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OPTIMAL-ORDER PARAMETER IDENTIFICATION IN SOLVING NONLINEAR SYSTEMS IN A BANACH SPACE*)

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Abstract

We study the sufficient and necessary conditions of the convergence for parameter-based rational methods in a Banach space. We derive a closed form of error bounds in terms of a real parameter λ $(1 \leq \lambda < 2)$. We also discuss some behaviors when the family is applied to abstract quadratic functions on a Banach space for $\lambda = 2$.

1. Introduction

We consider the problem of solving

$$F(x) = 0, \tag{1}$$

where $F : D \subset X \to Y$ is a nonlinear differential operator defined on some convex subset D of a Banach space X with values in a Banach space Y. Many problems of applied mathematics can be brought in the form of equation (1). (see Ortega and Rheinboldt [1970], Lancaster [1977], Dennis and Schnabel [1983], Cuyt and Rall [1985], Laub [1991], etc.) A well-known method for solving (1) is the third-order Halley. Given an approximation x_k, compute x_{k+1} by

$$x_{k+1} = x_k - [F'(x_k) - \frac{1}{2}F''(x_k)F'(x_k)^{-1}F(x_k)]^{-1}F(x_k), \tag{2}$$

Recent years, Kantorovich-type convergence (sufficient conditions for the convergence) of the Halley method in Banach space setting has been mentioned by many authors: Candela and Marquina [1990], and Kanno [1992]. In this paper, we introduce a real parameter λ and design a new parameter-based rational iterations in Banach spaces as follows:

$$y_k = x_k - F'(x_k)^{-1}F(x_k)$$

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$$H(x_k, y_k) = F'(x_k)^{-1}F''(x_k)(y_k - x_k)$$
$$x_{k+1} = y_k - \frac{1}{2}H(x_k, y_k)[I + \frac{\lambda}{2}H(x_k, y_k)]^{-1}(y_k - x_k), \tag{3}$$

which include the Halley method as a specific choice of the parameter. We will not only provide a complete Kantorovich-type convergence analysis as well as a local convergence for this one-parameter family for $1 \le \lambda < 2$ but also we point out that the maximum order of convergence for the iteration at $\lambda = 2$ is greater than the famous conjecture by Traub [15]. The conjecture states that their maximum order of convergence is three, but we will show that it is of order four.

2. Sufficient Conditions for the Convergence

We first need a lemma.

Lemma 2.1. *Let $F(x)$ be a nonlinear operator from an open convex domain D in a Banach space X to another Banach space Y. Suppose that F has 2nd order continuous Frechet derivatives on D. Then the $F(x_{k+1})$ together with the sequence $\{x_k\}_{k=0}^{\infty}$ generated by (3) has the following approximation for all $k \ge 0$ and $1 \le \lambda \le 2$,*

$$F(x_{k+1}) = \int_0^1 F''[y_k + t(x_{k+1} - y_k)](1-t)dt(x_{k+1} - y_k)^2 - \frac{1}{2}\int_0^1 [F''[x_k + t(y_k - x_k)]$$
$$[1 - \lambda(1-t)]dt(y_k - x_k)H(x_k, y_k)[I + \frac{\lambda}{2}H(x_k, y_k)]^{-1}(y_k - x_k)$$
$$+ \int_0^1 \{F''[x_k + t(y_k - x_k)](1-t) - \frac{1}{2}F''(x_k)\}dt(y_k - x_k)$$
$$\times [I + \frac{\lambda}{2}H(x_k, y_k)]^{-1}(y_k - x_k). \tag{4}$$

Now we can state our main result.

Theorem 2.1. *Let $F(x) : D \subset X \to Y$, X and Y are real or complex Banach spaces, and D is an open convex domain. Assume that F has 2nd order continuous Frechet derivatives on D and satisfies the following standard Newton-Kantorovich conditions:*

$$\| F''(x) \| \le M, \| F''(x) - F''(y) \| \le N \| x - y \|, \; for \; all \; x, y \in D. \tag{5}$$

For a given initial value $x_0 \in D$, assume that $F'(x_0)^{-1}$ exists and satisfies

$$\| F'(x_0)^{-1} \| \le \beta, \;\; \| F'(x_0)^{-1}F(x_0) \| \le \eta, \tag{6}$$

$$M[1 + \frac{2N}{3(2-\lambda)M^2\beta}]^{1/3} \le K, \; 1 \le \lambda < 2, \tag{7}$$

$$h = K\beta\eta \le 0.5, \tag{8}$$

$$\overline{S(x_0, t^*)} \subset D, \tag{9}$$

where $\overline{S(x,r)} = \{x' \in x \,|\| x' - x \| \leq r\}$ *and*

$$g(t) = \frac{1}{2}Kt^2 - \frac{1}{\beta}t + \frac{\eta}{\beta}, \tag{10}$$

$$t^* = \frac{1-\sqrt{1-2h}}{h}\eta,\ t^{**} = \frac{1+\sqrt{1-2h}}{h}\eta,\ \Theta = \frac{1-\sqrt{1-2h}}{1+\sqrt{1-2h}}, \tag{11}$$

where t^* *is the smallest root of equation (10).*

Then procedures (3) *are convergent for all* $1 \leq \lambda < 2$*, and* x_k*,* $y_k \in \overline{S(x_0, t^*)}$ *for all* $k \in N_0$*. The limit* x^* *is a unique solution of equation (1) in* $\overline{S(x_0, t^*)}$*. We have the following optimal error bound estimate for all* $k \geq 1$*:*

$$\| x_k - x^* \| \leq t^* - t_k \leq \frac{(1-\Theta^2)\eta}{1-\Theta^{3^k}}\Theta^{3^k-1}, \tag{12}$$

for all λ *in* $[1,2)$*, where* $\{t_k\}$ *and* $\{s_k\}_{k=0}^{\infty}$ *are defined as*

$$s_k = t_k - \frac{g(t_k)}{g'(t_k)},$$

$$h_g(t_k, s_k) = \frac{g''(t_k)(s_k - t_k)}{g'(t_k)},$$

$$t_{k+1} = s_k - \frac{1}{2}\frac{h_g(t_k,s_k)(s_k - t_k)}{1 + \frac{\lambda}{2}h_g(t_k,s_k)}. \tag{13}$$

Proof. It suffices to show that the following items are true for all $k \geq 0$ by mathematical induction.

$$x_k \in \overline{S(x_0, t_k)}; \tag{14}$$

$$\| F'(x_k)^{-1} \| \leq -g'(t_k)^{-1}; \tag{15}$$

$$\| y_k - x_k \| \leq s_k - t_k; \tag{16}$$

$$y_k \in \overline{S(x_0, s_k)}; \tag{17}$$

$$\| x_{k+1} - y_k \| \leq t_{k+1} - s_k. \tag{18}$$

It is easy to check in the case when $k = 0$ by the initial conditions. Now assume that the above statements are true for a fixed $k \geq 1$. Then

$$\begin{aligned}\| x_{k+1} - x_0 \| &\leq \| x_{k+1} - y_k \| + \| y_k - x_k \| + \| x_k - x_0 \| \\ &\leq (t_{k+1} - s_k) + (s_k - t_k) + (t_k - t_0) \\ &= t_{k+1}.\end{aligned}$$

It convinces that (14) is true when k is replaced by $k+1$.
Since

$$F'(x_{k+1}) - F'(x_0) = \int_0^1 F''[x_0 + t(x_{k+1} - x_0)]dt(x_{k+1} - x_0),$$

then

$$\begin{aligned}\| F'(x_{k+1}) - F'(x_0) \| &\le M \| x_{k+1} - x_0 \| \\ &\le K(t_{k+1} - t_0) \\ &= Kt_{k+1} \\ &< Kt^* \\ &= K\frac{1-\sqrt{1-2h}}{h}\eta \\ &= K\frac{1-\sqrt{1-2h}}{K\beta\eta}\eta \\ &= K\frac{1-\sqrt{1-2h}}{\beta} \\ &\le \frac{1}{\beta} \\ &\le \frac{1}{\| F'(x_0)^{-1} \|} \ ,\end{aligned}$$

and by the Banach lemma, $F'(x_{k+1})^{-1}$ exists and

$$\begin{aligned}\| F'(x_{k+1})^{-1} \| &\le \frac{\| F(x_0)^{-1} \|}{1- \| F'(x_0)^{-1} \| \ \| F'(x_{k+1}) - F'(x_0) \|} \\ &\le \frac{\beta}{1-\beta K \| x_{k+1} - x_0 \|} \\ &= \frac{1}{\frac{1}{\beta} - K \| x_{k+1} - x_0 \|} \\ &\le \frac{1}{\frac{1}{\beta} - K(t_{k+1} - t_0)} \\ &\le \frac{1}{\frac{1}{\beta} - Kt_{k+1}} \\ &= -\frac{1}{g'(t_{k+1})}.\end{aligned}$$

Which implies that (15) is true when k is replaced by $k+1$. By using the identity (4), we can estimate $F(x_{k+1})$ to obtain

$$\begin{aligned}\| F(x_{k+1}) \| &\le \frac{M}{2}\| x_{k+1} - y_k \|^2 + [\frac{1}{2} - \frac{\lambda}{4}]\frac{\frac{M^2}{\frac{1}{\beta}-M\|x_k-x_0\|} \| y_k - x_k \|^3}{1-\frac{\lambda}{2}\frac{M\|y_k-x_k\|}{\frac{1}{\beta}-M\|x_k-x_0\|}} \\ &+ \frac{N}{6}\frac{\| y_k - x_k \|^3}{1-\frac{\lambda}{2}\frac{M\|y_k-x_k\|}{\frac{1}{\beta}-M\|x_k-x_0\|}} \\ &\le \frac{M}{2} \| x_{k+1} - y_k \|^2 + \frac{\frac{\frac{(2-\lambda)M^2}{4}+\frac{N}{6\beta}}{\frac{1}{\beta}-M\|x_k-x_0\|} \| y_k - x_k \|^3}{1-\frac{\lambda}{2}\frac{M\|y_k-x_k\|}{\frac{1}{\beta}-M\|x_k-x_0\|}}\end{aligned}$$

$$\begin{aligned}
&\leq \frac{M}{2} \parallel x_{k+1} - y_n \parallel^2 + \frac{\frac{(2-\lambda)M^2/4+N/6\beta}{1/\beta - M\|x_k - x_0\|} \parallel y_k - x_k \parallel^3}{1 - \frac{\lambda}{2}\frac{M\|y_k - x_k\|}{1/\beta - M\|x_k - x_0\|}} \\
&\leq \frac{K}{2}(t_{k+1} - s_k)^2 + (2-\lambda)\frac{K^2}{4}\frac{\frac{(s_k - t_k)^3}{\frac{1}{\beta} - Kt_n}}{1 - \frac{\lambda}{2}\frac{K(s_k - t_k)}{\frac{1}{\beta} - Kt_k}} \\
&= g(t_{k+1}) \ ,
\end{aligned}$$

which yields

$$\begin{aligned}
\parallel y_{k+1} - x_{k+1} \parallel &= \parallel -F'(x_{k+1})^{-1} F(x_{k+1}) \parallel \\
&\leq \parallel F'(x_{k+1})^{-1} \parallel \parallel F(x_{k+1}) \parallel \\
&\leq -\frac{g(t_{k+1})}{g'(t_{k+1})} \\
&= s_{k+1} - t_{k+1}.
\end{aligned}$$

Then (16) is true when k is replaced by $k+1$. Now we conclude

$$\begin{aligned}
\parallel y_{k+1} - x_0 \parallel &\leq \parallel y_{k+1} - x_k \parallel + \parallel x_{k+1} - y_k \parallel + \parallel y_k - x_k \parallel + \parallel x_n - x_0 \parallel \\
&\leq (s_{k+1} - t_{k+1}) + (t_{k+1} - s_k) + (s_k - t_k) + (t_k - t_0) \\
&= s_{k+1}.
\end{aligned}$$

So is (17). Finally we use the fact that

$$\begin{aligned}
\parallel \frac{\lambda}{2} F'(x_{k+1})^{-1} F''(x_{k+1})(y_{k+1} - x_{k+1}) \parallel &\leq \frac{\lambda}{2} \parallel F'(x_{k+1})^{-1} \parallel \parallel F''(x_{k+1}) \parallel \parallel y_{k+1} - x_{k+1} \parallel \\
&\leq \frac{\lambda}{2}\frac{K(s_{k+1} - t_{k+1})}{-g'(t_{k+1})} \\
&\leq \frac{K(s_{k+1} - t_{k+1})}{\frac{1}{\beta} - Kt_{k+1}} \\
&< 1,
\end{aligned}$$

and conclude that $[1 + \frac{\lambda}{2} F'(x_{k+1})^{-1} F''(x_{k+1})(y_{k+1} - x_{k+1})]^{-1}$ exists and

$$\begin{aligned}
&\parallel [1 + \frac{\lambda}{2} F'(x_{k+1})^{-1} F''(x_{k+1})(y_{k+1} - x_{k+1})]^{-1} \parallel \\
&\leq [1 - \frac{\lambda}{2} \parallel F'(x_{k+1})^{-1} \parallel \parallel F''(x_{k+1}) \parallel \parallel y_{k+1} - x_{k+1} \parallel]^{-1} \\
&\leq [1 + \frac{\lambda}{2} g'(t_{k+1})^{-1} K(s_{k+1} - t_{k+1})]^{-1} \\
&= [1 + \frac{\lambda}{2} g'(t_{k+1})^{-1} g''(t_{k+1})(s_{k+1} - t_{k+1})]^{-1} \\
&= [1 + \frac{\lambda}{2} h_g(t_{k+1}, s_{k+1})]^{-1}.
\end{aligned}$$

From (3), we shall have

$$x_{k+2} - y_{k+1} = -\frac{1}{2}[I + \frac{\lambda}{2}H(x_{k+1}, y_{k+1})]^{-1}H(x_{k+1}, y_{k+1})(y_{k+1} - x_{k+1})$$

and then

$$\begin{aligned}\| x_{k+2} - y_{k+1} \| &\leq \frac{1}{2} \| [I + \frac{\lambda}{2}H]^{-1} \|\| F'(x_{k+1})^{-1} \|\| F''(x_{k+1}) \|\| y_{k+1} - x_{k+1} \|^2 \\ &\leq -\frac{1}{2}[1 + \frac{\lambda}{2}h_g(t_{k+1}, s_{k+1})]^{-1}h_g(t_{k+1}, s_{k+1})(s_{k+1} - t_{k+1}) \\ &= t_{k+2} - s_{k+1},\end{aligned}$$

which convinces that (18) is true when k is replaced by $k+1$. Now we are ready to prove (13). Notice that

$$g(t_k) = \frac{K}{2}(t^* - t_k)(t^{**} - t_k),$$

and

$$g'(t_k) = -\frac{K}{2}[(t^* - t_k) + (t^{**} - t_k)].$$

For convenience, we denote $a_k = t^* - t_k$, $b_k = t^{**} - t_k$. Then we have

$$g(t_k) = \frac{K}{2}a_k b_k,$$

$$g'(t_k) = -\frac{K}{2}(a_k + b_k),$$

$$b_k = a_k + \frac{(1 - \Theta^2)\eta}{\Theta}.$$

Now from (13), we have

$$\begin{aligned}a_k &= a_{k-1} - \frac{a_{k-1}b_{k-1}(a_{k-1} + b_{k-1})^2 + (1 - \lambda)a_{k-1}^2 b_{k-1}^2}{(a_{k-1} + b_{k-1})^3 - \lambda a_{k-1}b_{k-1}(a_{k-1} + b_{k-1})} \\ &= \frac{a_{k-1}^4 + (2 - \lambda)a_{k-1}^3 b_{k-1}}{(a_{k-1} + b_{k-1})^3 - \lambda a_{k-1}b_{k-1}(a_{k-1} + b_{k-1})}.\end{aligned}$$

By the similar way, we have an expression of b_k,

$$b_k = \frac{b_{k-1}^4 + (2 - \lambda)b_{k-1}^3 a_{k-1}}{(a_{k-1} + b_{k-1})^3 - \lambda a_{k-1}b_{k-1}(a_{k-1} + b_{k-1})}.$$

Hence we obtain

$$\begin{aligned}\frac{a_k}{b_k} &= \{\frac{a_{k-1}}{b_{k-1}}\}^3 \frac{a_{k-1} + (2 - \lambda)b_{k-1}}{b_{n-1} + (2 - \lambda)a_{k-1}} \\ &= \{\frac{a_{k-1}}{b_{k-1}}\}^3 \frac{\frac{a_{k-1}}{b_{k-1}} + (2 - \lambda)}{1 + (2 - \lambda)\frac{a_{k-1}}{b_{k-1}}}.\end{aligned}$$

Note that $0 \leq \frac{a_{k-1}}{b_{k-1}} \leq \theta \leq 1$, and if $1 \leq \lambda < 2$, then

$$\frac{\frac{a_{k-1}}{b_{k-1}} + (2-\lambda)}{1 + (2-\lambda)\frac{a_{k-1}}{b_{k-1}}} \leq 1.$$

Which implies

$$\frac{a_k}{b_k} \leq \{\frac{a_{k-1}}{b_{k-1}}\}^3,$$

and we solve this equation for a_k and obtain

$$a_k = t^* - t_k \leq \frac{(1-\Theta^2)\eta}{1-\Theta^{3^k}}\Theta^{3^k-1}.$$

Finally to show uniqueness, we assume that there exists a second solution y^* of the equation (1) in $\overline{S(x_0, t^*)}$. We now obtain the estimate:

$$\| F'(x_0)^{-1} \| \int_0^1 \| F'(x^* + t(y^* - x^*)) - F'(x_0) \| dt$$

$$\begin{aligned}
&\leq \beta M \int_0^1 \| x^* + t(y^* - x^*) - x_0 \| dt \\
&\leq M\beta \int_0^1 [(1-t) \| x^* - x_0 \| + t \| y^* - x_0 \|] dt \\
&\leq \frac{1}{2} M\beta(t^* + t^*) \\
&< 1.
\end{aligned}$$

Hence the linear operator $\int_0^1 F'(x^* + t(y^* - x^*))dt$ is invertible. It follows from the approximation

$$\int_0^1 F'(x^* + t(y^* - x^*))dt(y^* - x^*) = F(y^*) - F(x^*) = 0,$$

that $x^* = y^*$. The proof of the theorem is now completed.

3. Necessary Conditions for the Convergence

Assume that $F : D \subset X \to Y$ has the following property: there is an $x^* \in D$ such that $F(x^*) = 0$ and $F'(x^*)$ is nonsingular.

Theorem 3.1. *Assume that the oporator F satisfies* (6) *and* (7) *in Theorem* 2.1 *and there exist δ and β^* such that*

$$\| x_k - x^* \| \leq \delta, \ \| F'(x^*)^{-1} \| \leq \beta^*, \tag{19}$$

then x_{k+1} is well defined and converges to x^* with order of convergence four. We have also the following error estimate:

$$\| x_{k+1} - x^* \| \leq C^* \| x_k - x^* \|^3, \tag{20}$$

where C^* is an expression in terms of M, β^* , δ and λ.

Proof. Using the fact that

$$\| F'(x_k) - F'(x^*) \| \leq M \| x_k - x^* \| \leq M\delta,$$

we can choose $\delta > 0$ such that $\delta \leq \frac{1}{3M\beta^*}$. Thus $F'(x_k)^{-1}$ exists and

$$\| F'(x_k)^{-1} \| \leq \frac{\| F'(x^*)^{-1} \|}{1 - \| F'(x^*)^{-1} \| \| F'(x_k) - F'(x^*) \|}$$

$$\leq \frac{\beta^*}{1 - \beta^* M\delta} = \frac{1}{\frac{1}{\beta^*} - M\delta} < \frac{3}{2}\beta^*,$$

then

$$\| H(x_k, y_k) \| \leq \frac{3}{4} M\beta^* \| y_k - x_k \| .$$

Now we can estimate the distance between x_{k+1} and y_k.

$$\| x_{k+1} - y_k \| \leq \frac{1}{2} \| H(x_k, y_k) \| [1 - \frac{\lambda}{2} \| H(x_k, y_k) \|]^{-1} \| y_k - x_k \|$$

$$\leq \frac{\frac{2}{3} M\beta^*}{1 - \frac{2\lambda}{3} M\beta^* \| y_k - x_k \|} \| y_k - x_k \|^2 .$$

Here we need an identitity due to Dennis and Schnabel [6] that

$$x^* - y_k = -F'(x_k)^{-1} \int_0^1 F''(x^* + t(x_k - x^*))tdt(x^* - x_k)^2.$$

We have

$$\| y_k - x^* \| \leq \frac{3}{4} M\beta^* \delta^2,$$

and

$$\| y_k - x_k \| \leq (\frac{3}{4} M\beta^* \delta + 1)\delta,$$

this gives

$$\| H(x_k, y_k) \| \leq \frac{2}{3} M\beta^* \delta (\frac{3}{4} M\beta^* \delta + 1),$$

and

$$\| x_{k+1} - y_k \| \leq \frac{\frac{3}{4} M\beta^*}{1 - \frac{2\lambda}{3} M\beta^* \delta (\frac{3}{4} M\beta^* \delta + 1)} \| y_k - x_k \|^2 .$$

Following (3), we have

$$\| F(x_{k+1}) \| \leq \frac{M}{2} \| x_{k+1} - y_k \|^2 + (\frac{1}{2} - \frac{\lambda}{4}) \frac{\frac{2}{3} M\beta^*}{1 - \frac{2\lambda}{3} M\beta^* \delta (\frac{3}{4} M\beta^* \delta + 1)} \| y_k - x_k \|^3$$

$$+ \frac{\frac{N}{6} \| y_k - x_k \|^3}{1 - \frac{2\lambda}{3} M\beta^* \delta (\frac{3}{4} M\beta^* \delta + 1)}$$

$$
\begin{aligned}
&\leq \frac{\frac{M}{2}(\frac{3}{4})^2 M^2 \beta^{*2} \parallel y_k - x_k \parallel^4}{[1-\frac{2\lambda}{3}M\beta^*\delta(\frac{3}{4}M\beta^*\delta+1)]^2} \\
&\quad + \frac{\frac{2-\lambda}{6}M\beta^* \parallel y_k - x_k \parallel^3}{1-\frac{2\lambda}{3}M\beta^*\delta(\frac{3}{4}M\beta^*\delta+1)} + \frac{\frac{N}{6} \parallel y_k - x_k \parallel^3}{1-\frac{2\lambda}{3}M\beta^*\delta(\frac{3}{4}M\beta^*\delta+1)} \\
&\leq [\frac{\frac{9M^3\beta^{*2}}{32} \parallel y_k - x_k \parallel}{(1-\frac{2\lambda}{3}M\beta^*\delta(\frac{3}{4}M\beta^*\delta+1))^2} + \frac{\frac{2-\lambda}{6}M\beta^*}{1-\frac{2\lambda}{3}M\beta^*\delta(\frac{3}{4}M\beta^*\delta+1)} \\
&\quad + \frac{\frac{N}{6}}{1-\frac{2\lambda}{3}M\beta^*\delta(\frac{3}{4}M\beta^*\delta+1)}] \parallel y_k - x_k \parallel^3 \\
&= C \parallel y_k - x_k \parallel^3 .
\end{aligned}
$$

On the other hand, by the continuity of $F'(x^*)^{-1}$, there is an α such that

$$\parallel F(x_{k+1}) - F(x^*) \parallel \geq \alpha \parallel x_{k+1} - x^* \parallel,$$

and so

$$
\begin{aligned}
\frac{\parallel F(x_{k+1}) \parallel}{\parallel y_k - x_k \parallel^3} &= \frac{\parallel F(x_{k+1}) - F(x^*) \parallel}{\parallel y_k - x_k \parallel^3} \\
&\geq \frac{\alpha \parallel x_{k+1} - x^* \parallel}{\parallel y_k - x_k \parallel^3} \\
&\geq \frac{\alpha \parallel x_{k+1} - x^* \parallel}{[\parallel y_k - x^* \parallel + \parallel x_k - x^* \parallel]^3} \\
&= \alpha \frac{\frac{\|x_{k+1}-x^*\|}{\|x_k-x^*\|^3}}{[1+\frac{\|y_k-x^*\|}{\|x_k-x^*\|}]^3}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\frac{\parallel x_{k+1} - x^* \parallel}{\parallel x_k - x^* \parallel^3} &\leq \frac{1}{\alpha}[1+\frac{\parallel y_k - x^* \parallel}{\parallel x_k - x^* \parallel}]^3 \frac{\parallel F(x_{k+1}) \parallel}{\parallel y_k - x_k \parallel^3} \\
&= C\frac{1}{\alpha}[1+\frac{\parallel y_k - x^* \parallel}{\parallel x_k - x^* \parallel}]^3 ,
\end{aligned}
$$

which gives

$$\frac{\parallel x_{k+1} - x^* \parallel}{\parallel x_k - x^* \parallel^3} \leq \frac{C}{\alpha}[1 + M\beta^* \parallel x_k - x^* \parallel] \leq \frac{C}{\alpha}(1+M\beta^*\delta) = C^*.$$

That is (20). The theorem is now proved.

4. A New Method of Order Four

In order to increase the order of convergence, we select $\lambda = 2$ and construct a new Halley-type method as follows:

$$y_k = x_k - F'(x_k)^{-1}F(x_k)$$

$$H(x_k, y_k) = F'(x_k)^{-1}F''(x_k)(y_k - x_k)$$

$$x_{k+1} = y_k - \frac{1}{2}H(x_k, y_k)[I + H(x_k, y_k)]^{-1}(y_k - x_k). \tag{21}$$

We for the first time point out that the maximum order of convergence of this method could reach up to 4 when the function $F(x)$ is any quadractic function in a Banach space. Let $F(x)$ be a quadratic operator in an open convex domain D in a Banach space X to another Banach space Y with the form of

$$F(x) = \frac{M_2}{2}x^2 + M_1x + M_0, \tag{22}$$

where M_i is the i^{th} symmetric bilinear operator defined in [14]. Assume that $F'(x^*)^{-1}$ exists and satisfy

$$\| F'(x^*)^{-1} \| \leq \beta^*, \quad \| x_k - x^* \| \leq \delta. \tag{23}$$

From (4) and $\lambda = 2$, we obtain

$$F(x_{k+1}) = \int_0^1 F''(x_k + t(x_{k+1} - y_k))(1-t)dt(x_{k+1} - y_k)^2$$

$$-\frac{1}{2}\int_0^1 F''(x_k + t(y_k - x_k))(2t-1)dt(y_k - x_k)H(x_k, y_k)[I + H(x_k, y_k)]^{-1}$$

$$+\int_0^1 \{F''(x_k + t(y_k - x_k))(1-t) - \frac{1}{2}F''(x_k)\}(y_k - x_k)[I + H(x_k, y_k)]^{-1}dt(y_k - x_k).$$

We apply this approximation to a general quadratic operator (20) and obtain

$$\begin{aligned} F(x_{k+1}) &= \frac{M_2}{2}(x_{k+1} - y_k)^2 \\ &= \frac{M_2}{2}H(x_k, y_k)\{-\frac{1}{2}[I + H(x_k, y_k)]^{-1}(y_k - x_k)\}^2, \end{aligned}$$

and

$$\begin{aligned} \| F(x_{k+1}) \| &\leq \frac{\| M_2 \|}{2}\{\frac{1}{1 - \| H(x_k, y_k) \|}\frac{\| M_2 \|}{2} \| F'(x_k)^{-1} \| \| y_k - x_k \|^2\}^2 \\ &\leq \frac{\| M_2 \|^3}{8}\frac{\| F'(x_k)^{-1} \|}{(1 - \| H(x_k, y_k) \|)^2} \| y_k - x_k \|^4, \end{aligned}$$

that is

$$\frac{\| F(x_{k+1}) \|}{\| y_k - x_k \|^4} \leq \frac{\| M_2 \|}{8}\frac{\| F'(x_k)^{-1} \|^2}{(1 - \| H(x_k, y_k) \|)^2}.$$

Since

$$\begin{aligned} \| F'(x_k) - F'(x^*) \| &\leq \| M_2 \| \| x_k - x^* \| \\ &\leq \| M_2 \| \delta, \end{aligned}$$

and we can choose a $\delta > 0$ such that $\| M_2 \|\delta \leq \frac{1}{3\beta^*}$, then by the Banach lemma, $F'(x_k)^{-1}$ exists and

$$\begin{aligned}\| F'(x_k)^{-1} \| &\leq \frac{\| F'(x^*) \|}{1- \| F'(x^*) \|\| F'(x_k) - F'(x^*) \|}\\ &\leq \frac{\beta^*}{1- \| M_2 \| \beta^*\delta}\\ &\leq \frac{3}{2}\beta^*,\end{aligned}$$

and

$$\begin{aligned}\| H(x_k, y_k) \| &\leq \frac{3}{2} \| M_2 \| \beta^* \| y_k - x_k \|\\ &\leq \frac{3}{2} \| M_2 \| \beta^*(\| y_k - x^* \| + \| x_k - x^* \|).\end{aligned}$$

In order to estimate the distance between y_k and x^*, we need an identity that

$$x^* - y_k = -F'(x_k)^{-1} \int_0^1 F''(x^* + t(x_k - x^*))tdt(x^* - x_k)^2.$$

Now we shall have

$$\| y_n - x^* \|\leq \frac{3}{2} \| M_2 \| \beta^*\delta^2,$$

and

$$\| H(x_k, y_k) \|\leq \frac{3}{2} \| M_2 \| \beta^*\delta(\frac{3}{2} \| M_2 \| \beta^*\delta + 1).$$

this gives

$$\frac{\| F(x_{k+1}) \|}{\| y_k - x_k \|^4} \leq \frac{1}{8}\frac{\frac{9}{4} \| M_2 \| \beta^{*2}}{(1 - (\frac{3}{2} \| M_2 \| \beta^*\delta(\frac{3}{2} \| M_2 \| \beta^*\delta + 1))^2}.$$

On the other hand, by the continuity of $F'(x^*)^{-1}$, there is a $\gamma > 0$ such that

$$\| F(x_{k+1}) - F(x^*) \|\geq \gamma \| x_{k+1} - x^* \|,$$

and yields

$$\begin{aligned}\frac{\| F(x_{k+1}) \|}{\| y_k - x_k \|^4} &= \frac{\| F(x_{k+1}) - F(x^*) \|}{\| y_k - x_k \|^4}\\ &\geq \frac{\gamma \| x_{k+1} - x_* \|}{\| y_k - x_k \|^4}\end{aligned}$$

$$\geq \gamma\frac{\| x_{k+1} - x^* \|}{\{\| x_k - x^* \| + \| y_k - x^* \|\}^4}.$$

Hence

$$\frac{\| x_{k+1} - x^* \|}{\| x_k - x^* \|^4} \leq \frac{1}{\gamma}\{1 + \frac{\| y_k - x^* \|}{\| x_k - x^* \|}\}^4\frac{\| F(x_{k+1}) \|}{\| x_{k+1} - x_k \|^4}.$$

It follows immediately

$$\| x_{k+1} - x_* \| \leq C^* \| x_k - x^* \|^4, \tag{24}$$

where $C^* = \frac{1}{\gamma}(1 + \frac{3}{2} \| M_2 \| \beta^* \delta) \frac{9 \| M_2 \|^3 \beta^{*2}}{32[1 - \frac{3}{2} \| M_2 \| \beta^* \delta(\frac{3}{2} \| M_2 \| \beta^* \delta + 1)]^2}.$ 5

In this section, we first use the Theorem 2.1 to suggest some new approaches to the solution of quadratic integral equations of the forms:

$$x(s) = y(s) + \alpha x(s) \int_0^1 q(s,t) x(t) dt, \tag{25}$$

in the space $X = C[0,1]$ of all continuous functions on the interval $[0,1]$ with the norm

$$\| x \| = \max_{0 \leq s \leq 1} | x(s) | . \tag{26}$$

Here, we assume that α is a real number called the "albedo" for scattering and the kernal $q(s,t)$ is a continuous function of two variables with $0 \leq s,t \leq 1$ and satisfying

$$0 < q(s,t) < 1, 0 \leq s,t \leq 1, \tag{27}$$

$$q(s,t) + q(t,s) = 1, 0 \leq s,t \leq 1. \tag{28}$$

The function $y(s)$ is given by a continuous function defined on $[0,1]$, and $x(s)$ is the unknown function sought in $[0,1]$. Equations of this type are related with the work of S. Chandrasokhar [3], and arise in the theories of radiative transfer, neutron transport and in the kinetic theory of gases. There exists an extensive literature on equations like (25) under various assumptions on the kernel $q(s,t)$ and α is a real or complex number. One can refer to the recent work in [1] and the references there. Here, we demonstrate that the theorem via the iterative procedures (3) provide existence results for (23). Moreover, the iterative procedures (3) converge faster than the solution of all the previous known ones. Furthermore, a better information on the location of the solution is given. Note that the cost is not higher than the corresponding one of previous methods. For simplicity, we shall assume that

$$q(s,t) = \frac{s}{s+t}, 0 \leq s,t \leq 1. \tag{29}$$

Notice that $q(s,t)$ satisfies (27) and (28) above. Let us now choose $y(s) = 1$ for all s in $[0,1]$ and define the operator F on $X = C[0,1]$ by

$$F(x) = \alpha x(s) \int_0^1 \frac{s}{s+t} x(t) dt - x(s) + 1. \tag{30}$$

Note that every root of the equation $F(x) = 0$ satisfies the equation(25). Set $x_0(s) = 1$ and $\alpha = 0.25$, use the definition of the first and second Frechet derivatives of the operator F to obtain

$$M = 2 | \alpha | \max_{0 \leq s \leq 1} | \int_0^1 \frac{s}{s+t} dt | = (2 \ln 2) | \alpha | = 0.34657359,$$

$$N = 0, K = M, \beta = \| F'(1)^{-1} \| = 1.53039421,$$

$$t^* = 0.28704852, \Theta = 0.08239685$$

and

$$\| x_k(\lambda) - x^* \| \leq \frac{(1-\Theta^2)\eta}{1-\Theta^{3^k}} \Theta^{3^k-1} = \frac{0.26339662}{1-(0.08239685)^{3^k}} (0.08239685)^{3^k-1}$$

for $1 \leq \lambda < 2$ which shows that x^* is unique in $\overline{S(x_0, t^*)}$. We now discuss the determination of the parameter λ so that the iterative procedures (3) will produce better solutions by spending the same amount of computations. Our numerical example do convince the above theoretical conclusions. Let us consider $F(x) = x^3 - 2x - 5$, where $x^* = 2.094551481$, and

$$E_0(\lambda) = \| x_0(\lambda) - x^* \|, \quad E_1(\lambda) = \| x_1(\lambda) - x^* \| . \tag{31}$$

We have the following numerical results.

Table

λ	x_0	x_1	$E_0(\lambda)$	$E_1(\lambda)$
1.0	2.0	2.0943396	$0.95 * 10^{-1}$	$0.21 * 10^{-3}$
2.0	2.0	2.0946429	$0.95 * 10^{-1}$	$0.91 * 10^{-4}$
3.0	2.0	2.0949152	$0.95 * 10^{-1}$	$0.36 * 10^{-3}$
4.0	2.0	2.0951612	$0.95 * 10^{-1}$	$0.61 * 10^{-3}$

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