Journal of Computational Mathematics, Vol.13, No.3, 1995, 259–266.

UNCONSTRAINED OPTIMIZATION METHODS FOR NONLINEAR COMPLEMENTARITY PROBLEM *1)

J.M. Peng
(*Computing Centre, Academia Sinica, Beijing, China*)

Abstract

In this paper, we propose a class of new NCP functions and discuss their properties. By these function, we transfer the complementarity problem into unconstrained optimization problem and study the corresponding optimization problem. Numerical results are given.

1. Introduction

Since its introduction in mid-1960's and early 1970's, linear and nonlinear complementarity problems have been proven to be very useful both in optimization theories and real applications, such as the economic computation and game theoretic equilibria etc.

The standard nonlinear complementarity problem is to find a $x \in R^n$ such that:

$$F(x) \geq 0,\ x \geq 0,\ x^T F(x) = 0, \tag{1.1}$$

where $F:\ R^n \longrightarrow R^n$. For simplicity, we often call it NCP. Many authors have studied this problem and various methods for it are given. One can find an excellent summary for it in [1]. The methods mentioned in [1] are mainly based on some approximate equations and then use methods for equations to solve (1.1).

Recently, some new results about (1.1) are reported . For example, the authors of [4] consider (1.1) as unconstrained and constrained minimization. In [5], the authors propose a global Newton method for variational inequalities, which similar to the method in [4] to some extents. Both [4] and [5] transfer (1.1) into unconstrained problem which the objective function is differentiable everywhere. Kanzow[2],[3] have studied various NCP functions and give some methods, which mainly depend on the Newton equation and its local property, but some of these functions are not differentiable everywhere. For more details, see [2],[3].

In the following section, we propose a NCP function and discuss its optimal properties. In section 3, we introduce a class of functions which have the properties of the

* Received March 10, 1994.
1) This Work was Supported by the State Key Project "large scale scientific computing" of China.

function in section 2. Using these functions, we transfer (1.1) into an unconstrained minimization problem, we prove that the solution of (1.1) is a global minimizer of the optimization problem, the first order and second order optimal conditions at the global minimizer are considered. Finally, numerical results of our method are given in section 4.

2. A New NCP Function

First, we give a definition which is due to [2]:

Definition 2.1. *A function $\phi : R^2 \to R$ is called NCP-function if it satisfies the nonlinear complementarity condition*

$$\phi(a,b) = 0 \Longleftrightarrow a \geq 0,\ b \geq 0,\ ab = 0.$$

Many NCP functions have been introduced and studied by different authors. There is a complete comparison for these functions in [2]. We rewrite these NCP functions as follows:

$$\begin{aligned}
I.\quad \phi(a,b) &= -ab + \frac{1}{2}\min^2\{0, a+b\},\\
IIa.\quad \phi(a,b) &= -ab + \min^2\{0,a\} + \min^2\{0,b\},\\
IIb.\quad \phi(a,b) &= (a-b)^2 - a|a| - b|b|,\\
IIIa.\quad \phi(a,b) &= |a-b| - a - b,\\
IIIb.\quad \phi(a,b) &= \max\{0, a-b\} - a,\\
IIIc.\quad \phi(a,b) &= \min\{a,b\},\\
IV.\quad \phi(a,b) &= \sqrt{a^2+b^2} - a - b,\\
V.\quad \phi(a,b) &= \theta(|a-b|) - \theta(a) - \theta(b),
\end{aligned}$$

where $\theta(x)$ is a strictly increasing function with $\theta(0) = 0$.

It is easy to see that, except for IV, all NCP functions mentioned above mainly use the difference between a and $|a|$, or $|a-b|$ and a,b. In fact, the functions IIa and IIb and also the ones of $IIIa - c$ are identical, except for a multiplicative constant. The functions $I, IIa - b$ are globally differentiable, $IIIa - c$ and IV is not so, the function V depends on the definition of θ. By these functions , we can transfer (1.1) into some equations and then use Newton methods to solve the equations.

In [4], through an argumented Lagrangian formulation for mathematical programming, the authors construct NCP functions as follows:

$$\phi(a,b,\alpha) = ab + \frac{1}{2\alpha}(||(-\alpha b + a)_+||^2 - a^2 + ||(-\alpha a + b)_+||^2 - b^2), \alpha > 1, \qquad (2.1)$$

where the norm is the 2-norm and $(x)_+$ denotes $(x)_+ = \max\{0, x\}$. Through this function, (1.1) is cast as an unconstrained minimization problem. The properties of the corresponding optimization problem are also disscussed.

Our idea derives from function IV and [4], we mainly consider the difference between $\sqrt{a^2+b^2}$ and a, b. Consider the following function:

$$\phi(a,b) = (\sqrt{a^2+b^2} - a)(\sqrt{a^2+b^2} - b), \quad (a,b) \in R^2. \tag{2.2}$$

It is easy to show that $\phi(a,b)$ is a NCP-function. Furthermore, we can get result as follows:

Lemma 2.1. *let $\phi(a,b)$ is defined by (2.2), the partial derivative of $\phi(a,b)$ equals to 0 if and only if (a,b) satisfies the complementarity condition.*

proof. The if part of the above lemma follows from inductive algebraic calculus.

If $\frac{\partial\phi(a,b)}{\partial a} = 0$, then either $a = b = 0$ or that:

$$\frac{\partial\phi(a,b)}{\partial a} = (\sqrt{a^2+b^2} - a)(a + b - \sqrt{a^2+b^2})/\sqrt{a^2+b^2} = 0. \tag{2.3}$$

Thus either $\sqrt{a^2+b^2} - a = 0$ or $\sqrt{a^2+b^2} - a - b = 0$, each of the cases implies that (a,b) satisfies the complementarity condition. The same result holds for $\frac{\partial\phi(a,b)}{\partial b}$. Thus the lemma is true.

If (a,b) is strict complementarity, which means that $a+b > 0$, we have the following result:

Lemma 2.2. *Let $\phi(a,b)$ be defined by (2.2), then*

$$\frac{\partial^2\phi(a,b)}{\partial a^2} = 0, \quad \frac{\partial^2\phi(a,b)}{\partial a\partial b} = 0, \quad \frac{\partial^2\phi(a,b)}{\partial b^2} = 1 \tag{2.4}$$

holds for $b = 0$ and $a > 0$, and

$$\frac{\partial^2\phi(a,b)}{\partial a^2} = 1, \quad \frac{\partial^2\phi(a,b)}{\partial a\partial b} = 0, \quad \frac{\partial^2\phi(a,b)}{\partial b^2} = 0 \tag{2.5}$$

holds for $a = 0$ and $b > 0$.

This lemma shows that the Hessian of $\phi(a,b)$ is positive semi-definite if (a,b) satisfies the strict complementarity condition.

From (2.3) and the fact that $\partial\phi(0,0)/\partial a = \partial\phi(0,0)/\partial b = 0$, we can see that $\phi(a,b)$ is not twice differentiable at $(0,0)$.

3. Unconstrained Methods for NCP

In the last section, we have studied the properties of a NCP function. Now we consider the following functions:

$$\phi(a,b,\alpha,\beta) = (\sqrt{a^2+\alpha b^2} - a)(\sqrt{\beta a^2+b^2} - b), \quad (a,b) \in R^2, \alpha > 0, \beta > 0, \tag{3.1}$$

where α and β are two positive parameters. Similarly we have the following result:

Lemma 3.1. *let $\phi(a,b)$ is defined by (3.1), the gradient of $\phi(a,b)$ equals to 0 if and only if (a,b) satisfies the complementarity condition.*

proof. The if part of the above lemma follows from inductive algebraic calculus.

If the gradient of $\phi(a,b)$ equals to 0, then either $a=b=0$ or that:

$$\frac{\partial\phi(a,b)}{\partial a} = (\sqrt{a^2+\alpha b^2}-a)(\frac{\beta a}{\sqrt{b^2+\beta a^2}}+\frac{b-\sqrt{b^2+\beta a^2}}{\sqrt{a^2+\alpha b^2}})=0, \quad (3.2)$$

$$\frac{\partial\phi(a,b)}{\partial b} = (\sqrt{\beta a^2+b^2}-b)(\frac{\alpha b}{\sqrt{\alpha b^2+a^2}}+\frac{a-\sqrt{\alpha b^2+a^2}}{\sqrt{\beta a^2+b^2}})=0. \quad (3.3)$$

Because $a^2+b^2\neq 0$, if $\sqrt{a^2+\alpha b^2}-a$ or $\sqrt{\beta a^2+b^2}-b$ equals to 0, (a,b) satisfies complementarity condition. Otherwise , by (3.2) and (3.3) we have:

$$\beta a\sqrt{\alpha b^2+a^2}+b\sqrt{b^2+\beta a^2}-(\beta a^2+b^2)=0, \quad (3.4)$$

$$a\sqrt{\alpha b^2+a^2}+\alpha b\sqrt{b^2+\beta a^2}-(a^2+\alpha b^2)=0. \quad (3.5)$$

It follows that:

$$(1-\alpha\beta)b(\sqrt{b^2+\beta a^2}-b) = 0, \quad (3.6)$$

$$(1-\alpha\beta)a(\sqrt{a^2+\alpha b^2}-a) = 0. \quad (3.7)$$

If $\alpha\beta$ equals to 1 , it reduced to the case of Lemma 2.1 except for a constant factor. If $\alpha\beta\neq 1$, $a=b=0$, it is a contradiction. From the above discussion we know that, if the gradient of $\phi(a,b)$ equals to 0, (a,b) satisfies the complementarity condition. This proves the lemma.

Also, if $a>0, b=0$, we have:

$$\frac{\partial^2\phi(a,b)}{\partial a^2}=0,\quad \frac{\partial^2\phi(a,b)}{\partial a\partial b}=0,\quad \frac{\partial^2\phi(a,b)}{\partial b^2}=\alpha\sqrt{\beta}. \quad (3.8)$$

If $a=0, b>0$, then:

$$\frac{\partial^2\phi(a,b)}{\partial a^2}=\beta\sqrt{\alpha},\quad \frac{\partial^2\phi(a,b)}{\partial a\partial b}=0,\quad \frac{\partial^2\phi(a,b)}{\partial b^2}=0. \quad (3.9)$$

We can view $\sqrt{a^2+b^2}-a$ or $\sqrt{a^2+b^2}-b$ as a relative distance, α and β are scaling elements which affect the distance. Larger the scaling element is, heavilier it affects. So it can be viewed as penalty parameters to some extent.

Let ϕ is defined as above, $\theta(t)$ is a strict increasing function, one can show easily that $\theta(\phi)$ is a NCP function.

We consider the function defined by (2.2) or (3.1). Define:

$$\psi(x)=\sum_{i=1}^n\psi_i=\sum_{i=1}^n\phi(x_i,F_i(x)),\ x\in R^n, \quad (3.10)$$

where $\psi_i=\phi(x_i,F_i(x))$. Then, the following lemma is obvious:

Lemma 3.2. *Let ϕ is defined by* (2.2) *or* (3.1)*, ψ is defined as in* (3.10)*, then $\psi(x)$ equals to 0 if and only if x is a solution of NCP.*

In order to apply the methods for unconstrained minimization, we still need some assumptions. The follows is some general assumptions in NCP. Let x^* is a solution of NCP, $I = \{1, \ldots, n\}$:

Assumptions.

(i): *F is twice differentiable in a neighborhood of x^*.*

(ii): *The gradients $\nabla F_i(x^*)$ $(i \in I^{**} := \{i \in I | x_i^* > 0\})$ and $e_i (i \notin I^{**})$ are linearly independent, where e_i denotes the i-th column of the identity matrix I_n.*

(iii): *$x_i^* + F_i(x^*) > 0$ $(i \in I)$, which is strict complementarity or nondegeneracy condition.*

It follows from Lemma 3.1 that the gradient of ψ at x^* is 0, for the Hessian of ψ, we have the following result:

Theorem 3.1. *Let $x^* \in R^n$ be a solution of* (1.1)*. Suppose that the assumptions i-iii hold at x^*, let ψ is defined by* (3.10) *and ϕ is defined by* (2.2) *or* (3.1)*. Then the Hessian of ψ at x^* is positive definite.*

proof. First we consider the case that ϕ is defined by (2.2). From assumptions $i - iii$, one can see that ψ is twice differentiable in a neighborhood of x^*. Furthermore:

$$\begin{aligned}\nabla^2 \psi = \sum_{i=1}^{n} \nabla^2 \psi_i &= \sum_{i=1}^{n} \frac{\partial^2 \phi(x_i, F_i(x))}{\partial a^2} e_i e_i^T + \sum_{i=1}^{n} \frac{\partial^2 \phi(x_i, F_i(x))}{\partial b^2} \nabla F_i(x^*) \nabla F_i(x^*)^T \\ &= \sum_{i \notin I^{**}} e_i e_i^T + \sum_{i \in I^{**}} \nabla F_i(x^*) \nabla F_i(x^*)^T. \qquad (3.11)\end{aligned}$$

Let $A = (e_i, i \notin I^{**}, \nabla F_i(x^*), i \in I^{**})$, then it follows from assumption 2 that A is nonsingular. Because $\nabla^2 \psi(x^*) = AA^T$, it is positive definite. Similarly one can show the same conclusion holds for (3.1). This prove the theorem.

From the above theorem, we can see that x^* is not only a global minimizer of $\psi(x)$, it is also a strict local minimizer of $\psi(x)$. By the continuity, ψ is twice differentiable in a neighborhood of x^*, so if the initial point x^0 is chosen very near x^*, we can use Newton or Quasi-Newton method to search the minimizer of $\psi(x)$, the local convergence properties is obvious.

If we choose $\theta(t) = t$, $\phi_i = \theta(x_i, F_i(x))$, it is just the case discussed above. For any increasing function, if $\dfrac{\partial \theta(0)}{\partial t} > 0$, let $\phi_i = \theta(x_i, F_i(x))$ and ψ is defined as in (3.10), under assumptions 1-3, Lemma 3.2 and Theorem 3.1 still hold.

4. Numerical Results

Three small numerical examples from the literature were used in our test with a FORTRAN subroutine. The methods we take is BFGS with inexact line search, which satisfies Wolfe-conditions. The criterion for stop is that $||g|| \leq 10^{-8}$. The following are examples:

Example 4.1.

$$F_1(x) = -x_2 + x_3 + x_4,$$
$$F_2(x) = x_1 - (4.5x_3 + 2.7x_4)/x_2,$$
$$F_3(x) = 5 - x_1 - (0.5x_3 + 0.3x_4)/x_3,$$
$$F_4(x) = 3 - x_1.$$

Table 1

Initial point	solution obtained	$\alpha = \beta$	No. of. iter	Val. of. ψ
(1,1,1,1)	(3.0,6.54331,1.09055,5.45276)	1.0	20/29/23	1.304E-19
	(3.0,8.10471,1.35078,6.75392)	2.0	23/44/24	0.0
(2,2,2,2)	(3.0,6.24492,1.04082,5.20410)	1.0	20/29/22	9.725E-19
	(3.0,8.29561,1.38260,6.91300)	2.0	24/33/26	0
(1,2,3,4)	(3.0,7.15159,1.29193,6.45966)	1.0	22/35/26	1.752E-18
	(3.0,9.42643,1.57107,7.85536)	2.0	21/45/25	0
(10,10,10,10)	(3.0,7.40562,1.23427,6.75392)	1.0	23/47/30	2.37E-19
	(3.0,7.43234,1.23872,6.19361)	2.0	19/46/28	0

Example 4.2.

$$F_1(x) = 3x_1^2 + 2x_1x_2 + 2x_2^2 + x_3 + 3x_4 - 6,$$
$$F_2(x) = 2x_1^2 + x_1 + x_2^2 + 10x_3 + 2x_4 - 2,$$
$$F_3(x) = 3x_1^2 + x_1x_2 + 2x_2^2 + 2x_3 + 9x_4 - 9,$$
$$F_4(x) = x_1^2 + 3x_2^2 + 2x_3 + 3x_4 - 3.$$

Table 2

Initial point	solution obtained	$\alpha = \beta$	No. of. iter	Val. of. ψ
(1,1,1,1)	(1.22474,0,0,0.5)	1.0	30/50/40	8.243E-14
	(1,0,3,0)	2.0	26/43/33	0.0
(0,0,0,0)	(1.22474,0,0,0.5)	1.0	53/96/64	2.294E-12
	(1,0,3,0)	2.0	31/53/39	0
(-1,1,1,-1)	(0.30363E-2,2.12589,-0.27267,0.12826)	1.0	30/46/36	6.299E-2
	(-0.68278,-0.41662,6.27219E-2,1.25712)	2.0	29/52/34	2.6419
(10,10,10,10)	(1.22474,0,0,0.5)	1.0	39/72/52	1.529E-13
	(1,0,3,0)	2.0	32/64/37	0
(100,100,100,100)	(1.22474,0,0,0.5)	1.0	48/84/62	6.273E-13
	(1.22474,0,0,0.5)	2.0	43/73/50	1.977E-19

Example 4.3.

$$F_1(x) = x_1 - 5x_2 - 1,$$
$$F_2(x) = x_1 + x_2,$$
$$F_3(x) = -3x_1 - 3x_2 + x_3 + 2x_4 - x_5,$$

$$F_4(x) = -4x_1 - 4x_2 + 2x_3 + x_4 + 2x_5,$$
$$F_5(x) = -5x_1 - 5x_2 - x_3 + 4x_4 + 3x_5.$$

Table 3

Initial point	solution obtained	$\alpha = \beta$	No. of. iter	Val. of. ψ
(1,1,1,1,1)	(1,0,0,4,0)	1.0	31/42/35	0
	(1,0,0,4,0)	2.0	28/40/30	0.0
	(1,0,0,4,0)	4.0	25/49/28	0
(0,0,0,0,0)	(1,0,0,4,0)	1.0	29/37/30	0
	(1,0,0,4,0)	2.0	25/43/30	0
	(1,0,0,4,0)	4.0	25/40/29	0
(0,1,2,3,4)	(1,0,7,0,4)	1.0	32/52/40	0
	(1,0,7,0,4)	2.0	41/65/47	0
	(1,0,1,1.2,0.4)	4.0	35/64/45	0
(10,10,10,10,10)	fails	1.0		
	(1,0,0,4,0)	2.0	30/42/33	0
	(1,0,0,4,0)	4.0	30/57/34	0
(100,100,100,100,100)	(1,0,0,4,0)	1.0	32/49/39	0
	(1,0,1,1.2,0.4)	2.0	42/67/54	0

The example 4.1 has multiple nondegenerate solution. Table 1 summarizes the results for different initial point. From each of these points, the methods converge to different solution with various choice of α and β.

The numerical results of example 4.2 are interesting, because from an initial point, both methods converges to stationary points of the objective function, but the stationary points is not a solution for (1.1). It verifies the doubt of [4] (Question 6.2), when is a stationary point of ψ a solution of (1.1)? From a view of computation, we can use the scaling technique, which we use $\psi(x) = \sum_{i=1}^{n} c_i\psi_i$, c_i are positive constants. From Lemma 2.1, one can see that if a stationary point of (3.10) is not a solution of (1.1), it is not a stationary point of the scaling function with carefully chosen c_i. Much work is needed for such a method which will be our further studies.

The example 4.3 shows that, the choice of α and β may affect the number of iteration and solution obtained, sometimes it cause the sequence does not converge. We also notice that the function defined by (2.2) is similar to a function in [2], where $\eta = \sqrt{a^2+b^2} - a - b$ is used. It is easy to show that:

$$(\sqrt{a^2+b^2} - a) \times (\sqrt{a^2+b^2} - b) = \frac{1}{2}(\sqrt{a^2+b^2} - a - b)^2.$$

In [2], Newton's method without line search is used and better numerical results are shown, by the above equality, our method is trying to minimizing η^2. So if we using a partial BFGS method carefully, our numerical results should be comparable to that in [2].

Acknowledgements. We thank Professor Y. Yuan for his encouragements.

References

[1] P.T. Harker and J.S. Pang, Finite dimensional variational inequality and nonlinear complementarity problems: A survey of theory , algorithms and applications, *Math. Prog.*, 48 (1990), 161–220.

[2] C. Kanzow, Some equation-based methods for the nonlinear complementarity problem, Tech. Report, Reihe A, Preprint 63. Institute of applied mathematics, University of Hamburg, Hamburg, Germany, 1993.

[3] C. Kanzow, Newton-type methods for nonlinear constrained optimization, Tech. Report, Reihe A, Preprint 62. Institute of applied mathematics, University of Hamburg, Hamburg, Germany, 1993.

[4] O.L. Mangasarian and M.V. Solodov, Nonlinear complementarity as unconstrained and constrained minimization, *Math. Prog.*, 62 (1993), 277–297.

[5] K. Taji, M. Fukushima and T. Ibaraki, A globally convergent Newton method for solving strongly monotone variational inequalities, *Math. Prog.*, 58 (1993), 369–383.