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THE THEOREMS ON THE B-B POLYNOMIALS DEFINED ON A SIMPLEX IN THE BLOSSOMING FORM*

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Abstract

In this paper, an explicit expression of the blossom by the means of the B–form of B-B polynomial defined on a simplex is given. With its help, new but very short and simple blossom proofs of the most important theorems on B-B polynomials are derived, such as the degree elevation formula, the subdivision and the change of the underlying simplex procedure, the control points convergence property, the Marsden's identity.

1. Introduction

There are many approaches to polynomial and piecewise polynomial functions and curves. In the late '80 an elegant new approach has been developed by Ramshaw and others under the name of "blossoming"[7,8,2,3]. The idea applied can be traced to the algebraic geometry under the name *polar form*. As it turned out, the new approach successfully reconstructs and generalises the standard univariate polynomial and spline theory and makes it easier to understand and to explain the theorems and the computational algorithms involved. This is mainly due to the fact that it is possible to derive the main properties of splines (and polynomials) just from the recurrence relation that computes the B-spline basis. And this recurrence is a particular case of an algorithm that computes a blossom.

The basic idea in the blossom approach is the conclusion that there is a one–to–one correspondence between polynomials of degree at most n and a certain class of polynomials of n variables. Let us be precise.

A function $f : \mathbb{R} \to \mathbb{R}$ is called affine if it satisfies

$$f\left(\sum_i u_i x_i\right) = \sum_i u_i f(x_i)$$

for all affine combinations of $x_i \in \mathbb{R}$, i.e.

$$\sum_i u_i = 1, \quad u_i \in \mathbb{R}.$$

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A function $f : \mathbb{R}^m \to \mathbb{R}$ is called m–affine if it is affine with respect to each arguments. Also, a function $f : \mathbb{R}^m \to \mathbb{R}$ is called symmetric if it preserves its value under any permutation of its argument. Let us give now the definition of a blossom for the univariate polynomial case.

Definition 1.1 *Let f be a polynomial of degree $\leq n$. The blossom*

$$B_f(u_1, u_2, \cdots, u_n)$$

of the polynomial f is the symmetric n–affine multivariate polynomial satisfying the diagonal property

$$B_f(t, t, \cdots, t) = f(t) \ .$$

The blossom is well defined since it turns out to be unique. The definition can be straightforwardly extended to the splines, polynomial and spline curves, as well as to the multivariate polynomial case. In the recent years, many authors studied the problems on the univariate splines and on the spline or more general progressive curves using blossoming approach and produced fertile results[9,10,6,1,11,4]. The results were partly extended also to the multivariate polynomial case. But on the other hand, the analogies of blossoming for general multivariate splines are currently not known and this problem remains an important open question. In [5], some blossoming facts concerning splines on a simplicial partition can be found.

In this paper, we give an explicit expression of blossom

$$B_n(x^{(1)}, x^{(2)}, \cdots, x^{(n)}) := B_n(b_n; x^{(1)}, x^{(2)}, \cdots, x^{(n)})$$

by means of the B–form of B-B polynomial b_n on a simplex. Compared to the known one[5] it turns out to be much simpler to deal with, and gives the important dual functional property in a natural one row proof. It is also very easy to establish a necessary and sufficient condition (in the blossom form) for two B-B polynomials b_n and b_{n+k} to be identical. On this basis, the paper continues with very short blossom proofs of the most important theorems on B-B polynomials defined on simplex.

Although the majority of the theorems in this article is known, the proofs in the outline are new. The theorems are formulated in the blossoming form and the proofs are much shorter than the currently known ones and may give further insight into the basic theory of B-B surfaces and multivariate splines.

2. The Blossoming Proofs of the Facts on the B-B Polynomials

In the beginning, let us introduce the notation used throughout the paper. Let

$$V_m := < v^{(1)}, v^{(2)}, \cdots, v^{(m+1)} > := \left\{ \sum_{i=1}^{m+1} \lambda_i v^{(i)} : \lambda_i \geq 0, \quad \sum_{i=1}^{m+1} \lambda_i = 1 \right\}$$

denote a non–degenerate simplex in $\mathbb{R}^m$ with vertices

$$v^{(1)}, v^{(2)}, \cdots, v^{(m+1)} \in \mathbb{R}^m, \ \ \text{volume } V_m \neq 0.$$

Further, for any multindex

$$\alpha := (\alpha_1, \alpha_2, \cdots, \alpha_{m+1}) \in Z_+^{m+1},$$

let

$$\begin{aligned} |\alpha| &:= \alpha_1 + \alpha_2 + \cdots + \alpha_{m+1}, \\ \alpha! &:= \alpha_1!\alpha_2! \cdots \alpha_{m+1}!, \\ \lambda^\alpha &:= \lambda_1^{\alpha_1}\lambda_2^{\alpha_2} \cdots \lambda_{m+1}^{\alpha_{m+1}}, \quad \lambda = (\lambda_i)_{i=1}^{m+1} \in \mathbb{R}^{m+1}. \end{aligned}$$

For a pair of multiindices α, β the binomial coefficient is given as

$$\binom{\alpha}{\beta} := \begin{cases} \prod_{j=1}^{m+1} \binom{\alpha_j}{\beta_j} \ , & \text{if } \beta_j \leq \alpha_j, \text{ all } j \\ 0, & \text{otherwise .} \end{cases}$$

Any polynomial b_n of the total degree $\leq n$ in $\mathbb{R}^m$ can be expressed as

$$b_n(x) = \sum_{|\alpha|=n} c_\alpha B_\alpha^n(\lambda), \tag{1}$$

i.e. in a particular barycentric basis that depends on a given V_m, where

$$B_\alpha^n(\lambda) := \frac{n!}{\alpha!} \lambda^\alpha, \ \alpha \in Z_+^{m+1},$$

and

$$\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{m+1}) := \lambda(x, V_m)$$

are the barycentric coordinates of x satisfying

$$x = \sum_{i=1}^{m+1} \lambda_i v^{(i)}, \quad \sum_{i=1}^{m+1} \lambda_i = 1 \ .$$

(1) is called B–form of b_n with respect to V_m.

Let us recall the blossoming principle[7,8,5]. For any multivariate polynomial $b_n : \mathbb{R}^m \to \mathbb{R}$ of total degree $\leq n$, there exists a unique symmetric n–affine map

$$B_n : \underbrace{\mathbb{R}^m \times \mathbb{R}^m \cdots \times \mathbb{R}^m}_{n} \longrightarrow \mathbb{R}$$

satisfying the diagonal property

$$B_n(x, x, \cdots, x) = b_n(x), \quad x \in \mathbb{R}^m.$$

$B_n(x^{(1)}, x^{(2)}, \cdots, x^{(n)})$ is called the blossom of b_n.

The explicit representation of B_n in the original variables is well known[5]. However, the representation is rather clumsy to deal with. We shall show that the barycentric form is much simpler, and more adequate as a working tool in proving facts on the B-B surfaces. In the following we give the new explicit expression of B_n by means of the B–form of b_n.

Theorem 2.1 *Let $B_n(x^{(1)}, x^{(2)}, \cdots, x^{(n)})$ be the blossom of b_n, and let the points $x^{(k)}$ be expressed in the barycentric coordinates as follows*

$$x^{(k)} = \sum_{i=1}^{m+1} \sigma_i^{(k)} v^{(i)} \ , \quad \sum_{i=1}^{m+1} \sigma_i^{(k)} = 1 \ , \qquad k = 1, 2, \cdots, n.$$

Then

$$B_n(x^{(1)}, x^{(2)}, \cdots, x^{(n)}) = \prod_{k=1}^{n} \left(\sigma_1^{(k)} E_1 + \sigma_2^{(k)} E_2 + \cdots \sigma_{m+1}^{(k)} \ E_{m+1}\right) c_0. \tag{2}$$

Here E_j denotes the symbolic shift operator, i.e.

$$E_j c_\alpha := c_{\alpha+e_j}, \ j = 1, 2, \cdots, m+1,$$

and $e_j = (\delta_{j,\ell})_{\ell=1}^{m+1} \in \mathbb{R}^{m+1}$ denotes the j^{th} unit vector.

Proof. Quite clearly the right side of (2) is an affine symmetric multivariate polynomial. Thus one only has to verify the diagonal property: let

$$x^{(1)} = x^{(2)} = \cdots = x^{(n)} = x = \sum_{i=1}^{m+1} \lambda_i v^{(i)}.$$

Then the right hand reduces to

$$(\lambda_1 E_1 + \lambda_2 E_2 + \cdots + \lambda_{m+1} E_{m+1})^n c_0 = \sum_{|\alpha|=n} c_\alpha B_\alpha^n(\lambda) = b_n(x) \ .$$

Since the blossom is unique, the theorem is proved.

As an application of theorem 2.1, it is easy to get the dual functional property of the blossom $B_n(x^{(1)}, x^{(2)}, \cdots, x^{(n)})$.

Corollary 2.1 *Let $\alpha \in Z_+^{m+1}$, $|\alpha| = n$. Then*

$$B_n(\underbrace{v^{(1)}, \cdots, v^{(1)}}_{\alpha_1}, \underbrace{v^{(2)}, \cdots, v^{(2)}}_{\alpha_2}, \ \cdots, \ \underbrace{v^{(m+1)}, \cdots, v^{(m+1)}}_{\alpha_{m+1}}) = c_\alpha. \tag{3}$$

Proof. The representation (2) reduces the left side of (3) to

$$E_1^{\alpha_1} E_2^{\alpha_2} \ldots E_{m+1}^{\alpha_{m+1}} c_0 = c_\alpha \ .$$

The explicit representation given in the theorem 2.1 and the dual functional property of the blossom yield an easy way to the subdivision (or change of the underlying simplex) theorem in the blossom form. Let

$$\bar{V}_m := < \bar{v}^{(1)}, \bar{v}^{(2)}, \cdots, \bar{v}^{(m+1)} >$$

be another m–simplex in $\mathbb{R}^m$, and

$$\bar{v}_m^{(k)} = \sum_{j=1}^{m+1} \lambda_j^{(k)} v^{(j)} \ , \ \sum_{j=1}^{m+1} \lambda_j^{(k)} = 1, \qquad k = 1, 2, \cdots, m+1.$$

Corollary 2.2 *Let*

$$b_n(x) = \sum_{|\alpha|=n} c_\alpha B_\alpha^n(\lambda)$$

be a B-B surface on the simplex V_m, and let $B_n(x^{(1)}, x^{(2)}, \cdots, x^{(n)})$ be its blossom. Then on the simplex $\bar{V}_m$, b_n can be expressed as

$$b_n(x) = \sum_{|\alpha|=n} B_n(\underbrace{\bar{v}^{(1)}, \cdots, \bar{v}^{(1)}}_{\alpha_1}, \underbrace{\bar{v}^{(2)}, \cdots, \bar{v}^{(2)}}_{\alpha_2}, \ \cdots, \ \underbrace{\bar{v}^{(m+1)}, \cdots, \bar{v}^{(m+1)}}_{\alpha_{m+1}}) \ B_\alpha^n(\bar{\lambda}) \tag{4}$$

with

$$B_n(\underbrace{\bar{v}^{(1)}, \cdots, \bar{v}^{(1)}}_{\alpha_1}, \underbrace{\bar{v}^{(2)}, \cdots, \bar{v}^{(2)}}_{\alpha_2}, \cdots, \underbrace{\bar{v}^{(m+1)}, \cdots, \bar{v}^{(m+1)}}_{\alpha_{m+1}}) =$$

$$\prod_{k=1}^{m+1} \left(\lambda_1^{(k)} E_1 + \lambda_2^{(k)} E_2 + \cdots + \lambda_{m+1}^{(k)} E_{m+1}\right)^{\alpha_k} c_0.$$

Here $\bar{\lambda}$ denotes the barycentric coordinates of x with respect to the simplex $\bar{V}_m$.

Note that the coefficients B_n can be calculated efficiently by the de Casteljau algorithm, as already pointed out in [5]. The geometric meaning of (4) is the following: the Bézier coordinates of the restriction of a B-B surface b_n to the simplex $\bar{V}_m$ are equal to values of the blossom of b_n at the points

$$\underbrace{\bar{v}^{(1)}, \cdots, \bar{v}^{(1)}}_{\alpha_1}, \ \underbrace{\bar{v}^{(2)}, \cdots, \bar{v}^{(2)}}_{\alpha_2}, \ \cdots, \ \underbrace{\bar{v}^{(m+1)}, \cdots, \bar{v}^{(m+1)}}_{\alpha_{m+1}}, \quad |\alpha| = n.$$

It is well known that the degree elevation formula is an important tool in the study of B-B polynomials. With the blossom approach, it is straightforward to establish the degree elevation formula in its full generality. We first prove the following theorem.

Theorem 2.2 *Let $B_n(x^{(1)}, x^{(2)}, \cdots, x^{(n)})$ and $B_{n+k}(x^{(1)}, x^{(2)}, \cdots, x^{(n+k)})$ be the blossoms of B-B polynomials b_n and b_{n+k} respectively. Then*

$$b_n = b_{n+k}$$

if and only if

$$B_{n+k}(x^{(1)}, x^{(2)}, \cdots, x^{(n+k)}) = \frac{1}{\binom{n+k}{k}} \sum_{\pi_k} B_n\left(x^{\pi_k(1)}, x^{\pi_k(2)}, \cdots, x^{\pi_k(n)}\right), \tag{5}$$

where π_k denotes a map from $I_n := \{1, 2, \cdots, n\}$ to I_{n+k} such that $\pi_k(i) < \pi_k(j)$, if $i < j$.

Proof. Note that the right side of (5) is $(n+k)$–affine symmetric polynomial, given as an average of $\binom{n+k}{k}$ n-affine symmetric polynomials. If all the points coincide,

$$\begin{aligned} b_{n+k}(x) &= B_{n+k}(x, x, \cdots, x) \\ &= \tfrac{1}{\binom{n+k}{k}} \textstyle\sum_{\pi_k} B_n(x, x, \cdots, x) = b_n(x). \end{aligned}$$

But the blossoms are unique, and the theorem is confirmed.

The theorem 2.2 paves the way to the blossoming proof of the general degree elevation formula.

Theorem 2.3 *Let* $b_n(x) = \sum\limits_{|\alpha|=n} c_\alpha B^n_\alpha(\lambda)$*. Then*

$$b_n(x) = \sum_{|\alpha|=n+k} c^{(k)}_\alpha B^{n+k}_\alpha(\lambda) =: b_{n+k}(x). \tag{6}$$

The coefficients in this basis are given as

$$c^{(k)}_\alpha = \frac{k!\ n!}{(n+k)!} \sum_{|\beta|=n} \binom{\alpha}{\beta} c_\beta\ . \tag{7}$$

Proof. Let

$$B_n(x^{(1)}, x^{(2)}, \cdots, x^{(n)}),\quad B_{n+k}(x^{(1)}, x^{(2)}, \cdots, x^{(n+k)})$$

be the blossoms of b_n and b_{n+k} respectively. The dual functional property of the blossom B_{n+k}, and the theorem 2.2, used for a particular set of points, yield

$$\begin{aligned}
c^{(k)}_\alpha &= B_{n+k}(\underbrace{v^{(1)}, \cdots, v^{(1)}}_{\alpha_1}, \underbrace{v^{(2)}, \cdots, v^{(2)}}_{\alpha_2}, \cdots, \underbrace{v^{(m+1)}, \cdots, v^{(m+1)}}_{\alpha_{m+1}}) \\
&= \tfrac{k!\ n!}{(n+k)!} \sum_{|\beta|=n} B_n(\underbrace{v^{(1)}, \cdots, v^{(1)}}_{\beta_1}, \underbrace{v^{(2)}, \cdots, v^{(2)}}_{\beta_2}, \cdots, \underbrace{v^{(m+1)} \cdots, v^{(m+1)}}_{\beta_m}) \tbinom{\alpha}{\beta} \\
&= \tfrac{k!\ n!}{(n+k)!} \sum_{|\beta|=n} \tbinom{\alpha}{\beta} c_\beta
\end{aligned}$$

The theorem is proved.

From the theorems 2.2, 2.3, and the corollary 2.1, one obtains the blossoming form of Farin's theorem immediately.

Corollary 2.3 *Let*

$$b_{n+k} = b_n,\quad k = 1, 2, \ldots,$$

and $B_{n+k}(x^{(1)}, x^{(2)}, \cdots, x^{(n+k)})$ *be the blossom of* b_{n+k}*. Let* $\alpha \in Z^{m+1}_+,\ |\alpha| = n + k$*. If*

$$\lim_{k\to\infty} \frac{1}{k}\alpha = \lambda = \text{the barycentric coordinates of } x,$$

then

$$\lim_{k\to\infty} B_{n+k}\left(\underbrace{v^{(1)}, \cdots, v^{(1)}}_{\alpha_1}, \underbrace{v^{(2)}, \cdots, v^{(2)}}_{\alpha_2}, \cdots, \underbrace{v^{(m+1)}, \cdots, v^{(m+1)}}_{\alpha_{m+1}}\right) = b_n(x)$$

We conclude with the blossoming proof of Marsden's identity for our particular setup.

Theorem 2.4 *Let*

$$\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{m+1}), \quad \sum_{i=1}^{m+1} \lambda_i = 1.$$

Then

$$\lambda^\beta = \sum_{|\alpha|=n} \binom{\alpha}{\beta} B_\alpha^n(\lambda) \Bigg/ \binom{n}{\beta}, \qquad \text{for any} \quad |\beta| = k \le n \tag{8}$$

Proof. Let $B_k(x^{(1)}, x^{(2)}, \cdots, x^{(k)})$ be the blossom of λ^β, and $B_n(x^{(1)}, x^{(2)}, \cdots, x^{(n)})$ its n–blossom. The dual functional property of the blossom yields

$$\begin{aligned} \lambda^\beta &= \textstyle\sum_{|\alpha|=n} B_n(\underbrace{v^{(1)}, \cdots, v^{(1)}}_{\alpha_1}, \underbrace{v^{(2)}, \cdots, v^{(2)}}_{\alpha_2}, \cdots, \underbrace{v^{(m+1)}, \cdots, v^{(m+1)}}_{\alpha_{m+1}}) B_\alpha^n(\lambda) \\ &=: \textstyle\sum_{|\alpha|=n} c_\alpha B_\alpha^n(\lambda). \end{aligned}$$

An application of the theorem 2.2 and the corollary 2.1 gives

$$\begin{aligned} c_\alpha &= \tfrac{k!\ (n-k)!}{n!} \sum_{|\sigma|=k} B_k(\underbrace{v^{(1)}, \cdots, v^{(1)}}_{\sigma_1}, \underbrace{v^{(2)}, \cdots, v^{(2)}}_{\sigma_2}, \underbrace{v^{(m+1)}, \cdots, v^{(m+1)}}_{\sigma_{m+1}}) \tbinom{\alpha}{\sigma} \\ &= \sum_{|\sigma|=k} \delta_{\beta,\sigma} \tfrac{\beta!}{k!} \tbinom{\alpha}{\sigma} \tfrac{k!\ (n-k)!}{n!} \\ &= \tbinom{\alpha}{\beta} \Big/ \tbinom{n}{\beta} \end{aligned}$$

Hence the formula (8) is proved.

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