the discontinuous solutions.

1. Introduction

Many numerical methods mainly contribute to solve nonlinear hyperbolic conservation laws and efficiently make the methods to suit for solving discontinuous solutions, that is, numerical solutions have high resolution, high order accurate and non-oscillatory properties. Let us recall some methods to treat these things: First of all, we can say that the development of finite difference method is divided into two steps, the first step is called classical difference methods, such as, Lax-Friedrichs scheme, Godunov scheme, Lax-Wendroff scheme and so on; the other is called modern methods, for example, some TVD type schemes[1,2], MUSCL schemes[3], ENO (or UNO) schemes[4,5], PPM scheme[6], MmB schemes[7] etc.; The second of the methods should be finite element method and spectral method, and we can say that both finite element and spectral methods are far beyond finite difference methods to treat discontinuities for nonlinear hyperbolic equations, although some schemes, such as, characteristic Galerkin method [8] and modified characteristic Petrov-Galerkin method [9], discontinuity finite element method [10], finite element method based on stream lines [11] and so on, have been presented, and to solve discontinuous solutions, these methods are modified or combined with modern techniques from the work of finite difference methods.

In order to develop finite element method to be suit for solving nonlinear hyperbolic equations both widely and efficiently, in this paper a class of modified upwind Taylor

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finite element schemes are constructed for one dimensional linear hyperbolic equation. The methods is mainly to consider the constructions of base functions: firstly, form a base function, which depends on the characteristic (or upwind) property of the model; secondly, modify the base function obtained and give a nonlinear base function. Hence a nonlinear element is proposed as in [9]. Then we use Taylor expansion in time direction and finite element method in space direction. So in this paper, modified upwind finite element schemes are presented for two-point boundary problem in section 2, and in section 3, a class of modified upwind Taylor finite element schemes are derived for linear hyperbolic equation; Finally, some numerical experiments are given for Riemann initial value problems.

2. Modified Upwind FES for Two-Point Boundary Value Problem

Consider the following boundary value problem,

$$K\frac{d^2u}{dx^2} - V\frac{du}{dx} = 0, \quad x \in (0,1) \tag{2.1}$$

$$u(0) = u_0, \qquad u(1) = u_1. \tag{2.2}$$

When K/V is sufficient small, problem (2.1) (2.2) belongs to a singular perturbation problem and the solution of the problem produces boundary layer near point 0 or 1. The numerical solutions are required to have high resolution and non-oscillatory properties in the boundary layer regions, and have higher order approximate accuracy in smooth regions.

Let us see the weak form of (2.1) – (2.2),

$$K\int \frac{du}{dx}\frac{d\varphi}{dx} - V\int u\frac{d\varphi}{dx} = 0, \quad \forall \varphi \in C_0(I\!R) \tag{2.3}$$

Here we choose the following unit linear function as a base function

$$\varphi_1(x) = \begin{cases} 1 + \dfrac{x - x_i}{\Delta x}, & x \in (x_{i-1}, x_i) \\ 1 - \dfrac{x - x_i}{\Delta x} & x \in (x_i, x_{i+1}) \\ 0 & else \end{cases} \tag{2.4}$$

and set

$$u_h(x) = \sum_j u_j \varphi_j(x),$$

then we place $u_h(x)$ and φ_j in (2.3) and get

$$-\frac{K}{\Delta x}(u_{j+1} - 2u_j + u_{j-1}) - \frac{V}{2\Delta x}(u_{j+1} - u_{j-1}) = 0\ . \tag{2.5}$$

Scheme (2.5) is second order accurate, the solution will produce oscillatory phenomena near boundary layer when K/V is sufficient small.

In order to eliminate the oscillations, in [12] a modified base function is given by weighted function as follows,

$$\overline{\varphi}_j = \varphi_j + \alpha F_j(x)$$

where

$$F_j(x) = \begin{cases} \dfrac{3(x-x_j)(x-x_{j-1})}{\Delta x^2}, & x \in (x_{j-1}, x_j), \\ \dfrac{-3(x-x_{j+1})(x-x_j)}{\Delta x^2}, & x \in (x_j, x_{j+1}), \\ 0, & else, \end{cases} \tag{2.6}$$

then the modified schemes are obtained.

$$(1+\frac{\gamma}{2}(\alpha+1))u_{j-1} - (2+\gamma\alpha)u_j + (1+\frac{\gamma}{2}(\alpha-1))u_{j+1} = 0 \tag{2.7}$$

here $\gamma = V\Delta x/K$.

If $\gamma > 0$, we take $\alpha > 0$, $\gamma < 0$; take $\alpha < 0$, scheme (2.7) can be stable by suitably adjusting α. Then the upwind scheme is obtained, for example,

$$\begin{array}{ll} \alpha = 0, & if \quad \gamma < 2, \\ \alpha \geq 1 - \dfrac{2}{\gamma}, & if \quad \gamma \geq 2, \end{array}$$

scheme (2.7) is non-oscillatory.

From the above discussions, we know that it is impossible to get a non-oscillatory scheme as well as a high order accurate scheme. The reason is that α is a constant. In this section our main purpose is to choose α so that a scheme is adjustable by means of changing α. Here we present a modified weighted function, which contains the quantity of limiter, to unit linear base function mentioned above, that is

$$F_j = \begin{cases} \dfrac{3\alpha(1-Q_{j-\frac{1}{2}})(x-x_j)(x-x_{j-1})}{\Delta x^2}, & x \in (x_{j-1}, x_j), \\ -\dfrac{3\alpha(1-Q_{j+\frac{1}{2}})(x-x_j)(x-x_{j+1})}{\Delta x^2}, & x \in (x_j, x_{j+1}), \\ 0, & else. \end{cases} \tag{2.8}$$

According to (2.5)(2.8), we derive the following schemes

$$\begin{aligned} & (1+\frac{1}{2}\gamma\alpha(1-Q_{j+\frac{1}{2}}))(u_{j+1}-u_j) \\ & -(1+\frac{1}{2}\gamma\alpha(1-Q_{j-\frac{1}{2}}))(u_j-u_{j-1})) \\ & -\frac{1}{2}\gamma(u_{j+1}-u_{j-1}) \\ = \; & 0, \end{aligned} \tag{2.9}$$

where

$$Q_{j+\frac{1}{2}} = Q(r^-_{j+\frac{1}{2}}, r^+_{j+\frac{1}{2}}),$$
$$r^-_{j+\frac{1}{2}} = \frac{u_j - u_{j-1}}{u_{j+1} - u_j}, \qquad r^+_{j+\frac{1}{2}} = \frac{u_{j+2} - u_{j+1}}{u_{j+1} - u_j}.$$

There are many ways to choose $Q_{j+\frac{1}{2}}$, for example,

$$Q(r^-, r^+) = \max(0, \min(1, r^-, r^+)).$$

It is obvious that $Q_{j+\frac{1}{2}}$, $Q_{j-\frac{1}{2}}$ tend to 1 when u(x) smooth enough, then scheme (2.9) is a second order accurate scheme (except for $\frac{du}{dx} = 0$), and $Q_{j+\frac{1}{2}}$, $Q_{j-\frac{1}{2}}$ tend to 0, scheme (2.9) is a modified upwind scheme. There are non oscillations in the boundary layer region.

From (2.9), we construct the scheme which bases on five points. Here we remark that it is not our main ends how to solve equation (2.1) (2.2) in this section numerically. We just give the constructing method, then we shall use the technique to solve an initial value problem for one dimensional linear hyperbolic equation in the following section.

3. Modified Upwind Taylor FES for a Linear Hyperbolic Equation

Consider the initial value problem for a linear hyperbolic equation,

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \qquad (x,t) \in (0,T] \times I\!R, \tag{3.1}$$

$$u(x,0) = u_0(x), \qquad x \in I\!R. \tag{3.2}$$

Firstly, Taylor expansion in time direction gives

$$u^{n+1} = u^n + \Delta t\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial t^2} + O(\Delta t^3). \tag{3.3}$$

Eliminating the term of $O(\Delta t^3)$, from (3.1), we have

$$u^{n+1} = u^n - a\Delta t\frac{\partial u}{\partial x} + \frac{1}{2}a^2\Delta t^2\frac{\partial^2 u}{\partial x^2}\ . \tag{3.4}$$

The weak form of (3.4) is written to

$$\int u^{n+1}\varphi dx = \int u^n\varphi dx + a\Delta t\int u^n\frac{d\varphi}{dx}dx - \frac{1}{2}\Delta t^2 a^2\int\frac{du^n}{dx}\frac{d\varphi}{dx}dx. \tag{3.5}$$

Here we take linear elements φ_j as base functions, then we give the following approximation,

$$\begin{array}{l} u_h^n = \sum_j u_j^n\varphi_j \\ \int u_h^{n+1}\varphi_j dx = \int u_h^n\varphi_j dx + a\Delta\int u_h^n\frac{d\varphi_j}{dx} - \frac{1}{2}a^2\Delta t^2\int\frac{du_h^n}{dx}\frac{d\varphi_j}{dx}dx. \end{array} \tag{3.6}$$

Then a second order accurate scheme is derived to

$$\begin{aligned} \frac{1}{6}(u_{j+1}^{n+1} + 4u_j^{n+1} + u_{j-1}^{n+1}) = {} & \frac{1}{6}(u_{j+1}^n + 4u_j^n + u_{j-1}^n) \\ & -\frac{\Delta t}{2\Delta x}(u_{j+1}^n - u_{j-1}^n) + \frac{\Delta t^2}{2\Delta x^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n). \end{aligned} \tag{3.7}$$

When we think the approximation of (3.6) by using lumped mass methods, that is, $\int u\varphi_j dx = u_j$, a second order accurate Lax-Wendroff type scheme is obtained,

$$u_j^{n+1} = u_j^n - \frac{1}{2}\nu(u_{j+1}^n - u_{j-1}^n) + \frac{1}{2}\nu^2(u_{j+1}^n - 2u_j^n + u_{j-1}^n), \tag{3.8}$$

where $\nu = a\dfrac{\Delta t}{\Delta x}$. Scheme (3.8) is line stable by restriction $\nu \le 1$. It is well known that the solution of the scheme produces oscillatory phenomena near discontinuities. Therefore we take $\overline{\varphi}_j$ to the test function to be a modification of unit linear function as same as the previous discussions in section 2. Then a second order accurate scheme is derived to

$$\overline{\varphi}_j = \begin{cases} \varphi_j + \alpha_{j-\frac{1}{2}}F_j(x), & x \in (x_{j-1}, x_j) \\ \varphi_j + \alpha_{j+\frac{1}{2}}F_j(x), & x \in (x_j, x_{j+1}) \\ 0, & else \end{cases}\ , \tag{3.9}$$

where $F_j(x)$ is chosen as (2.6) and $\alpha_{j+\frac{1}{2}} = \alpha(\nu, Q_{j+\frac{1}{2}})$, and (3.6) is written to

$$\begin{array}{l} u_h^n = \sum_j u_j^n \varphi_j \\ \int u_h^{n+1}\overline{\varphi}_j dx = \int u_h^n \overline{\varphi}_j dx + a\Delta \int u_h^n \frac{d\overline{\varphi}_j}{dx} - \frac{1}{2}a^2\Delta t^2 \int \frac{du_h^n}{dx}\frac{d\overline{\varphi}_j}{dx} dx. \end{array} \quad (3.10)$$

Then we have the following scheme,

$$u_j^{n+1} = u_j^n - \frac{1}{2}\nu(u_{j+1}^n - u_{j-1}^n) + \frac{\nu}{2}[(\nu - \alpha_{j+\frac{1}{2}})(u_{j+1}^n - u_j^n) - (\nu - \alpha_{j-\frac{1}{2}})(u_j^n - u_{j-1}^n)]. \quad (3.11)$$

The MmB (or TVD) properties and the approximate accuracy were analyzed in [13] and [14]. The MmB property shows us how to choose $\alpha_{j+\frac{1}{2}}$. There are many choices to $\alpha_{j+\frac{1}{2}}$, for example, when $a > 0$, from the MmB (or TVD) schemes in [2] [7], take

$$\alpha_{j+\frac{1}{2}} = (1-\nu)(Q_{j+\frac{1}{2}} - 1),$$

scheme (3.11) can be rewritten to

$$u_j^{n+1} = u_j^n - \nu(u_j^n - u_{j-1}^n) - \frac{1}{2}\nu(1-\nu)[Q_{j+\frac{1}{2}}(u_{j+1}^n - u_j^n) - Q_{j-\frac{1}{2}}(u_j^n - u_{j-\frac{1}{2}}^n)]. \quad (3.12)$$

It is not difficult to prove that the solution of scheme (3.11) satisfies the following properties,

$$\min(u_j^n, u_{j-1}^n) \le u_j^{n+1} \le \max(u_j^n, u_{j-1}^n)$$

and when $\alpha < 0$, take

$$\alpha_{j+\frac{1}{2}} = -\nu(1+\nu)(1 - Q_{j+\frac{1}{2}})$$

scheme (3.10) can be rewritten to,

$$u_j^{n+1} = u_j^n - \nu(u_{j+1}^n - u_j^n) + \frac{1}{2}\nu(1+\nu)[Q_{j+\frac{1}{2}}(u_{j+1}^n - u_j^n) - Q_{j-\frac{1}{2}}(u_j^n - u_{j-1}^n)]. \quad (3.13)$$

The solution of scheme (3.13) has the property,

$$\min(u_j^n, u_{j+1}^n) \le u_j^{n+1} \le \max(u_j^n, u_{j+1}^n).$$

From [10] and [12], we know that scheme (3.12) and (3.13) are MmB. In the following section, we show some numerical examples by using the schemes to solve discontinuous solutions.

4. Numerical Experiments

In order to check the schemes in section 3, we consider the following discontinuous initial value problem for the two linear hyperbolic equations,

$$u_t + u_x = 0$$

with the initial data

$$u(x,0) = \begin{cases} 3, & x < 0 \\ 1, & x > 0 \end{cases},$$

Here we choose $\alpha_{j+\frac{1}{2}}$ in (3.12) as follows,

(i) $\alpha_1 := \alpha_{j+\frac{1}{2}}$=0, scheme (3.12) is a Lax-Wendroff type scheme.
(ii) $\alpha_2 := \alpha_{j+\frac{1}{2}} = \nu + Q_{j+\frac{1}{2}} - 1$, where $Q_{j+\frac{1}{2}}$ is a limiter described in [2][9].
(iii) $\alpha_3 := \alpha_{j+\frac{1}{2}} = (1-\nu)(Q_{j+\frac{1}{2}} - 1)$.

All the calculations are performed via $\nu = 0.9$ and time steps n=100. See Tab. 4.1.

x_j	α_1	α_2	α_3	x_j	α_1	α_2	α_3
i=89	2.998	3.000	3.000	i=101	3.299	3.000	3.000
i=90	2.997	3.000	3.000	i=102	3.187	3.000	2.999
i=91	2.999	3.000	3.000	i=103	2.878	2.999	2.998
i=92	3.005	3.000	3.000	i=104	2.433	2.998	2.936
i=93	3.012	3.000	3.000	i=105	1.961	2.751	2.331
i=94	3.009	3.000	3.000	i=106	1.561	1.249	1.552
i=95	2.985	3.000	3.000	i=107	1.282	1.000	1.141
i=96	2.949	3.000	3.000	i=108	1.121	1.000	1.032
i=97	2.934	3.000	3.000	i=109	1.044	1.000	1.006
i=98	2.981	3.000	3.000	i=110	1.013	1.000	1.001
i=99	3.099	3.000	3.000	i=111	1.003	1.000	1.000
i=100	3.239	3.000	3.000	i=112	1.000	1.000	1.000

Tab. 4.1

From the results, we can see that Lax-Wendroff type scheme produces oscillation near discontinuity, the others using limiters have high resolution and non-oscillatory properties.

Here we want you to know the differences about the modifications of base functions between paper [9] and this paper: in [9], we modified the try base function to get a nonlinear base function, so it was called nonlinear element, and in this paper, we modify the test base function, so it is called a nonlinear weighted test function space. We remember that constructing finite element methods depend on the structure of the model and the properties of the solutions. As the constructions of the modern finite difference methods, we must consider the discontinuous solutions when we derive the finite element schemes for hyperbolic equations. It is necessary to present nonlinear base functions (or nonlinear elements) to solve discontinuous solutions. In [15], we shall present the discussions of nonlinear finite elements, monotone interpolations and discontinuous solutions for hyperbolic equations in conservation laws.

References

[1] A. Harten, High resolution schemes for conservation laws, *J. Comput. Phys.*, 49 (1983), 357–393.

[2] P. K. Sweby, High resolution schemes using flux limiters for hyperbolic conservation laws, *SIAM J. Numer. Anal.*, 21 (1984), 995–1011.

[3] B. van Leer, Towards the ultimate conservative difference scheme, V. a second order sequel to Godunov's method, *J. Comput. Phys.*, 32 (1979), 101–136.

[4] A. Harten and S. Osher, Uniformly high-order accurate non-oscillatory scheme I., *SIAM J. Numer. Anal.*, 24 (1987), 279–309.

[5] A. Harten et al., Uniformly high order accurate essentially non-oscillatory schemes, I., *J. Comput. Phys.*, 71 (1987), 231–303.

[6] P. Collela and P. R. Woodward, The piecewise-parabolic method (PPM) for gas-dynamical simulations, *J. Comput. Phys.*, 54 (1984), 174–201.

[7] H-M Wu and S-L Yang, MmB — A new class of accurate high resolution schemes for conservation laws in two dimensions, *IMPACT of computing in science and engineering*, 1 (1989), 217–259.

[8] K. W. Morton, Generalized Galerkin methods for hyperbolic problems, *Computer Methods in Applied Mechanics and Engineering*, 52 (1985), 847–871.

[9] Shuli Yang, Characteristic Petrov-Galerkin methods for 2-D conservation laws, *IMPACT of Computing in Science and Engineering*, 5 (1993), 379–807.

[10] B. Cockburn and C.-W. Shu, TVB Range-Kutta local projection P1-discontinuous Galerkin finite element method for scalar conservation laws, *Mathematical Modelling and Numerical Analysis*, 25 (1991), 337–361.

[11] A. N. Brooks and T. J. R. Hughes, 'Streamline-upwind/ Petrov-Galerkin formulation for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations', *Computer Methods in Applied Mechanics and Engineering*, 32 (1982), 199–259.

[12] J. C. Heineich, P. S. Huyakorn, O. C. Zienkiewicz and A. R. Mitchell, An "upwind" finite element scheme for two-dimensional convective transport equation, *International Journal for Numerical Methods in Engineering*, 11 (1977), 131–143.

[13] Shuli Yang, Modified upwind Taylor finite element schemes for 1-D conservation laws, II. Scalar conservation law. (to appear).

[14] Shuli Yang, Modified upwind Taylor finite element schemes for 1-D conservation laws, III. Systems in conservation laws, Submitted to *IMPACT of Computing on Science and Engineering.*

[15] Shuli Yang, Nonlinear element, monotone interpolations and discontinuous solutions for conservation laws, (in preparation).