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MATRIX ANALYSIS TO ADDITIVE SCHWARZ METHODS*[1)]

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Abstract

Matrix analysis on additive Schwarz methods as preconditioners is given in this paper. Both cases of with and without coarse mesh are considered. It is pointed out that an advantage of matrix analysis is to obtain more exact upper bound. Our numerical tests access the estimations.

1. Introduction

We consider the following second order elliptic boundary value problem:

$$\mathcal{L}u = f, \quad \text{in} \quad \Omega, \tag{1}$$

$$u = 0, \quad \text{on} \quad \partial\Omega, \tag{2}$$

where $\mathcal{L}$ is a self-adjoint positive operator and

$$\Omega \subset \mathcal{R}^d \quad (1 \le d \le 3)$$

is a polyhedral domain.

A weak solution has the following form: Find u $\in H_1^0(\Omega)$ such that :

$$\mathcal{A}(u, v) = f(v), \quad \forall v \in H_1^0(\Omega)$$

$$\mathcal{A}(u, v) = \int_\Omega \mathcal{L}u(x)v(x)dx, \quad f(v) = \int_\Omega f(x)v(x)dx.$$

Let $V^h := \mathcal{M} = \text{Span}\ \{\phi_i\}$, where $\{\phi_i\}$ could be nodal basis consisting of piece-wise linear functions or other spline functions. Substituting the following solution

$$u^h = \sum u_i \phi_i$$

into the above weak form leads to a discrete equation

$$Au = f, \tag{3}$$

where

$$A = (\alpha_{ij}), \quad \alpha_{ij} = \mathcal{A}(\phi_i, \phi_j). \tag{4}$$

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It is well known that the coefficient matrix A is symmetry positive definite matrix with condition number

$$\kappa(A) = O(h^{-2}). \tag{5}$$

When we use conjugate gradient algorithm for solving the system, for a given tolerance, the iteration number will proportional to h^{-1}. This convergent rate is really slow for a large scale problems. It is our purpose, in this paper, to do some analysis on Additive Schwarz Methods (ASM) as preconditioners in detail. In order to obtain estimation on condition number of the preconditioner system more accuracy, we take 1-D case as a model problem. The related results on higher dimension will be reported later.

2. A Projector Preconditioner

Suppose a subspace

$$\mathcal{M}_c := \mathrm{Span}\{\psi_k\} \subset \mathcal{M}$$

with the basis transformation

$$\psi_k = \sum t_{ki}\phi_i, \quad \Psi = T\Phi, T = (t_{ki}).$$

Define a projector $P_c : \mathcal{M} \to \mathcal{M}_c$ such that for any given $u \in \mathcal{M}$

$$A(P_c u, v) = A(u, v), \quad \forall v \in \mathcal{M}_c. \tag{6}$$

Assume that

$$P_c\phi_j = \sum \beta_{kj}\psi_k \quad \text{or} \quad P_c\Phi = G\Psi, \quad G = (\beta_{kj}).$$

So

$$A(P_c\phi_j, \psi_l) = \sum A(\psi_k, \psi_l)\beta_{kj} \ .$$

Denote

$$A_c = (A(\psi_k, \psi_l)), \qquad Q = (A(\phi_j, \psi_l)),$$

then

$$A_c G = Q.$$

This means that as a linear operator from $\mathcal{M}$ to $\mathcal{M}_c$, the matrix representation of P_c from coordinate basis ϕ to ψ is as follows

$$P_c \sim \quad G = A_c^{-1}Q = A_c^{-1}TA.$$

When we back to the original space and take P_c as a linear operator from $\mathcal{M}$ to $\mathcal{M}$ itself, the corresponding matrix form becomes

$$P_c \sim \quad T'G = A_c^{-1}Q = T'A_c^{-1}TA \ . \tag{7}$$

Therefore, we may look the projector P_c as the result from a preconditioned operator of A, the related preconditioner is

$$B_c := T'A_c^{-1}T \ . \tag{8}$$

The above discuss provides us a way to construct an appropriate preconditioner via subspaces. First, it is easy to see that if $\mathcal{M}_c$ is a proper subspace of $\mathcal{M}$, the resulting matrix B_c is singular. It is impossible to get a good preconditioner by using one proper subspace only. So it is reasonable to add one subspace as a coarse mesh to the original space, this is just the technique called two-level multigrid method.

3. Subdomain Preconditioner

For parallel computing it is natural to consider subdomain solver first. Suppose

$$\Omega = \cup \Omega_i, \quad \text{with} \quad \Omega_i \cap \Omega_j \neq \phi \ \ \text{if} \ \ i \neq j$$

and denote A_i be a matrix representation which is the restriction of the original operator A over the subdomain Ω_i. If we permute the unknowns appropriately, then the related preconditioner is

$$B_i = \begin{bmatrix} A_i^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

and the corresponding matrix A has the following block form

$$A = \begin{bmatrix} A_i & A_{ij} \\ A_{ij}^t & A_j \end{bmatrix}, \qquad B_i A = \begin{bmatrix} I & A_i^{-1} A_{ij} \\ 0 & 0 \end{bmatrix}.$$

Because the term A_{ij} reflects the influence of subdomain Ω_i to Ω_j in the difference scheme, all elements are zero except on the boundary close of Ω_j. So all eigenvalues of the preconditioned matrix $B_i A$ are equal to one except those relating to the inner boundary.

Hence, in general we obtain a whole subdomain preconditioner B_s as follows:

$$B_s = \sum B_i, \quad \text{with} \quad B_i = T_i^t A_i^{-1} T_i \ , \tag{9}$$

where T_i is a truncated permutation matrix from Ω to the subdomain Ω_i.

In particular, let us take

$$B_s = \begin{bmatrix} A_i^{-1} & 0 \\ 0 & A_j^{-1} \end{bmatrix} ,$$

then

$$AB_S A = \begin{bmatrix} A_i + A_{ij} A_j^{-1} A_{ij}^t & 2A_{ij} \\ 2A_{ij}^t & A_j + A_{ij}^t A_i^{-1} A_{ij} \end{bmatrix} ,$$

$$AB_s A = 2A - Q, \quad Q = \mathrm{diag}\{A_i - A_{ij} A_j^{-1} A_{ij}^t, A_j - A_{ij}^t A_i^{-1} A_{ij}\}. \tag{10}$$

Note that the matrix Q is symmetric positive definite if A is. In this case we have an upper bound for eigenvalues of $B_s A$

$$\lambda(B_s A) < 2 \ . \tag{11}$$

4. Coarse Mesh Correction

It is not difficult to imagine that the preconditioned effect is not good if the number of subdomain increase. Because the block matrix A_i represents the same difference scheme with the original matrix A for internal points of Ω_i with zero boundary, we have to add the boundary effect and take these inner boundaries as coarse mesh. Consider

$$B = B_s + B_c \ . \tag{12}$$

We may interpret the coarse mesh preconditioner as global correction from another point of view. We may ask the following question: which kind of matrices B_s could be as a good preconditioner? For an answer let us do observation further. For sake of simplicity, we suppose

$$B_s = \mathrm{diag}\{A_{11}^{-1}, A_{22}^{-1}, ..., A_{mm}^{-1}\}, \tag{13}$$

$$BA = (B_s + B_c)A = B_s\hat{A} \ ,$$

where

$$\hat{A} = A + B_s^{-1} B_c A = A + A_d T' A_c^{-1} T A \ ,$$

$$A_d = \mathrm{diag}\{A_{11}, A_{22}, ..., A_{mm}\} \ ,$$

and

$$\mathrm{Rank}((TA_d)' A_c^{-1} (TA)) \leq \mathrm{Rank}(A_c) \ .$$

Moreover,

$$(AB_cAu, u) = \mathcal{A}(P_c u, u) = \mathcal{A}(P_c u, P_c u) \leq (u, u) \ ,$$

then we have an upper bound for eigenvalues of BA

$$\lambda(BA) < 3 \ . \tag{14}$$

5. 1-D Model Problem

To exploit the core of Additive Schwarz Method, as first example we discuss the following simple two-points boundary value problem in detail.

$$-u'' = f, \quad u(0) = u(1) = 0 \ . \tag{15}$$

For a given partition

$$\Delta : 0 = x_0 < x_1 < ... < x_{N+1} = 1 \ ,$$

by using linear finite element method, the resulting stiffness matrix A is tridiagonal of order N and its coefficients are equal to

$$\alpha_{ij} = \mathcal{A}(\phi_i, \phi_j) = \begin{cases} 0 \ , & \text{if } |i-j| > 1 \\ -h_i^{-1} \ , & \text{if } j = i-1 \\ h_i^{-1} + h_{i+1}^{-1} \ , & \text{if } j = i \\ -h_{i+1}^{-1} \ , & \text{if } j = i+1 \end{cases} \ . \tag{16}$$

and it is easy to find its inverse matrix explicitly as follows

$$A^{-1} = (\gamma_{ij}) \text{ where } \gamma_{ij} = \begin{cases} \dfrac{(x_i - x_0)(x_{N+1} - x_j)}{x_{N+1} - x_0} , & \text{if } i \le j \\ \gamma_{ji} , & \text{if } i < j \end{cases} .$$

At first, we consider Block-Jacobi preconditioner (22). Let

$$s_0 = 0, s_1 = n_1, s_2 = n_1 + n_2, s_m = \sum_{k=1}^{m} n_k = N, \quad \text{with } n_k > 1, k = 1, 2, ..., m,$$

$$B_s A = \begin{bmatrix} I & A_{11}^{-1} A_{12} & & ... & \\ A_{22}^{-1} A_{12}^t & I \ ... & & A_{22}^{-1} A_{23} \ ... & \\ & ... & & ... & \\ & & & ..A_{mm}^{-1} A_{m,m-1}^t & I \end{bmatrix},$$

where $\text{rank}(A_{kk}) = n_k, k = 1, 2, ..., m$.

Denote

$$A_{kk}^{-1} = (\gamma_{ij}^{[k]}), \quad H_k = x_{s_k+1} - x_{s_{k-1}},$$

then

$$A_{11}^{-1} A_{12} = -h_{n_1+1}^{-1} \begin{bmatrix} \gamma_{1,n_1}^{[1]} & 0... & 0 \\ \gamma_{2,n_1}^{[1]} & 0... & 0 \\ & ... & \\ \gamma_{n_1,n_1}^{[1]} & 0... & 0 \end{bmatrix},$$

$$A_{22}^{-1} A_{12}^t = -h_{n_1+1}^{-1} \begin{bmatrix} 0 & ... & 0 & \gamma_{n_1+1,n_1+1}^{[2]} \\ 0 & ... & 0 & \gamma_{n_1+2,n_1+1}^{[2]} \\ 0 & ... & 0 & ... \\ 0 & ... & 0 & \gamma_{s_2,n_1+1}^{[2]} \end{bmatrix}$$

and so on.

Hence, in this case all eigenvalues of the preconditioned matrix $B_s A$ are equal to 1 except 2(m-1) eigenvalues which are ones of the following matrix with 2(m-1) order:

$$A_s = \begin{bmatrix} 1 & -(1-\alpha_1) & & & ... & \\ -(1-\beta_1) & 1 & & -\beta_1 & ... & \\ -\alpha_2 & 0 & 1 & -(1-\alpha_2) & ... & \\ & & -(1-\beta_2) & 1 & ... & \\ & ... & & & ... & \\ & & ... & & & -\beta_m \\ & & -\alpha_m & 0 & 1 & -(1-\alpha_m) \\ & & & & -(1-\beta_m) & 1 \end{bmatrix},$$

where

$$\alpha_k = h_{s_k+1} H_k^{-1}, \qquad \beta_k = h_{s_k+1} H_{k+1}^{-1}.$$

In particularly, for a special case $m = 2$, the eigenvalues are the roots of polynomial

$$(\lambda - 1)^2 = (1 - \alpha_1)(1 - \beta_1)$$

and for $m = 3$

$$(\lambda-1)^4-(\lambda-1)^2[(1-\alpha_1)(1-\beta_1)+(1-\alpha_2)(1-\beta_2)]+(1-\alpha_1)(1-\beta_2)(1-\alpha_2-\beta_1) = 0 \ .$$

In general, it is not difficult to verify A_s is a positive real matrix. Moreover, there is an estimation of the largest eigenvalue:

$$\lambda_{Max}(A_s) < 2 \ .$$

In order to estimate the lowest eigenvalue, by using so called "Red-Black" order, the above matrix can be rewritten as

$$\hat{A}_s = \begin{bmatrix} S_{RR} & S_{RB} \\ S_{BR} & S_{BB} \end{bmatrix}, \qquad (17)$$

where S_{RB} and S_{BR} both are diagonal matrices:

$$S_{RB} = \text{diag}\{-(1-\alpha_1), ..., -(1-\alpha_m)\}, S_{BR} = \text{diag}\{-(1-\beta_1), ..., -(1-\beta_m)\}$$

and S_{RR} and S_{BB} both are bi-diagonal matrices

$$S_{RR} = \begin{bmatrix} 1 & & & \\ -\alpha_2 & 1 & & \\ & ... & & \\ & & -\alpha_m & 1 \end{bmatrix}, \quad S_{BB} = \begin{bmatrix} 1 & -\beta_1 & & \\ & 1 & & \\ & ... & & -\beta_{m-1} \\ & & & 1 \end{bmatrix}.$$

Now we turn to consider the following eigen-problem:

$$\begin{bmatrix} S_{RR} - \lambda I_m & S_{RB} \\ S_{BR} & S_{BB} - \lambda I_m \end{bmatrix} \begin{bmatrix} X_R \\ X_B \end{bmatrix} = 0 \ . \qquad (18)$$

Hence

$$S_{RB} X_B = (\lambda I - S_{RR}) X_R = (\lambda I - S_{RR}) S_{BR}^{-1} (\lambda I - S_{BB}) X_B \ ,$$

or

$$S_B(\lambda) X_B = 0 \ ,$$

where

$$S_B = S_{RB} - (\lambda I - S_{RR}) S_{BR}^{-1} (\lambda I - S_{BB}).$$

Therefore, all eigenvalues are the roots of the following polynomial

$$\Delta_m(\lambda) = \det(S_B(\lambda)).$$

In the uniform case $(m+1)H = 1, \quad \alpha = hH^{-1}$, it reduces to

$$\Delta_m = \begin{vmatrix} (\lambda-1)^2 - (1-\alpha)^2 & \alpha(\lambda-1) & ... & \\ \alpha(\lambda-1) & (\lambda-1)^2 - (1-\alpha)^2 + \alpha^2 & \alpha(\lambda-1) & ... \\ & ... & ... & \\ & ..\alpha(\lambda-1) & (\lambda-1)^2 - (1-\alpha)^2 + \alpha^2 & \end{vmatrix},$$

There is a three-term recurrence

$$\Delta_0 = 1, \Delta_1 = (\lambda - 1)^2 - (1-\alpha)^2,$$

$$\Delta_k = \{(\lambda-1)^2 - (1-\alpha)^2 + \alpha^2\}\Delta_{k-1} - \alpha^2(\lambda-1)^2\Delta_{k-2}.$$

Hence, we have the following representation

$$\Delta_m = \frac{\xi_2^{m+1} - \xi_1^{m+1} - \alpha^2(\xi_2^m - \xi_1^m)}{\xi_2 - \xi_1} \ ,$$

where ξ_1 and ξ_2 are two roots of the following characteristic quadratic equation

$$\xi^2 - \{(\lambda-1)^2 - (1-\alpha)^2 + \alpha^2\}\xi + \alpha^2(\lambda-1)^2 = 0 \ .$$

Let

$$(\lambda-1)^2 - (1-\alpha)^2 + \alpha^2 = 2\alpha(\lambda-1)\cos\theta \ ,$$

then

$$\Delta_m(\lambda) = \frac{\alpha^m(\lambda-1)^{m-1}}{\sin\theta}\{(\lambda-1)\sin(m+1)\theta - \alpha\sin m\theta\} \ .$$

Note that $\lambda = 1$ is not a root of the above polynomial. So for small $\alpha = h/H$, it is reasonable to take the roots of $\sin(m+1)\theta = 0$ as initial guess, then the Newton's iteration could be used to get more accuracy roots. Hence, corresponding to the smallest eigenvalue we may find

$$\theta = H(1 + O(h))\pi,$$

or

$$\lambda_{min} = \frac{hH\pi^2}{2} + O(h^2). \tag{19}$$

Finally, we have computed the condition number of the preconditioned system and got one order improvement comparing with A as follows:

$$\kappa(B_s A) = \frac{4}{\pi^2}(Hh)^{-1}(1 - \frac{h}{2H}) + O(H^{-2}). \tag{20}$$

A numerical result is shown in section 7 Table A.

Therefore, for getting better preconditioner it is necessary to introduce some global corrections adding to the above subdomain solver. As a choice, we introduce a coarse mesh as follows:

$$\Delta_c : 0 = x_{c_0} < x_{c_1} < ... < x_{c_{m+1}} = 1 \quad (m << N) \ .$$

In this case new basis in the coarse mesh subspace becomes

$$\psi_k(x) = \begin{cases} 0 \ , & \text{if} \ \ x < x_{c_{k-1}} or x > x_{c_{k+1}} \\ (x - x_{c_{k-1}})H_k^{-1} \ , & \text{if} \ \ x_{c_{k-1}} \le x \le x_{c_k} \\ (x_{c_{k+1}} - x)H_{k+1}^{-1} \ , & \text{if} \ \ x_{c_k} \le x \le x_{c_{k+1}} \end{cases} ,$$

where

$$H_k = x_{c_k} - x_{c_{k-1}}.$$

And the coordinate transformation is

$$\psi_k = \phi_{c_k} + \sum_{i \neq c_k} t_{ki}\phi_i \ , \qquad \text{where} \qquad t_{ki} = \psi_k(x_i) \ .$$

Hence

$$(TA)_{kj} = \sum_i t_{ki} < A\phi_i, \phi_j >=< A\psi_k, \phi_j >$$

or

$$TA = [A_c, 0]P_r \ ,$$

where P_r is a truncated permutation matrix

$$(P_r)_{ki} = \delta_{l_k,i}, \quad (1 \leq i \leq N, \quad 1 \leq k \leq m).$$

So, it means that introducing coarse mesh will only modify the matrix B_sA at the columns which are related to the coarse mesh points and the contribution correction matrix will have the following form:

$$B_cA = T'A_c^{-1}TA = P_r' \begin{bmatrix} T_r & 0 \\ T_b & 0 \end{bmatrix} P_r.$$

Now the left question is how to match the coarse mesh to a given subdomain partition. As an example, one natural choice is to set

$$x_{c_k} = x_{s_k} \quad k = 1, ..., m-1 \ . \tag{21}$$

The contribution of this coarse mesh to the preconditioner is in terms of the following $2(m-1)$ order matrix:

$$A_{cs} = \begin{bmatrix} 1 & 0 & & & ... & \\ (1-\hat{\beta}_1) & 0 & \hat{\beta}_1 & 0 & ... & \\ 0 & 0 & 1 & 0 & ... & \\ & & (1-\hat{\beta}_2) & 0 & ... & \\ & ... & & & ... & \\ & & ... & & \hat{\beta}_m & 0 \\ & & 0 & 0 & 1 & 0 \\ & & & & (1-\beta_m) & 0 \end{bmatrix},$$

where

$$\hat{\beta}_k = \frac{x_{s_k+1} - x_{s_k}}{x_{s_{k+1}} - x_{s_k}}, \qquad \beta_k = \frac{x_{s_k+1} - x_{s_k}}{x_{s_{k+1}+1} - x_{s_k}}.$$

Adding the above to A_s leads to

$$A_s + A_{cs} = \begin{bmatrix} 2 & -(1-\alpha_1) & & & ... & \\ \beta_1 - \hat{\beta}_1 & 1 & \hat{\beta}_1 & -\beta_1 & ... & \\ -\alpha_2 & 0 & 2 & -(1-\alpha_2) & ... & \\ & & \beta_2 - \hat{\beta}_2 & 1 & ... & \\ & ... & & & ... & \\ & & ... & & \hat{\beta}_m & -\beta_m \\ & & -\alpha_m & 0 & 2 & -(1-\alpha_m) \\ & & & & 0 & 1 \end{bmatrix}.$$

It follows that the resulting preconditioned matrix $(B_s+B_c)A$ has $N+1-2m$ unit eigenvalues, the lest $2m-1$ eigenvalues are the same to the above matrix. Moreover, using the Gershgoring's disk theorem gives localization for others:

$$|\lambda - 2| < 1 \quad \cup \quad |\lambda - 1| < 2\beta \ ,$$

where $\beta = \mathrm{Max}(\hat{\beta}_k, \beta_k)$.

Finally, we obtain the following estimation:

$$1 - 2\beta < \lambda((B_s + B_c)A) < 3 \tag{22}$$

and

$$\kappa((B_s + B_c)A) < \frac{3}{1-2\beta}. \tag{23}$$

In some particular cases, we may obtain more exact estimation. For $m = 2$ and $m = 3$ with $x_{c_1} = x_{s_1}$, $x_{c_2} = x_{s_2+1}$, it is easy to verify that

$$A(B_c + B_s)A = A + Q_1 = 2A - Q_2,$$

where Q_1 and Q_2 both are non-negative definite matrices with zero eigenvalue. Hence, in this case we have

$$\lambda_{Max}((B_s + B_c)A) = 2, \quad \lambda_{Min}((B_s + B_c)A) = 1$$

The numerical tests are listed in section 7 (Table B and C) to match the above results.

6. Operator Approach

To extend the related conclusion from 1-D to higher dimension later, we need take more general operator approach instead of the above pure matrix approach.

For the given subdomain partition, we define subspaces consisting of piecewise linear function as follows:

$$\mathcal{M}_k := \mathrm{Span}_{l_k+1 \le l \le l_{k+1}}\{\phi_l\} \subset \mathcal{M}, \quad k = 0, 1, 2, ..., m,$$

and

$$\mathcal{M}_c := \mathrm{Span}_{1 \le k \le m}\{\psi_k\} \subset \mathcal{M}$$

Hence, ψ_k can be written as a linear combination of $\{\phi_j\}$

$$\psi_k = \phi_{l_k} + \sum_{j=l_{k-1}+1}^{l_k - 1} \xi_{kj}\phi_j + \sum_{j=l_{k1}+1}^{l_{k+1}-1} (1 - \xi_{k+1,j})\phi_j,$$

where

$$\xi_{kj} = (x_j - x_{l_{k-1}})H_k^{-1}.$$

For any given $u^h \in \mathcal{M}$, there is a decomposition $u^h = \sum_{j=1}^N u_j\phi_j$. Because

$$\sum_{k=1}^m u_{l_k}\phi_{l_k} = \sum_{k=1}^m u_{l_k}\{\psi_k - \sum_{j=l_{k-1}+1}^{l_k-1} \xi_{kj}\phi_j - \sum_{j=l_{k1}+1}^{l_{k+1}-1} (1 - \xi_{k+1,j})\phi_j\},$$

hence

$$u^h = \sum_{k=1}^{m} u_{l_k}\psi_k + \sum_{k=1}^{m+1}\{ \sum_{j=l_{k-1}+1}^{l_k-1}(u_j - \xi_{kj})\phi_j + \sum_{j=l_{k1}+1}^{l_{k+1}-1}(u_j - (1-\xi_{k+1,j}))\phi_j\}.$$

So we may decompose u^h in the following way:

$$u^h = \sum_{k=0}^{m+1} u^h_{[k]} \tag{24}$$

where

$$u^h_{[0]} = \sum_{\nu=1}^{m} u_{l_\nu}\psi_\nu, \quad u^h_{[k]} = \sum_{j=l_{k-1}+1}^{l_k-1}(u_j - \hat{u}_{jk})\phi_j,$$

$$\hat{u}_{jk} = u_{l_k}\xi_{kj} + u_{l_{k-1}}(1-\xi_{kj}), \quad k = 1, ..., m+1.$$

Then

$$\begin{aligned}
\|u^h_{[0]}\|^2_{\mathcal{A}} &= \mathcal{A}(u^h_{[0]}, u^h_{[0]}) = \sum_{\nu,\mu=1}^{m}\mathcal{A}(\psi_\mu,\psi_\nu)u_{l_\mu}u_{l_\nu},\\
\|u^h_{[k]}\|^2_{\mathcal{A}} &= \sum_{i,j=l_{k-1}+1}^{l_k-1}\mathcal{A}(\phi_i,\phi_j)(u_i-\hat{u}_{ik})(u_j-\hat{u}_{jk})\\
&= \sum_{i,j=l_{k-1}+1}^{l_k-1}\mathcal{A}(\phi_i,\phi_j)u_iu_j - 2\sum_{i=l_{k-1}+1}^{l_k-1}u_i\sum_{j=l_{k-1}+1}^{l_k-1}\mathcal{A}(\phi_i,\phi_j)\hat{u}_{jk}\\
&\quad + \sum_{i,j=l_{k-1}+1}^{l_k-1}\mathcal{A}(\phi_i,\phi_j)\hat{u}_{jk}\hat{u}_{ik}.
\end{aligned}$$

A straightforward computation leads to

$$\sum_{j=l_{k-1}+1}^{l_k-1}\mathcal{A}(\phi_i,\phi_j)\hat{u}_{jk} = \begin{cases} 0\ , & \text{if} \quad i \neq l_{k-1}+1 \quad \text{and} \quad i \neq l_k - 1\\ u_{l_{k-1}}h^{-1}_{l_{k-1}+1}\ , & \text{if} \quad i = l_{k-1}+1\\ u_{l_k}h^{-1}_{l_k}\ , & \text{if} \quad i = l_{k-1}-1 \end{cases}.$$

Moreover, we may have

$$\|u^h_{[k]}\|^2_{\mathcal{A}} = \sum_{i,j=l_{k-1}+1}^{l_k-1}\mathcal{A}(\phi_i,\phi_j)u_iu_j - 2(u_{l_{k-1}}u_{l_{k-1}+1}h^{-1}_{l_{k-1}+1} + u_{l_k}u_{l_k-1}h^{-1}_{l_k})$$

$$+u^2_{l_{k-1}}(h^{-1}_{l_{k-1}+1} - H^{-1}_k) + 2u_{l_{k-1}}u_{l_k}H^{-1}_k + u^2_{l_k}(h^{-1}_{l_k} - H^{-1}_k),$$

and

$$\|u^h_{[0]}\|^2_{\mathcal{A}} = \sum_{\nu=1}^{m}(H^{-1}_\nu + H^{-1}_{\nu+1})u^2_{l_\nu} - 2\sum_{\nu=1}^{m}H^{-1}_{\nu+1}u_{l_\nu}u_{l_\nu+1}.$$

Summation all above terms of $\mathcal{A}$ norm gives

$$\|u_{[0]}^h\|_{\mathcal{A}}^2 + \sum_{k=1}^{m} \|u_{[k]}^h\|_{\mathcal{A}}^2 = \sum_{i,j=1}^{N} \mathcal{A}(\phi_i, \phi_j) u_i u_j - 2\sum_{k=1}^{m} \mathcal{A}(\phi_{l_k}, \phi_{l_k-1}) u_{l_k} u_{l_k-1}$$

$$-2\sum_{k=1}^{m} \mathcal{A}(\phi_{l_k}, \phi_{l_k+1}) u_{l_k} u_{l_k+1} + 2\sum_{k=1}^{m} \{-(u_{l_{k-1}} u_{l_{k-1}+1} h_{l_{k-1}+1}^{-1} + u_{l_k} u_{l_k-1} h_{l_k}^{-1})$$

$$+u_{l_{k-1}}^2 (h_{l_{k-1}+1}^{-1} - H_k^{-1}) + 2u_{l_{k-1}} u_{l_k} H_k^{-1} + u_{l_k}^2 (h_{l_k}^{-1} - H_k^{-1})\}$$

$$+\sum_{\nu=1}^{m} (H_\nu^{-1} + H_{\nu+1}^{-1}) u_{l_\nu}^2 - 2\sum_{\nu=1}^{m} H_{\nu+1}^{-1} u_{l_\nu} u_{l_\nu+1}.$$

Finally we may get the following result:

$$\|u_{[0]}^h\|_{\mathcal{A}}^2 + \sum_{k=1}^{m+1} \|u_{[k]}^h\|_{\mathcal{A}}^2 = \mathcal{A}(u^h, u^h)\ . \tag{25}$$

By using the well-known Lion-Lemma[3], we obtain a good estimation of the lowest eigenvalue for the preconditioned system:

$$\lambda_{min}(BA) \geq 1\ . \tag{26}$$

Therefore, we get a final estimation on the condition number once again

$$\kappa(BA) \leq 3\ . \tag{27}$$

7. Numerical Results

Three numerical tests are listed here to show they access our above analysis.

Table A

Condition number of ASM for equidistant mesh

m	3		5		9	
1/h	$(B_c + B_s)$A	B_cA	$(B_c + B_s)$A	B_cA	$(B_c + B_s)$A	B_c A
32	2.25	22.95	2.46	36.07	2.79	68.29
64	2.13	44.92	2.26	71.31	2.49	135.24
128	2.07	86.98	2.13	137.55	2.24	240.21
256	2.03	172.98	2.07	274.12	2.13	479.57
512	2.02	343.00	2.03	538.85	2.07	958.18

Table B

Condition number of ASM for an non-uniform mesh $x(i) = \frac{(i-1)i}{n(n+1)}$

m	3	5	9	11
n=32	2.2764	2.4697	2.2897	2.9973
n=64	2.1506	2.2675	2.4908	2.5650
n=128	2.08	2.134	2.2453	2.3038
n=256	2.04	2.07	2.13	2.1580
n=512	2.02	2.04	2.0677	2.08

Table C

Eigenvalues test for ASM with 3 sub-intervals Non-uniform mesh $x(i) = \frac{(i-1)i}{n(n+1)}$

n		$(B_c + B_s)A$			B_cA	
	λ_M	λ_m	κ	λ_M	λ_m	κ
8	1.7961	0.2029	8.8076	2.0000	1.0000	2.0000
16	1.8842	0.1158	16.2779	2.0000	1.0000	2.0000
32	1.9386	0.0615	31.5301	2.0000	1.0000	2.0000
64	1.9682	0.0318	61.9696	2.0000	1.0000	2.0000
128	1.9837	0.0163	121.7075	2.0000	1.0000	2.0000
256	1.9981	0.0082	242.3839	2.0000	1.0000	2.0000
512	1.9959	0.0041	482.2293	2.0000	1.0000	2.0000

References

[1] J.H. Bramble, J.E. Pasciak and J.Xu, Parallel multilevel preconditioners, *Math. Comp.*, 55 (1990), 1–20.

[2] T.F. Chan, W. E and J.C. Sun, Domain decomposition interface preconditioner for fourth order elliptic problems, *Applied Numerical Mathematics*, 8 (1991), 317–331.

[3] P. Lions, On Schwarz Alternating Method I, R. Glowinski et al eds., Proceeding of First International Symposium on Domain Decomposition Methods for Partial Differential Equations, *SIAM*, Philadelphia, PA, 1988.

[4] P. Lions, On Schwarz Alternating Method II, T.F. Chan et al eds., Proceeding of Second International Symposium on Domain Decomposition Methods for Partial Differential Equations, *SIAM*, Philadelphia, PA, 1989.