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UNCONSTRAINED METHODS FOR GENERALIZED NONLINEAR COMPLEMENTARITY AND VARIATIONAL INEQUALITY PROBLEMS*1)

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Abstract

In this paper, we construct unconstrained methods for the generalized nonlinear complementarity problem and variational inequalities. Properties of the correspondent unconstrained optimization problem are studied. We apply these methods to the subproblems in trust region method, and study their interrelationships. Numerical results are also presented.

1. Introduction

Linear and nonlinear complementarity problems have many important applications in various fields such as economics, transportation etc., they have attracted much attention since early 1960's. A standard nonlinear complementarity problem is to find a $x \in R^n$ such that:

$$F(x) \geq 0,\ x \geq 0,\ x^T F(x) = 0, \tag{1.1}$$

where $F:\ R^n \longrightarrow R^n$. For simplicity, we often call it NCP. Many authors have studied this problem and encouraging results have been reported. One can find an excellently complete summary for it in [2]. For recent works, see [7], [9], [3], [8].

The generalized complementarity problem, denoted by $GCP(X, F)$, is to find a vector $x^* \in X$ such that:

$$F(x^*) \in X^*, \quad \text{and} \quad F(x^*)^T x^* = 0, \tag{1.2}$$

where X^* denotes the dual cone of X at x^*:

$$X^* = \{y \in R^n :\ y^T x \geq 0,\ \forall\ x \in X\}. \tag{1.3}$$

It is well known that, problem (**??**) is a special case of variational inequality problem, which takes the following form:

$$x^* \in X, \quad \text{and} \quad F(x^*)^T (y - x^*) \geq 0, \forall\ y \in X\ . \tag{1.4}$$

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For simplity, we called it $VI(X,F)$. But in general, a variational inequality problem does not equal to a complementarity problem. However, under certain conditions, a variational problem may be considered as a mixed nonlinear complementarity problem.

The purpose of this paper is to construct unconstrained method for (**??**) and (**??**). In the following section, we first describe some notations and concepts. Some results which will be used in this paper are also stated. In Section 3, we consider problem (**??**) as unconstrained optimization problem and study its optimal properties. The subproblem in trust region method is discussed in Section 4. We also explore the relations between them and show a new view of trust region method. Some numerical results are also reported in the last section.

2. Preliminaries

First, we give a definition which is due to [3]:

Definition 2.1. *We call a function $\phi : R^2 \to R$ NCP-function if it satisfies the nonlinear complementarity condition*

$$\phi(a,b) = 0 \Longleftrightarrow a \geq 0,\ b \geq 0,\ ab = 0.$$

Consider the function defined as follows:

$$\phi(a,b) = (\sqrt{a^2+b^2} - a)(\sqrt{a^2+b^2} - b), \quad (a,b) \in R^2. \tag{2.1}$$

It is obvious that it is a NCP-function. Furthermore, we have the following result[8]:

Lemma 2.1. *let $\phi(a,b)$ is defined by* (**??**)*, the partial derivative of $\phi(a,b)$ equals to 0 if and only if (a,b) satisfies the complementarity condition. If (a,b) is strict complementarity, which means that $a+b>0$, we have:*

$$\frac{\partial^2\phi(a,b)}{\partial a^2} = 0, \quad \frac{\partial^2\phi(a,b)}{\partial a\partial b} = 0, \quad \frac{\partial^2\phi(a,b)}{\partial b^2} = 1 \tag{2.2}$$

hold for $b=0$ and $a>0$, and

$$\frac{\partial^2\phi(a,b)}{\partial a^2} = 1, \quad \frac{\partial^2\phi(a,b)}{\partial a\partial b} = 0, \quad \frac{\partial^2\phi(a,b)}{\partial b^2} = 0 \tag{2.3}$$

hold for $a=0$ and $b>0$.

Karamardian[4] first established the following basic relation between $GCP(X,F)$ and $VI(X,F)$.

Theorem 2.1. *Let X be a convex cone. Then $x^* \in X$ solves the problem VI(X,F) if and only if x^* solves the GCP(X,F).*

In the case where the set X is defined by the inequalities of the form

$$X = \{x \in R^n : g_i(x) \leq 0, i = 1,2,\ldots,m;\ h_j(x) = 0, j = 1,2,\ldots,p\}, \tag{2.4}$$

provided that the functions $g : R^n \to R^m$ and $h : R^n \to R^p$ satisfy some standard constraint qualification of the type often imposed in nonlinear programming[1], a variational inequality problem can also be cast as a generalized complementaeity problem. The following result summarized this conversion and has been used by several authors in different literatures[2].

Theorem 2.2. *Let $g : R^n \to R^m$ and $h : R^n \to R^p$ be continuously differentiable and let X be defined by (**??**).*

(a) *If x^* solves the VI(X,F) and if a certain constraint qualification holds for the set X at the point x^*, then for some $\lambda^* \in R^m$ and $\mu^* \in R^p$, (x^*, λ^*, μ^*) solves the $GCP(R^n \times R^m_+ \times R^p, H)$ where $H: \ R^{n+m+p} \to R^{n+m+p}$ is defined by*

$$H\begin{pmatrix} x \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} F(x) + \sum_{i=1}^m \lambda_i \nabla g_i(x) + \sum_{j=1}^p \mu_j \nabla h_j(x) \\ -g(x) \\ h(x) \end{pmatrix}. \tag{2.5}$$

(b) *Conversely, if g_i is convex for $i = 1, 2, \ldots, m$ and h_j is affine for $j = 1, 2, \ldots, p$, and if (x^*, λ^*, μ^*) solves the $GCP(R^n \times R^m_+ \times R^p, H)$, then x^* solves the VI(X,F).*

The problem (**??**) is usually called a mixed nonlinear complementarity problem, it is often used in the analysis of sensitivity. Now we state some assumptions which will be employed in this paper.

Assumptions.

(i) *Let $F : R^n \to R^n$ and $g : R^n \to R^m, h : R^n \to R^p$ be twice continuously differentiable. Let X be defined in (**??**) and x^* be a solution of the VI(X,F). Let $I_1 = \{i : \ \lambda_i > 0\}, I_2 = \{i : \ g_i(x^*) < 0\}, I_0 = \{i : \ g_i(x^*) = 0, \lambda_i = 0\}$.*

(ii) *There exists vector $\lambda^* \in R^m$ and $\mu^* \in R^p$ such that:*

$$\begin{aligned} F(x^*) \quad &+ \quad \sum_{i=1}^m \lambda_i \nabla g_i(x^*) + \sum_{j=1}^p \mu^* \nabla h_j(x^*) = 0, &(2.6)\\ \lambda_i \geq 0, \lambda_i g_i(x^*) \quad &= \quad 0, \quad i = 1, 2, \ldots, m. \ \ h_j(x^*) = 0, \ j = 1, \ldots, p. &(2.7) \end{aligned}$$

The gradients of the binding constraints $(\nabla g_i(x^*) : \ i \in I_1, \nabla h_j(x^*) : \ j \in \{1, \ldots, p\})$ are linearly independent.

(iii)

$$z^T[\nabla F(x^*) + \sum_{i \in I_1} \lambda_i \nabla^2 g_i(x^*) + \sum_{j=1}^p \mu_j \nabla^2 h_j(x^*)]z > 0, \tag{2.8}$$

for all $z \neq 0$ such that:

$$\begin{aligned} z^T \nabla g_i(x^*) = 0, \qquad & \forall i \in I_1. \\ z^T \nabla g_i(x^*) \leq 0, \qquad & \forall i \in I_0. \\ z^T \nabla h_j(x^*) = 0, \qquad & \forall j = 1, \cdots, p. \end{aligned}$$

These assumptions are general in the sensitivity analysis, usually, if the solution is a regular solution, all these assumptions are satisfied. In fact, (**??**) is the first-order condition at the solution x^*, (**??**) is a complementarity condition for $\lambda \in R^m_+$ and the constraints $g(x) \leq 0$. Assumption (iii) is the generalized second-order sufficient condition for an optimization problem which does not demand $\nabla F(x^*)$ be positive definite. From theorem 3.11 in [2] one can see that x^* is a locally unique solution of the $VI(X, F)$. If $g_i(x)$ is convex and $h_j(x)$ is affine, x^* is also a locally unique solution of the $GCP(R^n \times R^m_+, H)$ where H is defined in (**??**), for details, see [5],[6]. Obviously, $I_0 + I_1 + I_2 = \{1, 2, \cdots, m\}$ holds.

3. Unconstrained Methods for GCP and VI

In last section, we have stated some assumptions and results. Now we consider the problem (**??**). Assume assumptions (i)–(iii) hold, from discussions in Section 2, we know that x^* is a locally unique solution of mixed complementarity problem (**??**). Now we define:

$$H_1 \begin{pmatrix} x \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} F(x) + \sum_{i=1}^m \lambda_i \nabla g_i(x) + \sum_{j=1}^p \mu_j \nabla h_j(x) \\ h(x) \end{pmatrix}. \tag{3.1}$$

Let ϕ is defined as in (**??**), $H_1(x, \lambda, \mu)$ is defined in (**??**) . Define:

$$\psi(y) = H_1(y)^T H_1(y) + \sum_{i=1}^m \psi_i = H_1(y)^T H_1(y) + \sum_{i=1}^m \phi(\lambda_i, -g_i(x)),\ y = \begin{pmatrix} x \\ \lambda \\ \mu \end{pmatrix} \in R^{n+m+p}. \tag{3.2}$$

Now we have the following results:

Lemma 3.2. *Let ϕ is defined by* (**??**)*, $\psi(x, \lambda, \mu)$ is defined as in* (**??**)*, X is a convex set. If x^* is a solution of $VI(X, F)$ and assumption* (ii) *holds, then (x^*, λ^*, μ^*) is a global minimizer of $\psi(x, \lambda, \mu)$ and $\psi = 0$. Conversely, if $\psi(x, \lambda, \mu)$ equals 0, x is a solution of $VI(X, F)$.*

Proof. The first part of the lemma follows from the definition of ψ and assumptions. If $\psi(x, \lambda, \mu)$ equals 0, by lemma 2.1, it must hold $\lambda_i \geq 0$. By (**??**), one can easily verify that x is a solution of $VI(X, F)$.

It is noticed that for the above lemma, we do not suppose all assumptions (i)-(iii) hold. If all of the assumptions in Section 2 are true and a strict complementarity condition such that:

$$\lambda_i \geq 0,\ \lambda_i g_i(x^*) = 0,\ \lambda_i - g_i(x^*) > 0, \quad i = 1, 2, \cdots, m \tag{3.3}$$

hold, which implies that $I_0 = \emptyset$. Then we can get the following result:

Theorem 3.3. *Let $x^* \in R^n$ be a solution of* (**??**)*. Suppose that the assumptions* (i)-(iii) *and* (**??**) *hold at x^*, let $\psi(x, \lambda, \mu)$ is defined by* (**??**) *and ϕ is defined by* (**??**)*. Then (x^*, λ^*, μ^*) is a global minimizer of the function $\psi(x, \lambda, \mu)$, the gradients of $\psi(x, \lambda, \mu)$ at (x^*, λ^*) equals to 0, furthermore, the Hessian of $\psi(x, \lambda, \mu)$ at (x^*, λ^*, μ^*) is positive definite.*

Proof. The first statement of the theorem follows from Lemma 3.1. The second conclusion can be derived by inductive algebraic calculus and Lemma 2.1. Now we consider the Hessian of ψ. From assumptions (i)-(iii), one can see that ψ is twice differentiable in a neighborhood of (x^*, λ^*, μ^*). Also we have:

$$\begin{aligned}
\nabla^2\psi &= \nabla^2 H_1(x^*, \lambda^*, \mu^*)^T H_1(x^*, \lambda^*, \mu^*) + \sum_{i=1}^{m} \nabla^2\psi_i \\
&= \nabla H_1(x^*, \lambda^*, \mu^*)\nabla H_1(x^*, \lambda^*, \mu^*)^T + \sum_{i=1}^{m} \frac{\partial^2\phi(\lambda_i^*, -g_i(x^*))}{\partial a^2} e_{n+p+i}e_{n+p+i}^T \\
&\quad + \sum_{i=1}^{m} \frac{\partial^2\phi(\lambda_i^*, -g_i(x^*))}{\partial b^2} \begin{pmatrix} \nabla g_i(x^*) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \nabla g_i(x^*) & 0 \\ 0 & 0 \end{pmatrix}^T \\
&= \nabla H_1(x^*, \lambda^*, \mu^*)\nabla H_1(x^*, \lambda^*, \mu^*)^T + \sum_{i\in I_2} e_{n+p+i}e_{n+p+i}^T \\
&\quad + \sum_{i\in I_1} \begin{pmatrix} \nabla g_i(x^*) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \nabla g_i(x^*) & 0 \\ 0 & 0 \end{pmatrix}^T \\
&= A^T A, \qquad (3.4)
\end{aligned}$$

where e_{n+p+i} denotes a $n+m+p$ dimension unit vector whose $n+p+i$-th component is 1, A is a block matrix such that:

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \\ A_{41} & A_{42} & A_{43} \end{pmatrix},$$

$$\begin{aligned}
A_{11} = [\nabla F + \sum_{i=1}^{m} \lambda_i^* \nabla^2 g_i + \sum_{j=1}^{p} \mu_j \nabla^2 h_j] \in R^{n\times n},\ A_{12} = [\nabla g_1, \nabla g_2, \ldots, \nabla g_m] \in R^{n\times m}; \\
A_{13} = [\nabla h_1, \nabla h_2, \ldots, \nabla h_p] \in R^{n\times p}, A_{21} = [\nabla g_i : i \in I_1]^T \in R^{m1\times n}, A_{22} = 0_{m1\times m}; \\
A_{23} = 0_{m1\times p}, A_{31} = 0_{m2\times n}, A_{32} = [e_i : i \in I_2]^T \in R^{m2\times m}, A_{33} = 0_{m2\times p}; \\
A_{41} = [\nabla h_j : j = 1, \ldots, p]^T \in R^{p\times n}, A_{42} = 0_{p\times m}, A_{43} = 0_{p\times p}
\end{aligned}$$

where $m1, m2$ denote the number of components in set I_1, I_2 separately, e_i is a m dimension unit vector whose i-th component equals 1. Assume there exists a vector $y = (x^T, \lambda^T, \mu^T)^T \in R^{n+m+p}$ such that $Ay = 0$, then:

$$A_{11}x + A_{12}\lambda + A_{13}\mu = 0; \qquad (3.5)$$

$$A_{21}x = 0, A_{32}\lambda = 0, A_{41}x = 0. \qquad (3.6)$$

Thus for such $\lambda = (\lambda_1, \ldots, \lambda_m)^T$, we have that $\lambda_i = 0, \forall i \in I_2$. Furthermore, for all $i \in I_1, \nabla g_i^T x = 0$ and for $j = 1, 2, \cdots, p$ $\nabla h_j^T x = 0$. It follows that

$$A_{11}x + \sum_{i\in I_1} \lambda_i \nabla g_i + \sum_{j=1}^{p} \mu_j \nabla h_j = 0. \qquad (3.7)$$

So

$$x^T A_{11} x + \sum_{i \in I_1} \lambda_i x^T \nabla g_i + \sum_{j=1}^{p} \mu_j x^T \nabla h_j = 0 \ . \tag{3.8}$$

By assumption (iii), it holds $x = 0$. It follows that $\sum_{i \in I_1} \lambda_i \nabla g_i + \sum_{j=1}^{p} \mu_j \nabla h_j = 0$. From assumption (ii), $\lambda_i = 0, \forall i \in I_1$ and $\mu_j = 0, j = 1, 2, \ldots, p$. So if $Ay = 0, y = 0$, which means A is nonsingular. This proves the last conclusion.

From the above theorem, we can see that x^* is not only a global minimizer of $\psi(x)$, but also a strict local minimizer of $\psi(x)$. By the continuity, ψ is twice differentiable in a neighborhood of x^*, so if the initial point x^0 is chosen sufficiently near x^*, we can use Newton or Quasi-Newton method to search the minimizer of $\psi(x)$, the local convergence properties is obvious.

4. Application to Subproblem in Trust Region Method

Trust region methods has obtained great success in optimization. One of its main advantages is the global convergence. Typically, at each iteration, we solve a subproblem which is reliable in a near neighborhood of the current point. Now we consider the subproblem in trust region methods for constrained optimization. The problem has the form

$$\min_{d \in R^n} \Phi(d) = g^T d + \frac{1}{2} d^T B d, \tag{4.1}$$

s.t.

$$d^T d - \alpha_1 \leq 0, \tag{4.2}$$

$$||A^T d + c||_2^2 - \alpha_2 \leq 0, \tag{4.3}$$

where $g \in R^n, B \in R^{n \times n}, A \in R^{n \times m}, c \in R^m, \alpha_1 > 0, \alpha_2 > 0$. Many authors have studied this problem, for examples, Y. Yuan[10] propose a dual algorithm for it, Y. Zhang reform this problem and give some results which include the case when the two constraints are non-convex quadratics[12].

Usually, this problem has a global minimizer which satisfies the assumptions in section 2. From the discussions in last section, we can cast this problem as unconstrained optimization in a space of $n + 2$ dimensions. It is worth mentioning that when we do this, the objective function and constraints can be non-quadratic functions, thus some subproblem with non-quadratic constraints can also use our methods[11]. On the other hand, if the original constrained problem has a global minimizer and some qualifications are satisfied at the minimizer, then it can also be considered as unconstrained problem in a space of $n + m$ dimensions, where m is the number of constraints. Thus we can view the trust region method as a method which decreases the number of space dimensions. Because all the subproblem are locally reliable , by the local properties of our method, one can wish the results is promising.

We now consider the trust region method for variational inequality problem $VI(X, F)$ with convex set X. Under mild conditions, one can convert $VI(X, F)$ into unconstrained problem. If a feasible interior point is given, we can solve the problem

$VI(X \cap S, F)$ where S denotes a neighorhood of the current point. This subproblem often has a local solution when the original problem is solvable. From the results in last section, the subproblem can be relatively easy to solve. If a solution of the subproblem is the current point, it is also a solution of the original problem. Much work is needed for such a method which will be on our further study.

5. Numerical Results

A FORTRAN subroutine is designed to test our method. The update we take is BFGS with inexact line search, which satisfies Wolfe-conditions. The stop criterion is that $||g|| \leq 10^{-8}$. The first five problem are the same as in [10], all of them consider the problem (**??**) with $n = 4$ and $\alpha_1 = 1.0$, we restate them as follows:

Problem 1. $m = 1, \alpha_2 = 0.25, g = (0.5, 1, 1, 1)^T, B = I, c = -1$ and $A = (1, 0, 0, 0)^T$.

Problem 2. $m = 2, \alpha_2 = 2, g = (0, 0.5, 0, 0)^T, B = diag[1, 2, 3, 4], c = (-2, 0)^T$ and $A = (I_{2\times2} 0_{2\times2})^T$.

Problem 3. $m = 3, \alpha_2 = 0.36, g = (-3, -4, -5, 0)^T, B = I, c = (-0.3, -0.4, -0.5)^T$ and $A = (I_{3\times3} 0_{3\times1})^T$.

Problem 4. $m = 4, \alpha_2 = 3, g = (-5, -1, -1, -1)^T, B = diag[1, 1/2, 1/3, 1/4], c = (1, 1, 1, -0.5)^T$ and $A = (A_{i,j})_{4\times4}$ where $A_{i,j} = 1$ for all $i, j = 1, 2, 3, 4$.

Problem 5. As problem 4, except that $A_{i,j} = 0.1$ for all $i \neq j$.

The following is our result:

Table 1

Problem	Initial point	No. of. iter	Val. of. ψ
prob 1	(0,0,0,0,0,0)	24/31/25	0.4916976E-18
	(-10,-10,-10,-10,1,1)	48/64/53	0.734398E-19
prob 2	(0,0,0,0,0,0)	14/19/15	0.0
	(-10,-10,-10,-10,1,1)	63/80/65	0
prob 3	(0,0,0,0,0,0)	28/43/32	0.414645E-17
	(-10,-10,-10,-10,1,1)	61/95/74	0
prob 4	(0,0,0,0,0,0)	30/52/36	0.302106E-18
	(-10,-10,-10,-10,1,1)		fails
prob 5	(0,0,0,0,0,0)	22/33/24	0.2188219E-17
	(-10,-10,-10,-10,1,1)	45/61/49	0.568738E-19

All the solution we get is the same as that in [10] except for problem 5, where our solution is $(0.5827114, -0.4720780, -0.4955691, 0.4381793)$, it seems to be a editorial mistake in [10] where the solution is $(0.5827114, -0.4720780, -0.4955691, -0.4381793)$.

From the table, one can see that more iterations are needed to obtain a solution of the problem, one of them even fails. But our method is using BFGS method while Y.Yuan[10] using Newton method. In fact, if we view our method as sums of least square of some equations and partial Newton method is employed, then our method will be similar to that in [10].

It is worth mentioning that all of our initial points are infeasible. We also try various choices of the initial point, when it is not too far from the feasible set, our method converges in all cases. In fact, if a feasible initial point is given, the method converges rapidlier. For the subproblem in trust region methods, at each iteration, we can choose the current point as a feasible initial point.

We also use our methods for variational inequality problem, our choice is $F(x) = Bx + g$, where g and the constrained set are the same as stated above, for simplicity, we still call it problem 1-5 separately. the following is the result with

$$B = \begin{pmatrix} 10.5 & -10 & 8.5 & 13.6 \\ 3.5 & 6.88 & 1.45 & -2.8 \\ -11 & 1.25 & 2.8 & 0.5 \\ 1.25 & 15.3 & 6.5 & 10 \end{pmatrix} \tag{5.1}$$

and all the initial points is $(0, 0, 0, 0, 0, 0)$.

Table 2

Problem	solution obtained	No. of. iter	Val. of. ψ
prob 1	(0.5,-0.38287,0.77458,-0.05868,3.6654,34.394)	73/97/80	0.217E-18
prob 2	(0.59276,-0.14028,0.76197,-0.21991,6.1317,10.48)	37/50/40	0.203E-19
prob 3	(0.16436,0.07274,0.86996,-0.4592,5.18496,0)	25/38/29	0.0
prob 4	(0.06854,-0.12322,-0.14963,0.15214,0,0.98236)	27/46/29	0.128E-16
prob 5	(0.04959,-0.03899,-0.10487,0.22057,0,1.62654)	17/28/19	0.0

From the above table, it can be seen that for the variational inequality problem defined by (**??**), our method converges for various constrained set. We also notice that the behavior of our method is different according to various constrained set, sometimes our method may fail. One of the reasons is due to the ineffectiveness of our method, another reason is that when the constrained set is large, the original variational inequality problem may have no solution. The following are two examples with $F(x) = B_1x + g$ and $F(x) = B_2x + g$, where g is the same as stated before and that:

$$B_1 = (b_{i,j}), b_{i,j} = {x_i}^3 - {x_j}^2 + x_ix_j - x_i + x_j + i/2,$$
$$B_2 = (b_{i,j}), b_{i,j} = \exp(x_i) - \cos x_j + x_ix_j - x_i + x_j + i/2,$$

the constrained set is defined in problem 1-5 with different α_1, α_2.

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Table 3

Problem point	solution obtained	(α_1, α_2)	No. of. iter	Val. of. ψ
Prob 1(B_1)	(0,-0.45190,-0.21390,-0.00562)	(0.25,1.0)	29/43/31	0.117E-19
B_2	(0,-0.44908,-0.21596,0.04114)	(0.25,1.0)	25/36/27	0.765E-17
Prob 2(B_1)	(0.28096,0.21189,-0.25369,-0.24860)	(0.25,3)	31/42/34	0.130E-18
B_2	(0.28097,0.21197,-0.25360,-0.24862)	(0.25,3.0)	30/43/32	0.188E-17
Prob 3 (B_1)	(0.21669,0.27387,0.33074,-0.13655)	(0.25,0.36)	34/52/38	0
B_2	(0.21304,0.27193,0.33055,-0.14632)	(0.25,0.36)	33/53/41	0.279E-17
Prob 4 (B_1)	(0.41935,-0.16099,-0.15718,-0.15337)	(0.25,3)	39/54/43	0.163E-19
B_2	(0.41935,-0.16098,-0.15718,-0.15338)	(0.25,3)	38/51/40	0.343E-18
(B_1)	(0.32049,-0.2440,-0.24402,0.16794)	(0.25,3)	28/47/32	0.403E-19
B_2	(0.32078,-0.24431,-0.24423,0.16663)	(0.25,3)	27/45/31	0.244E-18

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