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ON BOUNDARY TREATMENT FOR THE NUMERICAL SOLUTION OF THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS WITH FINITE DIFFERENCE METHODS*

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1. Introduction

Consider the incompressible Navier-Stokes equations (INSE)

$$\frac{\partial \mathbf{w}}{\partial t} + u\frac{\partial \mathbf{w}}{\partial x} + v\frac{\partial \mathbf{w}}{\partial y} + \text{grad } p = \alpha \text{ div grad } \mathbf{w} \tag{1}$$

$$\text{div } \mathbf{w} = 0 \tag{2}$$

on region Ω, where $\mathbf{w} = (u, v)'$, with initial condition

$$\mathbf{w}(x, y, 0) = \mathbf{w}^0(x, y) \quad \text{on } \Omega$$

satisfying (2) and boundary conditions satisfying

$$\oint w_n ds = 0 \quad \text{on } \partial\Omega \tag{3}$$

Specific boundary conditions for the INSE and numerical boundary conditions for its numerical solution have been controversial issues in computational fluid dynamics. An attempt is made to clarify some of the problems in this work, based on the author's experience, see Huang et al. [1] for example, and a paper of Perot[2]. In the latter, the issue on numerical boundary conditions is resolved with the linear algebra approach. This approach will be used here on the 'delta' form of the finite difference equation, leading to $O(\Delta t^2)$ results and will be extended to dimensional split and uniform boundary treatment. It is found that no numerical boundary conditions are needed, but numerical boundary conditions similar to those of Kim and Moin[3] and Yanenko[4], for auxiliary velocity and for intermediate velocity with dimensional split respectively, are desirable for uniform boundary treatment.

It is the author's belief that interior and boundary schemes should be developed together such that their properties match as much as possible. This, amongst other

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reasons, leads to a change from the staggered mesh to the half-staggered mesh shown in Fig. 1. On this mesh, there is no half interval differencing near the boundary and pressure boundary condition remains unnecessary, which is mathematically correct as we see in §2 on boundary conditions for the INSE. In §3, we derive as [1] the pressure correction projection method via approximate factorization (AF) as fractional step method with the Crank-Nicolson scheme. We know that INSE, upon spatial discretization, forms a differential algebraic system, and the local second order temporal accuracy of the Crank-Nicolson scheme implies global second order temporal accuracy, see Hairer [5]. Van Kan [6] has shown that this scheme with pressure correction preserves its global accuracy. In §3 and §4, we show that all numerical boundary conditions considered are local second order approximations for the auxiliary velocity, which should not effect the global accuracy. In §5, results of preliminary numerical experiment are presented confirming some of the conclusions.

2. Boundary Conditions for INSE

In this section, we state some 'proper' boundary conditions for the INSE. Now, normal mode analysis applied to the 'frozen coefficient' systems for small perturbation of hyperbolic systems can lead to significant results. Here, normal mode analysis applied to the corresponding INSE system

$$\begin{gathered}\frac{\partial \dot{\mathbf{w}}}{\partial t} + u\frac{\partial \dot{\mathbf{w}}}{\partial x} + v\frac{\partial \dot{\mathbf{w}}}{\partial y} + \text{grad } \dot{p} - \alpha \text{ div grad } \dot{\mathbf{w}} = 0 \\ \text{div}\dot{\mathbf{w}} = 0\end{gathered} \tag{4}$$

yields only that the number of boundary conditions is two. So we turn to the energy method and obtain

$$\frac{d}{dt}(\dot{\mathbf{w}}, \dot{\mathbf{w}}) \le -\oint w_n \ \dot{\mathbf{w}}.\dot{\mathbf{w}} ds + 2\alpha \oint \dot{\mathbf{w}}.\frac{\partial \dot{\mathbf{w}}}{\partial n} ds - 2\oint \dot{p}\dot{w}_n ds$$

which we set ≤ 0 as sufficient condition for the boundary conditions to be 'proper'. In the above inequality, $(\mathbf{u}, \mathbf{v}) = \int\int \mathbf{u}.\mathbf{v} dxdy$. On the left boundary, say, setting the integrand to be ≤ 0 everywhere, i.e.

$$u(\dot{u}^2 + \dot{v}^2) + 2\dot{u}\dot{p} - 2\alpha\dot{u}\frac{\partial \dot{u}}{\partial x} - 2\alpha\dot{v}\frac{\partial \dot{v}}{\partial x} \le 0 \tag{5}$$

we deduce the following 'proper' boundary conditions:

$$\begin{array}{rcl} \text{solid wall} & : & \dot{u} = 0 \text{ and } \dot{v} = 0 \ (u \text{ and } v \text{ given}) \\ \text{inflow}(u > 0) & : & \dot{u} = 0 \text{ and } \dot{v} = 0 \ (u \text{ and } v \text{ given}) \\ \text{outflow}(u < 0) & : & \dot{u} = 0 \text{ or } \dot{p} - \alpha\dfrac{\partial \dot{u}}{\partial x} = 0 \ (u \text{ or } p - \alpha\dfrac{\partial u}{\partial x} \text{ given}) \\ & \text{and} & \dot{v} \text{ or } \dfrac{\partial \dot{v}}{\partial x} = 0 \ (v \text{ or } \dfrac{\partial v}{\partial x} \text{ given}) \end{array}$$

We note that if $\frac{\partial p}{\partial x} = 0$ at the outflow, then Halpern's absorbing boundary condition[7] for the advection-diffusion equation, now the x-momentum equation, can be used to predict the u distribution on the boundary and then corrected to satisfy the constraint (2).

In the following discussion, we consider just the solid wall boundary condition.

3. Fractional Step Boundary Conditions

Without further ado, we discretisize the INSE and take the Crank-Nicholson centered difference scheme on the half-staggered mesh as shown in Fig.1 as an example of second order accurate approximation of the INSE. With $\mathbf{w} = (u, v)', \mathbf{f} = (u^2, uv)'$, $\mathbf{g} = (vu, v^2)'$ and Newton linearization $\mathbf{f}^{n+1} = \mathbf{f}^n + A^n \dot{\Delta}\mathbf{w} + O(\Delta t^2), \mathbf{g}^{n+1} = \mathbf{g}^n + B^n \dot{\Delta}\mathbf{w} + O(\Delta t^2)$ with $\dot{\Delta}\mathbf{w} = \mathbf{w}^{n+1} - \mathbf{w}^n$, and

$$A = \begin{bmatrix} 2u & 0 \\ v & u \end{bmatrix}, \quad B = \begin{bmatrix} v & u \\ 0 & 2v \end{bmatrix}$$

the momentum equation is approximated by the following finite difference equation in 'delta' form

$$\begin{aligned} &\frac{\dot{\Delta}\mathbf{w}}{\Delta t} + \frac{1}{2}\frac{\delta}{\delta x}(A^n \dot{\Delta}\mathbf{w}) + \frac{1}{2}\frac{\delta}{\delta y}(B^n \dot{\Delta}\mathbf{w}) + \frac{1}{2}\nabla(\dot{\Delta}p) - \frac{\alpha}{2}\left(\frac{\delta^2 \dot{\Delta}\mathbf{w}}{\delta x^2} + \frac{\delta^2 \dot{\Delta}\mathbf{w}}{\delta y^2}\right) \\ &\qquad = -\left[\frac{\delta \mathbf{f}^n}{\delta x} + \frac{\delta \mathbf{g}^n}{\delta y} + \nabla p^n - \alpha\left(\frac{\delta^2 \mathbf{w}^n}{\delta x^2} + \frac{\delta^2 \mathbf{w}^n}{\delta y x^2}\right)\right] \equiv rhs \end{aligned} \tag{6}$$

for interior-interior points I. For ease of presentation, here $\frac{\delta}{\delta x}, \frac{\delta}{\delta y}$ denote centered differences on a uniform mesh. For boundary-interior points BI, the operator is split into interior and boundary parts and put respectively onto the left and right hand sides, for example

$$\frac{\delta}{\delta x}(A^n \dot{\Delta}\mathbf{w}) = \frac{A^n_{j+1}\dot{\Delta}\mathbf{w}_{j+1}}{2\Delta x} - \frac{A^n_{j-1}\dot{\Delta}\mathbf{w}_{j-1}}{2\Delta x} \equiv (l_I + l_B)\dot{\Delta}\mathbf{w}\ .$$

Similarly, the continuity equation is approximated by

$$\nabla \cdot (\dot{\Delta}\mathbf{w}) \tag{7}$$

for interior-interior points and split into interior and boundary parts for boundary-interior points. In the above ∇ and $\nabla\cdot$ are difference approximations of the *grad* and *div* operators, each involving 4 neighboring points with an average, see Huang [8].

We assume the spatial mesh size is fixed and write the discretized equations in matrix form and use AF to form the corresponding fractional step method. Equations (6) and (7) yield respectively

$$(I + \tfrac{\Delta t}{2}L)\dot{\Delta}W + \tfrac{\Delta t}{2}G\dot{\Delta}P = \Delta t\ (R - B^1) \tag{8}$$

$$D\dot{\Delta}W = -B^2 \tag{9}$$

where vector $W = (...u_{j+1/2\ k+1/2}, v_{j+1/2\ k+1/2}...)'$ with u, v on all interior points (I and BI), $P = (...p_{jk}...)'$ with p on interior points. Matrix L, G and D correspond respectively to the convection-diffusion operator, the gradient operator and the divergence operator; R holds elements rhs; B^1 and B^2 contain the known boundary values. We write (8) and (9) in the form

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\Delta}W \\ \dot{\Delta}P \end{bmatrix} + \begin{bmatrix} \frac{\Delta t}{2}L & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\Delta}W \\ \dot{\Delta}P \end{bmatrix} + \begin{bmatrix} 0 & \frac{\Delta t}{2}G \\ \frac{\Delta t}{2}D & 0 \end{bmatrix} \begin{bmatrix} \dot{\Delta}W \\ \dot{\Delta}P \end{bmatrix} = \begin{bmatrix} \Delta t(R - B^1) \\ \frac{\Delta t}{2}(-B^2) \end{bmatrix}$$

or with $\dot{\Delta}U = (\dot{\Delta}W, \dot{\Delta}P)'$, as

$$(E + \tfrac{\Delta t}{2}\ M + \tfrac{\Delta t}{2}\ N\ )\ \dot{\Delta}U = \Delta t\ RHS\ . \tag{10}$$

First, we use AF on the matrix of the above equation and get

$$(I + \tfrac{\Delta t}{2}\ M)\ (E + \tfrac{\Delta t}{2}\ N\ )\ \dot{\Delta}U = \Delta t\ RHS$$

which is a second order approximation of (10). This results in the following fractional step method

$$(I + \tfrac{\Delta t}{2}\ M)\ \Delta U = \Delta t\ RHS, \quad \Delta U = \tilde{U} - U^n$$
$$(E + \tfrac{\Delta t}{2}\ N)\ \dot{\Delta}U = \Delta U, \quad \dot{\Delta}U = U^{n+1} - U^n$$

which is simply the pressure correction method

$$(I + \tfrac{\Delta t}{2}L)\Delta W = \Delta t(R - B^1), \quad \Delta W = \tilde{W} - W^n \tag{11}$$

$$W^{n+1} + \tfrac{\Delta t}{2}G\Phi = \tilde{W}, \quad \Phi = \dot{\Delta}P \tag{12}$$

and (9); $(\tilde{u}, \tilde{v})$ in $\tilde{W}$ is called the auxiliary velocity. We note that (11) for $\tilde{W}$ is just (8) for W^{n+1}, without Φ, but with the same right hand side, in particular, with the same boundary conditions – no approximation is involved.

Now applying D to (12), we form the following system of linear algebraic equations for Φ,

$$D\ G\ \Phi = (D\ \tilde{W} + B^2)/\tfrac{\Delta t}{2}\ . \tag{13}$$

The computational steps are: calculate ΔW with (11), find Φ from (13), and update W^{n+1} with (12). Note that (13) does not involve boundary Φ; indeed, here pressure is not even defined on the boundary, and no Poisson equation is referred to. Equation (13) has been in effect since Easton[9], but presented via the linear algebraic approach, should leave no room for confusion and should end the discussion on pressure boundary condition for the pressure Poisson equation.

We now use the same approach for the dimensioanl split on the auxiliary velocity equation. Writing (11) as

$$(I + \tfrac{\Delta t}{2}L_x + \tfrac{\Delta t}{2}L_y)\Delta W = \Delta t(R - B^1) \tag{14}$$

and using AF on the above matrix, we get

$$(I + \tfrac{\Delta t}{2} L_x)\Delta W^1 = \Delta t(R - B^1) \tag{15a}$$

$$(I + \tfrac{\Delta t}{2} L_y)\Delta W^2 = \Delta W^1 \tag{15b}$$

where $\Delta W^2 = \Delta W$. Again only interior values of ΔW^1 and ΔW^2 are involved. We will call (15) MD as it involves just matrix decomposition. Note that in the x sweep solution of ΔW^1, the right hand side takes into account boundary values on all sides.

Writing the right hand side of (14) as $\Delta t R - \Delta t B_x^1 - \Delta t B_y^1$, we form

$$(I + \tfrac{\Delta t}{2} L_x)\Delta W^1 = \Delta t R - \Delta t B_x^1 \tag{16a}$$

$$(I + \tfrac{\Delta t}{2} L_y)\Delta W^2 = \Delta W^1 - \Delta t B_y^1 \tag{16b}$$

where B_x^1 and B_y^1 depend on $\dot{\Delta} W$. We will call (16) AV; it is the usual dimensional split method for the familiar auxiliary velocity introduced at the finite difference equation stage, and is equivalent to

$$(I + \tfrac{\Delta t}{2} L_x)(I + \tfrac{\Delta t}{2} L_y)\Delta W = \Delta t R - \Delta t B_x^1 - (I + \tfrac{\Delta t}{2} L_x)\Delta t B_y^1\ .$$

We see that there is an extra $O(\Delta t^3)$ term in the approximation, but second order accuracy is retained.

Figure 1. Half-staggered mesh

Figure 2. Sample region

4. Uniform Boundary Conditions

The AF for (10) introduces an error $\frac{\Delta t^2}{4} MN\dot{\Delta} U$, which reduces to $\frac{\Delta t^2}{4} LG\dot{\Delta} P$. It is desirable to have uniform error distribution, i.e. the error for boundary-interior points the same as that for the interior-interior points. To look into this problem, we put all u and v – on interior and boundary points, and all p – including a rim of points outside of the boundary into W and study the structure of L and G. On a sample mesh shown in Fig. 2, L and G are of forms

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ D_{21} & C_{22} & D_{23} & 0 \\ 0 & D_{32} & C_{33} & D_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad G = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & H_{22} & \tilde{H}_{23} & 0 & 0 \\ 0 & 0 & H_{33} & \tilde{H}_{34} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where D, C, H and $\tilde{H}$ are of forms

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ x & x & x & 0 \\ 0 & x & x & x \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad H, \tilde{H} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & x & x & 0 & 0 \\ 0 & 0 & x & x & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then

$$LG = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ (D_{21}H_{11}) & (D_{21}\tilde{H}_{12}+)C_{22}H_{22} & C_{22}\tilde{H}_{23} + D_{23}H_{33} & D_{23}\tilde{H}_{34} & 0 \\ 0 & D_{32}H_{22} & D_{32}\tilde{H}_{23} + C_{33}H_{33} & C_{33}\tilde{H}_{34}(+D_{34}H_{44}) & (D_{34}\tilde{H}_{45}) \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

without the terms in parentheses. But only with terms in the parentheses do we have uniform error distribution. This is achieved by adding $H_{11}, \tilde{H}_{12}$ and $H_{44}, \tilde{H}_{45}$ to G, and similar elements at corresponding positions to H and $\tilde{H}$. So G is modified and equations (8) to (12) hold with all vectors and matrices extended to and beyond the boundary and with B^1 and B^2 equal to 0. The 'Poisson' equation for Φ (13) is unchanged, as for its formation we use simply

$$\mathbf{w}^{n+1} = \mathbf{w}_B^{n+1}$$

on the boundary, here subscript B stands for boundary data.

No error is introduced if the corresponding right hand side is modified, i.e. boundary values of $\tilde{\mathbf{w}}$ is modified according to

$$\mathbf{w}_B^{n+1} + \tfrac{\Delta t}{2}\nabla\phi = \tilde{\mathbf{w}}_B \quad \phi = \dot{\Delta}p \tag{17}$$

which is the boundary part of (12) and is an $O(\Delta t^2)$ modification. But $\nabla\phi$ is not known at the auxiliary velocity step, $\nabla\phi$ at the previous time level can be used and it is obtained with linear extrapolation of ϕ to a rim of points outside of the computational region.

With the same approach for the intermediate velocity with dimensional split, we obtain

$$\Delta\mathbf{w}_B^1 = (I + \tfrac{\Delta t}{2} l_y)\Delta\mathbf{w}_B^2 \tag{18}$$

for the left and right boundary value modification for the x sweep. Here l_y denote the part of L_y at these boundaries, (18) is the boundary part of (16b). Relations (17) and (18) are respectively similar to those of [3] and [4] for improving 1st order accurate boundary values, but here the goal is uniform boundary error with uniform boundary treatment.

5. Numerical Experiment

Consider INSE with exact solution

$$\begin{aligned} u &= e^t sinxcosy \\ v &= -e^t cosxsiny \\ p &= e^t sinxsiny \end{aligned} \tag{19}$$

with corresponding nonhomogeneous terms in the momentum equations, on a square $0 \le x \le \pi, 0 \le y \le \pi$. Initial and boundary values of u and v are taken from (19), so the boundary values are time-dependent and $\frac{\partial p}{\partial n} \neq 0$.

Nonuniform meshes (8x8, 16x16, 32x32, 64x64) were generated with smooth transformation functions and centered differences on uniform meshes in the computational region appear as differences on nonuniform mesh in the physical region, see Wu [10]. Scheme MD – (15)+(13)+(12), scheme AV – (16)+(13)+(12), and scheme AVb – AV with boundary treatment (17) were tested. The l_2 errors of u and v, $\|eu\|$ and $\|ev\|$, at $t = 1$ obtained with different Δt for $\alpha = 1$ are given in Table 1. To avoid confusion, only relevant results are shown. For sufficiently small Δt ($= 0.0125$), second order spatial accuracy is obvious from the last column. We note here that for each mesh the decrease in error with the decrease of Δt is limited, as shown by the 1st line of the MD block. For sufficiently small mesh size (64x64, $\Delta x = \Delta y \simeq 0.03 - 0.08$), second order temporal accuracy is seen from the 4th line of the MD block, errors of $\Delta t = 0.1$ to errors of $\Delta t = 0.05$ being about 3.89, and errors of $\Delta t = 0.05$ to errors of $\Delta t = 0.025$ being about 3.99. From the 4th line of the AV block, we see that the corresponding error quotients for AV are 3.83 and 3.57 respectively. The trend of errors for $\alpha = 0.01$, also given here in the 5th line of the blocks, is less apparent. It is not clear how various errors of various terms interact and accumulate with time.

As for AVb, second order temporal accuracy is not expected due to the present realization of (17). We note from the first two columns of the AVb block that for large Δt the error is greatly reduced for $\alpha = 1$ with this method. Fig. 3 shows the $\frac{\partial p}{\partial x}$ distibution at $y = \pi/2$ where $\frac{\partial p}{\partial x} = e^t cosx$, obtained with MD (or AV) and AVb. The improvement with uniform boundary treatment is obvious.

Figure 3.
$\dfrac{\partial p}{\partial x}$ at $y = \pi/2$
MD or AV : dotted line
AVb: solid line

References

Table 1. Errors $\|eu\|$ and $\|ev\|$

Scheme		$\Delta t = 0.1$		$\Delta t = 0.05$		$\Delta t = 0.025$		$\Delta t = 0.0125$	
$(\alpha = 1)$		$\|eu\|$	$\|ev\|$	$\|eu\|$	$\|ev\|$	$\|eu\|$	$\|ev\|$	$\|eu\|$	$\|ev\|$
MD	8x8	4.80-2	4.83-2	5.11-2	5.12-2	5.21-2	5.21-2	5.23-2	5.24-2
	16x16							1.27-2	1.27-2
	32x32							3.08-3	3.08-3
	64x64	2.08-2	1.86-2	5.35-3	4.78-3	1.34-3	1.20-3	7.40-4	7.24-4
$(\alpha = 0.01)$	64x64	3.15-2	2.75-2	8.57-3	7.34-3	3.47-3	3.17-3	3.00-3	2.94-3
AV	8x8							5.24-2	5.24-2
	16x16							1.27-2	1.27-2
	32x32							3.16-3	3.17-3
	64x64	1.74-2	2.05-2	4.53-3	5.38-3	1.28-3	1.49-3	7.95-4	8.22-4
$(\alpha = 0.01)$	64x64	3.11-2	2.71-2	8.59-3	7.31-3	3.52-3	3.21-3	3.01-3	2.96-3
AVb	8x8							5.25-2	5.25-2
	16x16							1.28-2	1.28-2
	32x32							3.23-3	3.23-3
	64x64	7.08-3	6.92-3	2.11-3	2.08-3	1.01-3	1.01-3	8.34-4	8.34-4
$(\alpha = 0.01)$	64x64	2.95-2	2.60-2	8.21-3	7.08-3	3.47-3	3.19-3	3.01-3	2.96-3

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