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DIFFERENCE SCHEMES WITH NONUNIFORM MESHES FOR NONLINEAR PARABOLIC SYSTEM*1)

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Abstract

The boundary value problem for the nonlinear parabolic system is solved by the finite difference method with nonuniform meshes. The existence and a priori estemates of the discrete vector solutions for the general difference schemes with unequal meshsteps are established by the fixed point technique. The absolute and relative convergence of the discrete vector solution are justified by a series of a priori estimates. The analysis of mentioned problems are based on the assumption of heuristic character concerning the existence of the unique smooth solution for the original problem of the nonlinear parabolic system.

1. Introduction

1. From the very beginning of sixties to the late eighties, there are many works contributed to the studies of the boundary problems and initial value problems for the ordinary differential equations by the method of difference schemes with nonuniform meshes[1−4]. But it is extremely rare on the works concerning to the analysis of finite difference schemes with nonuniform meshes for the problems of partial differential equations. By using of the difference schemes with nonuniform meshes approximation for the problems of partial differential equations there are many unexpected phenomenon and self-contradictive things both in computations and in analysis.

In this work, we are going to study the difference schemes with nonuniform meshes approximated to the boundary problem for nonlinear parabolic systems of partial differential equations under the assumption of the heuristic character for the existence and uniqueness of the smooth solution of the mentioned problem.

Let us now consider the boundary problem of the nonlinear parabolic systems of second order

$$u_t = A(x,t,u,u_x)u_{xx} + f(x,t,u,u_x), \tag{1}$$

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where $u = (u_1, \cdots, u_m)$ is the m-dimensional unknown vector function ($m \geq 1$), $A(x,t,u,p)$ is a $m \times m$ matrix function, $f(x,t,u,p)$ is a m-dimensional vector function and $u_t = \frac{\partial u}{\partial t}$, $u_x = \frac{\partial u}{\partial x}$ and $u_{xx} = \frac{\partial^2 u}{\partial x^2}$ are the corresponding m-dimensional vector derivatives. The coefficient matrix $A(x,t,u,p)$ is positive definite, hence the system is parabolic. Let us consider in the rectangular domain $Q_T = \{0 \leq x \leq l, \quad 0 \leq t \leq T\}$ with the given positive constants $l > 0$ and $T > 0$, the boundary problem for the nonlinear parabolic system (1) of partial differential equations with the boundary conditions

$$\begin{aligned} u(0,t) &= \psi_0(t), \\ u(l,t) &= \psi_1(t) \end{aligned} \tag{2}$$

and the initial condition

$$u(x,0) = \phi(x), \tag{3}$$

where $\psi_0(t), \quad \psi_1(t)$ and $\phi(x)$ are given m-dimensional vector functions of variables $t \in [0, \ T]$ and $x \in [0, \ l]$ respectively.

In this work the existence and the estimates of the discrete solutions for the finite difference schemes with nonuniform meshes are established by the fixed point technique[5]. The absolute and relative convergence of the discrete solutions of difference schemes with unequal meshsteps are justified by means of a series of a priori estimates. It is notice that in the present, the existence of the unique smooth solution for original problem of the nonlinear parabolic system is assumed to be valid. This is the fundamental assumption of heurustic character in the present study.

In the present investigation for the difference schemes with nonuniform meshes we are repeatedly using the methods and treatments similar to the study of analogous problems for the cases of difference schemes with equal meshstep.

2. Now suppose that for the boundary problem (2) and (3) of the nonlinear parabolic system (1) of second order, the following conditions are fulfilled.

(I) The boundary problem (2) and (3) for the nonlinear parabolic system (1) has a unique smooth m-dimensional vector solution $u(x,t) \in C^{(4,2)}_{x,t}(Q_T)$.

(II) The coefficient matrix $A(x,t,u,p)$ is positively definite, that is, there is a positive constant $\sigma_0 > 0$, such that for any $\xi \in R^m$,

$$(\xi, \quad A(x,t,u,p)\xi) \geq \sigma_0 |\xi|^2, \tag{4}$$

where $(x,t) \in Q_T$ and $u, \quad p \in R^m$.

(III) The m-dimensional vector function $f(x,t,u,p)$ and the $m \times m$ matrix function $A(x,t,u,p)$ are continuous with respect to variables $(x,t) \in Q_T$ and continuously differentiable with respect to m -dimensional vector variables $u, \quad p \in R^m$.

(IV) The boundary vector functions $\psi_0(t)$ and $\psi_1(t)$ are continuously differentiable with respect to $t \in [0, \quad T]$. The initial vector function $\phi(x)$ is continuously differentiable with respect to $x \in [0,\ l]$. Furthermore, the equalities

$$\begin{aligned} \psi_0(0) &= \phi(0), \\ \psi_1(0) &= \phi(l) \end{aligned} \tag{5}$$

hold at the corners of the bottom of the rectangular domain Q_T.

2. Difference Schemes

3. Let us divide the rectangular domain $Q_T = \{0 \le x \le l,\quad 0 \le t \le T\}$ into the small rectangular grids $\overline{Q}_\Delta = \{\overline{Q}^{n+\frac{1}{2}}_{j+\frac{1}{2}} = (x_j \le x \le x_{j+1};\ \ t^n \le t \le t^{n+1}) \big| j = 0,1,\cdots,J-1;\ \ n = 0,1,\cdots,N-1\}$ by the parallel lines $x = x_j\ \ (j = 0,1,\cdots,J)$ and $t = t^n\ \ (n = 0,1,\cdots,N)$ with the integers J and N, where

$$\begin{aligned} &0 = x_0 < x_1 < \cdots < x_{J-1} < x_J = l, \\ &0 = t^0 < t^1 < \cdots < t^{N-1} < t^N = T. \end{aligned}$$

The meshsteps $h = \{h_{j+\frac{1}{2}} = x_{j+1} - x_j > 0 | j = 0,1,\cdots,J-1\}$ and $\tau = \{\tau^{n+\frac{1}{2}} = t^{n+1} - t^n > 0 | n = 0,1,\cdots,N-1\}$ are in general assumed to be unequal.

Let us denote by τ^* or simply by τ the maximum of the meshsteps $\tau = \{\tau^{n+\frac{1}{2}} \big| n = 0,1,\cdots,N-1\}$, that is,

$$\tau = \max_{n=0,1,\cdots,N-1} \tau^{n+\frac{1}{2}}.$$

Also let us denote for the unequal meshsteps $h = \{h_{j+\frac{1}{2}} \big| j = 0,1,\cdots,J-1\}$ by h^* or simply by h and by h_* the maximum and minimum of the meshsteps respectively

$$\begin{aligned} h &= \max_{j=0,1,\cdots,J-1} h_{j+\frac{1}{2}}, \\ h_* &= \min_{j=0,1,\cdots,J-1} h_{j+\frac{1}{2}}. \end{aligned}$$

And also denote the ratio constant

$$M_h^* = \frac{h}{h_*}.$$

Denote by $v_\Delta = v_h^\tau = \{v_j^n \big| j = 0,1,\cdots,J;\ \ n = 0,1,\cdots,N\}$ the m-dimensional discrete vector function defined on the discrete rectangular domain $Q_\Delta = \{(x_j,\ t^n) | j = 0,1,\cdots,J;\quad n = 0,1,\cdots,N\}$ of the grid points. The difference quotient $\delta v_h^\tau = \{\delta v^n_{j+\frac{1}{2}} = \frac{v^n_{j+1} - v^n_j}{h_{j+\frac{1}{2}}} | j = 0,1,\cdots,J-1;\ \ n = 0,1,\cdots,N\}$ of first order for the discrete vector function $v_\Delta = v_h^\tau = \{v_j^n | j = 0,1,\cdots,J;\ \ n = 0,1,\cdots,N\}$ is a discrete vector function defined on the grid points $\{(x^{(1)}_{j+\frac{1}{2}},\ \ t^n) | j = 0,1,\cdots,J-1;\ \ n =$

$0, 1, \cdots, N\}$, where $x^{(1)}_{j+\frac{1}{2}} = \frac{1}{2}(x_{j+1} + x_j)$ $(j = 0, 1, \cdots, J-1)$. The difference quotient $\delta^2 v^\tau_h = \{\delta^2 v^n_j = \frac{\delta v^n_{j+\frac{1}{2}} - \delta v^n_{j-\frac{1}{2}}}{h^{(2)}_j} | j = 1, 2, \cdots J-1; \quad n = 0, 1, \cdots, N\}$ of second order can be regarded as a discrete vector function defined on the grid points $\{(x^{(2)}_j, \ t^n) | j = 0, 1, \cdots, J-1; \ \ n = 0, 1, \cdots, N\}$, where $x^{(2)}_j = \frac{1}{2}(x^{(1)}_{j+\frac{1}{2}} + x^{(1)}_{j-\frac{1}{2}})$ and $h^{(2)}_j = \frac{1}{2}(h_{j+\frac{1}{2}} + h_{j-\frac{1}{2}})$, $(j = 0, 1, \cdots, J-1)$.

Let us now construct the general difference schemes with nonuniform meshes for the above mentioned nonlinear parabolic system (1) of second order as follows:

$$\begin{aligned} \frac{v^{n+1}_j - v^n_j}{\tau^{n+\frac{1}{2}}} = \ & A^{n+\alpha}_j \delta^2 v^{n+\alpha}_j + f^{n+\alpha}_j, \\ & (j = 1, 2, \cdots, J-1; \ \ n = 0, 1, \cdots, N-1), \end{aligned} \tag{$(1)_\Delta$}$$

where

$$\begin{aligned} \delta^2 v^{n+\alpha}_j &= \frac{1}{h^{(2)}_j}\left[\frac{v^{n+\alpha}_{j+1} - v^{n+\alpha}_j}{h_{j+\frac{1}{2}}} - \frac{v^{n+\alpha}_j - v^{n+\alpha}_{j-1}}{h_{j-\frac{1}{2}}}\right], \\ A^{n+\alpha}_j &= A\left(x_j, \ t^{n+\alpha}, \ \overline{\delta}^0 v^{n+\alpha}_j, \ \overline{\delta}^1 v^{n+\alpha}_j\right), \\ f^{n+\alpha}_j &= f\left(x_j, \ t^{n+\alpha}, \ \tilde{\delta}^0 v^{n+\alpha}_j, \ \tilde{\delta}^1 v^{n+\alpha}_j\right) \end{aligned} \tag{6}$$

and

$$\begin{aligned} \overline{\delta}^0 v^{n+\alpha}_j &= \alpha\left[\overline{\beta}^{n+1}_{1j} v^{n+1}_{j+1} + \overline{\beta}^{n+1}_{2j} v^{n+1}_j + \overline{\beta}^{n+1}_{3j} v^{n+1}_{j-1}\right] \\ &\quad + \left[\overline{\beta}^n_{4j} v^n_{j+1} + \overline{\beta}^n_{5j} v^n_j + \overline{\beta}^n_{6j} v^n_{j-1}\right], \\ \overline{\delta}^1 v^{n+\alpha}_j &= \alpha\left[\overline{\gamma}^{n+1}_{1j} \delta v^{n+1}_{j+\frac{1}{2}} + \overline{\gamma}^{n+1}_{2j} \delta v^{n+1}_{j-\frac{1}{2}}\right] \\ &\quad + \left[\overline{\gamma}^n_{3j} \delta v^n_{j+\frac{1}{2}} + \overline{\gamma}^n_{4j} \delta v^n_{j-\frac{1}{2}}\right], \\ \tilde{\delta}^0 v^{n+\alpha}_j &= \alpha\left[\tilde{\beta}^{n+1}_{1j} v^{n+1}_{j+1} + \tilde{\beta}^{n+1}_{2j} v^{n+1}_j + \tilde{\beta}^{n+1}_{3j} v^{n+1}_{j-1}\right] \\ &\quad + \left[\tilde{\beta}^n_{4j} v^n_{j+1} + \tilde{\beta}^n_{5j} v^n_j + \tilde{\beta}^n_{6j} v^n_{j-1}\right], \\ \tilde{\delta}^1 v^{n+\alpha}_j &= \alpha\left[\tilde{\gamma}^{n+1}_{1j} \delta v^{n+1}_{j+\frac{1}{2}} + \tilde{\gamma}^{n+1}_{2j} \delta v^{n+1}_{j-\frac{1}{2}}\right] \\ &\quad + \left[\tilde{\gamma}^n_{3j} \delta v^n_{j+\frac{1}{2}} + \tilde{\gamma}^n_{4j} \delta v^n_{j-\frac{1}{2}}\right] \end{aligned} \tag{7}$$

with

$$\begin{aligned}
&\alpha\left[\overline{\beta}_{1j}^{n+1}+\overline{\beta}_{2j}^{n+1}+\overline{\beta}_{3j}^{n+1}\right] + \left[\overline{\beta}_{4j}^{n}+\overline{\beta}_{5j}^{n}+\overline{\beta}_{6j}^{n}\right] = 1\ ,\\
&\alpha\left[\overline{\gamma}_{1j}^{n+1}+\overline{\gamma}_{2j}^{n+1}\right] + \left[\overline{\gamma}_{3j}^{n}+\overline{\gamma}_{4j}^{n}\right] = 1\ ,\\
&\alpha\left[\tilde{\beta}_{1j}^{n+1}+\tilde{\beta}_{2j}^{n+1}+\tilde{\beta}_{3j}^{n+1}\right] + \left[\tilde{\beta}_{4j}^{n}+\tilde{\beta}_{5j}^{n}+\tilde{\beta}_{6j}^{n}\right] = 1\ ,\\
&\alpha\left[\tilde{\gamma}_{1j}^{n+1}+\tilde{\gamma}_{2j}^{n+1}\right] + \left[\tilde{\gamma}_{3j}^{n}+\tilde{\gamma}_{4j}^{n}\right] = 1
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
&\alpha\left[\left|\overline{\beta}_{1j}^{n+1}\right|+\left|\overline{\beta}_{2j}^{n+1}\right|+\left|\overline{\beta}_{3j}^{n+1}\right|\right] + \left[\left|\overline{\beta}_{4j}^{n}\right|+\left|\overline{\beta}_{5j}^{n}\right|+\left|\overline{\beta}_{6j}^{n}\right|\right] \le \overline{\delta}_0\ ,\\
&\alpha\left[\left|\overline{\gamma}_{1j}^{n+1}\right|+\left|\overline{\gamma}_{2j}^{n+1}\right|\right] + \left[\left|\overline{\gamma}_{3j}^{n}\right|+\left|\overline{\gamma}_{4j}^{n}\right|\right] \le \overline{\delta}_1\ ,\\
&\alpha\left[\left|\tilde{\beta}_{1j}^{n+1}\right|+\left|\tilde{\beta}_{2j}^{n+1}\right|+\left|\tilde{\beta}_{3j}^{n+1}\right|\right] + \left[\left|\tilde{\beta}_{4j}^{n}\right|+\left|\tilde{\beta}_{5j}^{n}\right|+\left|\tilde{\beta}_{6j}^{n}\right|\right] \le \tilde{\delta}_0\ ,\\
&\alpha\left[\left|\tilde{\gamma}_{1j}^{n+1}\right|+\left|\tilde{\gamma}_{2j}^{n+1}\right|\right] + \left[\left|\tilde{\gamma}_{3j}^{n}\right|+\left|\tilde{\gamma}_{4j}^{n}\right|\right] \le \tilde{\delta}_1
\end{aligned} \tag{9}$$

and here $\overline{\beta}$, $\overline{\gamma}$, $\tilde{\beta}$, $\tilde{\gamma}'$s with indices are constants and $\overline{\delta}_0$, $\overline{\delta}_1$, $\tilde{\delta}_0$, $\tilde{\delta}_1 \ge 1$ and $0 \le \alpha \le 1$ are also constants. Here we also have

$$\begin{aligned}
\delta^2 v_j^{n+\alpha} = \ & \alpha\delta^2 v_j^{n+1} + (1-\alpha)\delta^2 v_j^n\ ,\\
& (j=1,2,\cdots,J-1;\ \ n=0,1,\cdots,N-1).
\end{aligned}$$

The finite difference boundary conditions are

$$\begin{aligned}
& v_0^n = \psi_0^n\ ,\\
& v_J^n = \psi_1^n\ ,\quad (n=0,1,\cdots,N)\ ,
\end{aligned} \tag{2}_\Delta$$

where $\psi_0^n = \psi_0(t^n)$ and $\psi_1^n = \psi_1(t^n)$ $(n=0,1,\cdots,N)$. The finite difference initial condition is

$$v_j^0 = \phi_j\ ,\quad (j=0,1,\cdots,J)\ , \tag{3}_\Delta$$

where $\phi_j = \phi(x_j)$, $(j=0,1,\cdots,J)$.

4. Let us state some lemmas as follows, which are useful in later discussion. Some lemmas can be obtained by direct calculations and some can be found in [9-11].

Lemma 1. *For any* $u_h = \{u_j \,|\, j=0,1,\cdots,J\}$ *and* $v_h = \{v_j \,|\, j=0,1,\cdots,J\}$, *there* *are*

$$\begin{aligned} & \sum_{j=0}^{J-1} u_j\left(v_{j+1}-v_j\right) \\ = & -\sum_{j=1}^{J} v_j\left(u_j-u_{j-1}\right)-u_0 v_0+u_J v_J \quad . \\ & \sum_{j=1}^{J-1} u_j\left(\delta v_{j+\frac{1}{2}}-\delta v_{j-\frac{1}{2}}\right) \\ = & -\sum_{j=0}^{J-1} \delta u_{j+\frac{1}{2}} \delta v_{j+\frac{1}{2}} h_{j+\frac{1}{2}}-u_0 \delta v_{\frac{1}{2}}+u_J \delta v_{J-\frac{1}{2}} \quad . \end{aligned}$$

Lemma 2. *For any $w_h = \{w_j|\, j = 0, 1, \cdots, J\}$ defined on the grid points $\{x_j|\, j = 0, 1, \cdots, J\}$ with unequal meshsteps $h = \{h_{j+\frac{1}{2}}\big|\, j = 0, 1, \cdots, J-1\}$, there are relations*

$$\begin{aligned} \|w_h\|_\infty & \leq h_*^{-\frac{1}{2}}\|w_h\|_2 \quad , \\ \|\delta w_h\|_\infty & \leq h_*^{-\frac{1}{2}}\|\delta w_h\|_2 \quad . \end{aligned}$$

Proof. For the first estimatate, we have

$$\begin{aligned} \|w_h\|_\infty & = \max_{j=0,1,\cdots,J}|w_j| = |w_{j_0}| \\ & \leq \frac{|w_{j_0}|}{\sqrt{h_*}}\sqrt{\frac{1}{2}\left(h_{j_0+\frac{1}{2}}+h_{j_0-\frac{1}{2}}\right)} \leq \frac{1}{\sqrt{h_*}}\|w_h\|_2 \quad , \end{aligned}$$

where

$$\|w_h\|_2^2 = \sum_{j=1}^{J-1}|w_j|^2 \frac{1}{2}\left(h_{j+\frac{1}{2}}+h_{j-\frac{1}{2}}\right).$$

The second estimate follows from

$$\begin{aligned} \|\delta w_h\|_\infty & = \max_{j=0,1,\cdots,J}\left|\delta w_{j+\frac{1}{2}}\right| = \left|w_{j_0+\frac{1}{2}}\right| \\ & \leq \left|\delta w_{j_0+\frac{1}{2}}\right| \frac{\sqrt{h_{j_0+\frac{1}{2}}}}{\sqrt{h_*}} \leq \frac{1}{\sqrt{h_*}}\|\delta w_h\|_2 \quad , \end{aligned}$$

where $\delta w_{j+\frac{1}{2}} = \frac{w_{j+1}-w_j}{h_{j+\frac{1}{2}}}$ $(j = 0, 1, \cdots, J-1)$ and

$$\|\delta w_h\|_2^2 = \sum_{j=0}^{J-1}\left|\delta w_{j+\frac{1}{2}}\right|^2 h_{j+\frac{1}{2}}.$$

Lemma 3. *For any $u_h = \{u_j|\, j = 0, 1, \cdots, J\}$ defined on the grid points $\{x_j|\, j = 0, 1, \cdots, J\}$ with unequal meshsteps $h = \{h_{j+\frac{1}{2}}\big|\, j = 0, 1, \cdots, J-1\}$, there are relations*

$$\|u_h\|_2^2 \leq l^2\|\delta u_h\|_2^2 + 2l|u_0|^2 \quad ,$$

where $0 = x_0 < x_1 < \cdots < x_{J-1} < x_J = l$.

Proof. For any $m = 0, 1, \cdots, J$, there is

$$u_m - u_0 = \sum_{j=0}^{m-1} \delta u_{j+\frac{1}{2}} h_{j+\frac{1}{2}} \quad ,$$

$$|u_m| \le |u_0| + \sqrt{x_m} \, \|\delta u_h\|_2 \quad ,$$

then

$$\|u_h\|_2^2 = \sum_{j=0}^{J-1} \tfrac{1}{2} \left(|u_j|^2 + |u_{j+1}|^2 \right) h_{j+\frac{1}{2}}$$

$$\le \tfrac{1}{2} \sum_{j=0}^{J-1} 4 \, |u_0|^2 \, h_{j+\frac{1}{2}} + 2 \sum_{j=0}^{J-1} (x_j + x_{j+1}) \, h_{j+\frac{1}{2}} \, \|\delta u_h\|_2^2$$

$$\le l^2 \, \|\delta u_h\|_2^2 + 2l \, |u_0|^2 \quad .$$

Lemma 4. *Suppose that the discrete function* $w^\tau = \{ w^n | \, n = 0, 1, \cdots, N \}$ *defined on the grid points* $\{ t^n | \, n = 0, 1, \cdots, N \}$ *with unequal meshsteps* $\tau = \{ \tau^{n+\frac{1}{2}} = t^{n+1} - t^n > 0 | n = 0, 1, \cdots, N-1 \}$ *satisfies the recurring relation*

$$w^{n+1} - w^n \le A \tau^{n+\frac{1}{2}} \left(w^{n+1} + w^n \right) + C \tau^{n+\frac{1}{2}} \quad .$$

then there is

$$w^n \le e^{3At^n} w^0 + 2Ct^n e^{3At^n} \quad ,$$

where the meshsteps $0 = t^0 < t^1 < \cdots < t^{N-1} < t^N = T$ *are sufficiently small that* $2A\tau^* < 1$ *and* A*,* C *are constants.*

Proof. From the recurring formula, there is

$$w^{n+1} \le \left(\frac{1 + A\tau^{n+\frac{1}{2}}}{1 - A\tau^{n+\frac{1}{2}}} \right) w^n + \frac{C\tau^{n+\frac{1}{2}}}{1 - A\tau^{n+\frac{1}{2}}} \quad .$$

Then we have

$$w^{n+1} \le \left[\prod_{k=0}^{n} \left(\frac{1+A\tau^{k+\frac{1}{2}}}{1-A\tau^{k+\frac{1}{2}}} \right) \right] w^0$$

$$+ \sum_{k=0}^{n} \frac{C\tau^{k+\frac{1}{2}}}{1-A\tau^{k+\frac{1}{2}}} \prod_{j=k+1}^{n} \left(\frac{1+A\tau^{j+\frac{1}{2}}}{1-A\tau^{j+\frac{1}{2}}} \right) \quad .$$

We take the meshsteps $\tau = \{ \tau^{n+\frac{1}{2}} \big| \, n = 0, 1, \cdots, N-1 \}$ so small that $A\tau^* < \frac{1}{2}$. Thus we have

$$\prod_{k=0}^{n} \left(\frac{1 + A\tau^{k+\frac{1}{2}}}{1 - A\tau^{k+\frac{1}{2}}} \right) \le e^{3At^{n+1}}$$

and

$$\sum_{k=0}^{n} \frac{C\tau^{k+\frac{1}{2}}}{1 - A\tau^{k+\frac{1}{2}}} \prod_{j=k+1}^{n} \left(\frac{1 + A\tau^{j+\frac{1}{2}}}{1 - A\tau^{j+\frac{1}{2}}} \right) \le 2Ct^{n+1} e^{3At^{n+1}} \quad .$$

Hence the lemma is proved.

Lemma 5. *For any* $v_h = \{v_j | j = 0, 1, \cdots, J\}$ *defined on the grid points* $\{x_j | j = 0, 1, \cdots, J\}$ *with unequal meshsteps* $h = \{h_{j+\frac{1}{2}} = x_{j+1} - x_j > 0 \Big| j = 0, 1, \cdots, J-1\}$ *and for any* $\epsilon > 0$*, there exists a constant* $K(\epsilon, n)$ *dependent on* ϵ *and* n*, such that*

$$\left\|\delta^k v_h\right\|_2 \leq \epsilon \left\|\delta^n v_h\right\|_2 + K(\epsilon, n) \left\|v_h\right\|_2$$

and

$$\left\|\delta^k v_h\right\|_\infty \leq \epsilon \left\|\delta^n v_h\right\|_2 + K(\epsilon, n) \left\|v_h\right\|_2 \quad ,$$

where $0 \leq k < n$ *and* $K(\epsilon, n)$ *is independent of* v_h *and the unequal meshsteps, but dependent on the ratio constant* M_h^* *.*

3. Existence

5. We are going now to prove the existence of the discrete solutions for the finite difference system $(1)_\Delta$, $(2)_\Delta$ and $(3)_\Delta$.

Since $u(x,t) \in C_{x,t}^{(4,2)}(Q_T)$ is the unique smooth vector solution of the original boundary problem (2) and (3) for the nonlinear parabolic system (1) of second order, then for the discrete vector function $u_\Delta = \{u_j^n = u(x_j, t^n) \Big| j = 0, 1, \cdots, J; \quad n = 0, 1, \cdots, N\}$, we have the difference system

$$\begin{aligned} \frac{u_j^{n+1} - u_j^n}{\tau^{n+\frac{1}{2}}} = \quad & \overline{A}_j^{n+\alpha} \delta^2 u_j^{n+\alpha} + \overline{f}_j^{n+\alpha} + R_j^{n+\alpha}, \\ & (j = 1, 2, \cdots, J-1; \quad n = 0, 1, \cdots, N-1), \end{aligned} \qquad (\bar{1})_\Delta$$

where

$$\begin{aligned} \overline{A}_j^{n+\alpha} &= A(x_j,\ t^{n+\alpha},\ \overline{\delta}^0 u_j^{n+\alpha},\ \overline{\delta}^1 u_j^{n+\alpha}), \\ \overline{f}_j^{n+\alpha} &= f(x_j,\ t^{n+\alpha},\ \tilde{\delta}^0 u_j^{n+\alpha},\ \tilde{\delta}^1 u_j^{n+\alpha}) \end{aligned} \qquad (\bar{6})$$

$$\begin{aligned} \overline{\delta}^0 u_j^{n+\alpha} &= \alpha \left[\overline{\beta}_{1j}^{n+1} u_{j+1}^{n+1} + \overline{\beta}_{2j}^{n+1} u_j^{n+1} + \overline{\beta}_{3j}^{n+1} u_{j-1}^{n+1}\right] \left[\overline{\beta}_{4j}^{n} u_{j+1}^{n} + \overline{\beta}_{5j}^{n} u_j^{n} + \overline{\beta}_{6j}^{n} u_{j-1}^{n}\right], \\ \overline{\delta}^1 u_j^{n+\alpha} &= \alpha \left[\overline{\gamma}_{1j}^{n+1} \delta u_{j+\frac{1}{2}}^{n+1} + \overline{\gamma}_{2j}^{n+1} \delta u_{j-\frac{1}{2}}^{n+1}\right] \left[\overline{\gamma}_{3j}^{n} \delta u_{j+\frac{1}{2}}^{n} + \overline{\gamma}_{4j}^{n} \delta u_{j-\frac{1}{2}}^{n}\right], \\ \tilde{\delta}^0 u_j^{n+\alpha} &= \alpha \left[\tilde{\beta}_{1j}^{n+1} u_{j+1}^{n+1} + \tilde{\beta}_{2j}^{n+1} u_j^{n+1} + \tilde{\beta}_{3j}^{n+1} u_{j-1}^{n+1}\right] \left[\tilde{\beta}_{4j}^{n} u_{j+1}^{n} + \tilde{\beta}_{5j}^{n} u_j^{n} + \tilde{\beta}_{6j}^{n} u_{j-1}^{n}\right], \\ \tilde{\delta}^1 u_j^{n+\alpha} &= \alpha \left[\tilde{\gamma}_{1j}^{n+1} \delta u_{j+\frac{1}{2}}^{n+1} + \tilde{\gamma}_{2j}^{n+1} \delta u_{j-\frac{1}{2}}^{n+1}\right] \left[\tilde{\gamma}_{3j}^{n} \delta u_{j+\frac{1}{2}}^{n} + \tilde{\gamma}_{4j}^{n} \delta u_{j-\frac{1}{2}}^{n}\right] . \end{aligned} \qquad (\bar{7})$$

It is clear that the trunction error $R_j^{n+\alpha}$ is of order $\tau + h$, that is,

$$R_j^{n+\alpha} = O(\tau + h) \ .$$

In fact we can easy see that

$$\begin{aligned}
\frac{u_j^{n+1}-u_j^n}{\tau^{n+\frac{1}{2}}} &= u_t\left(x_j, t^{n+\alpha}\right) + O(\tau) \\
&= u_t\left(x_j, t^{n+\frac{1}{2}}\right) + O(\tau^2) \ , \\
\delta^2 u_j^{n+\alpha} &= u_{xx}\left(x_j, t^{n+\alpha}\right) + O(\tau+h) \\
&= u_{xx}\left(x_j^{(2)}, t^{n+\alpha}\right) + O(\tau+h^2) \ , \\
\bar{\delta}^0 u_j^{n+\alpha} &= u\left(x_j, t^{n+\alpha}\right) + O(\tau+h) \ , \\
\bar{\delta}^1 u_j^{n+\alpha} &= u_x\left(x_j, t^{n+\alpha}\right) + O(\tau+h) \ , \\
\tilde{\delta}^0 u_j^{n+\alpha} &= u\left(x_j, t^{n+\alpha}\right) + O(\tau+h) \ , \\
\tilde{\delta}^1 u_j^{n+\alpha} &= u_x\left(x_j, t^{n+\alpha}\right) + O(\tau+h) \ ,
\end{aligned}$$

then

$$\begin{aligned}
\overline{A}_j^{n+\alpha} &= A\left(x,t,u,u_x\right)\Big|_{\substack{x=x_j \\ t=t^{n+\alpha}}} + O(h+\tau) \ , \\
\overline{f}_j^{n+\alpha} &= f\left(x,t,u,u_x\right)\Big|_{\substack{x=x_j \\ t=t^{n+\alpha}}} + O(h+\tau) \ .
\end{aligned}$$

The discrete boundary conditions are

$$\begin{aligned}
u_0^n &= \psi_0^n \ , \\
u_J^n &= \psi_1^n \ , \ (n=0,1,\cdots,N)
\end{aligned} \qquad (\bar{2})_\Delta$$

and the initial condition is

$$u_j^0 = \phi_j \ , \quad (j=0,1,\cdots,J) \ . \qquad (\bar{3})_\Delta$$

Let $w_\Delta = u_\Delta - v_\Delta = \{w_j^n = u_j^n - v_j^n \big| j=0,1,\cdots,J; \ \ n=0,1,\cdots,N\}$. Then from

$(1)_\Delta$, $(2)_\Delta$, $(3)_\Delta$ and $(\bar{1})_\Delta$, $(\bar{2})_\Delta$, $(\bar{3})_\Delta$, we get

$$\begin{aligned}\frac{w_j^{n+1}-w_j^n}{\tau^{n+\frac{1}{2}}} = \; & A(v)_j^{n+\alpha}\delta^2 w_j^{n+\alpha} \\ & +B(u,v)_j^{n+\alpha}\overline{\delta}^0 w_j^{n+\alpha} + C(u,v)_j^{n+\alpha}\overline{\delta}^1 w_j^{n+\alpha} \\ & +D(u,v)_j^{n+\alpha}\tilde{\delta}^0 w_j^{n+\alpha} + E(u,v)_j^{n+\alpha}\tilde{\delta}^1 w_j^{n+\alpha} \\ & +R_j^{n+\alpha}, \\ & \qquad (j=1,\cdots,J-1; \quad n=0,1,\cdots,N-1)\end{aligned} \tag{10}$$

and

$$\begin{aligned} w_0^n &= w_J^n = 0 \;, \quad (n=0,1,\cdots,N)\;, \\ w_j^0 &= 0 \;, \quad (j=0,1,\cdots,J)\;, \end{aligned} \tag{11}$$

where

$$\begin{aligned} A(v)_j^{n+\alpha} &= A_j^{n+\alpha} \;, \\ B(u,v)_j^{n+\alpha} &= \left(\tilde{A}_u\right)_j^{n+\alpha} \delta^2 u_j^{n+\alpha} \;, \\ C(u,v)_j^{n+\alpha} &= \left(\tilde{A}_p\right)_j^{n+\alpha} \delta^2 u_j^{n+\alpha} \;, \\ D(u,v)_j^{n+\alpha} &= \left(\tilde{f}_u\right)_j^{n+\alpha} \;, \\ E(u,v)_j^{n+\alpha} &= \left(\tilde{f}_p\right)_j^{n+\alpha} \;, \end{aligned} \tag{12}$$

and

$$\begin{aligned} \left(\tilde{A}_u\right)_j^{n+\alpha} &= \int_0^1 A_u(x_j, t^{n+\alpha}, \; \lambda\overline{\delta}^0 u_j^{n+\alpha} + (1-\lambda)\overline{\delta}^0 v_j^{n+\alpha}, \\ &\qquad \lambda\overline{\delta}^1 u_j^{n+\alpha} + (1-\lambda)\overline{\delta}^1 v_j^{n+\alpha})d\lambda \;, \\ \left(\tilde{A}_p\right)_j^{n+\alpha} &= \int_0^1 A_p(x_j, t^{n+\alpha}, \; \lambda\overline{\delta}^0 u_j^{n+\alpha} + (1-\lambda)\overline{\delta}^0 v_j^{n+\alpha}, \\ &\qquad \lambda\overline{\delta}^1 u_j^{n+\alpha} + (1-\lambda)\overline{\delta}^1 v_j^{n+\alpha})d\lambda \;, \\ \left(\tilde{f}_u\right)_j^{n+\alpha} &= \int_0^1 f_u(x_j, t^{n+\alpha}, \; \lambda\tilde{\delta}^0 u_j^{n+\alpha} + (1-\lambda)\tilde{\delta}^0 v_j^{n+\alpha}, \\ &\qquad \lambda\tilde{\delta}^1 u_j^{n+\alpha} + (1-\lambda)\tilde{\delta}^1 v_j^{n+\alpha})d\lambda \;, \\ \left(\tilde{f}_p\right)_j^{n+\alpha} &= \int_0^1 f_p(x_j, t^{n+\alpha}, \; \lambda\tilde{\delta}^0 u_j^{n+\alpha} + (1-\lambda)\tilde{\delta}^0 v_j^{n+\alpha}, \\ &\qquad \lambda\tilde{\delta}^1 u_j^{n+\alpha} + (1-\lambda)\tilde{\delta}^1 v_j^{n+\alpha})d\lambda \;. \end{aligned} \tag{12$'$}$$

6. Let us construct a mapping $\Phi : R^* \to R^*$ of $m(J+1)(N+1)$-dimensional Euclidean space $R^* = R^{m(J+1)(N+1)}$ into itself as follows:

For any $z_\Delta \in R^*$, the corresponding image $w_\Delta = \Phi(z_\Delta)$ is the solution of the linear system

$$\begin{aligned} \frac{w_j^{n+1}-w_j^n}{\tau^{n+\frac{1}{2}}} = \;& A(u-z)_j^{n+\alpha}\delta^2 w_j^{n+\alpha} \\ &+B(u,u-z)_j^{n+\alpha}\overline{\delta}^0 w_j^{n+\alpha} \\ &+C(u,u-z)_j^{n+\alpha}\overline{\delta}^1 w_j^{n+\alpha} \\ &+D(u,u-z)_j^{n+\alpha}\tilde{\delta}^0 w_j^{n+\alpha} \\ &+E(u,u-z)_j^{n+\alpha}\tilde{\delta}^1 w_j^{n+\alpha} + R_j^{n+\alpha} \\ &(j=1,2,\cdots,J-1;\;\; n=0,1,\cdots,N-1) \end{aligned} \tag{13}$$

with homogeneous boundary and initial conditions (11). The solution $w_\Delta = \Phi(z_\Delta) \in R^*$ exists and is unique for any $z_\Delta \in R^*$.

Let G be a positive constant, which dominantes the maximum modulo of the unique vector solution $u(x,t)$ of the original boundary problem (2) and (3) for the nonlinear parabolic system (1) and its vector derivatives $u_x(x,t)$, $u_{xx}(x,t)$ and $u_t(x,t)$, that is, $|u(x,t)|$, $|u_x(x,t)|$, $|u_{xx}(x,t)|$, $|u_t(x,t)| \le G$, for $(x,t) \in Q_T$.

Let $\Omega \subset R^*$ be a bounded closed set, given by

$$\Omega = \left\{ z_\Delta \left| \max_{\substack{0\le j\le J\\ 0\le n\le N}} \left|z_j^n\right| \right., \max_{\substack{0\le j\le J-1\\ 0\le n\le N}} \left|\delta z_{j+\frac{1}{2}}^n\right| \le G \right\}. \tag{14}$$

It is clear that Ω is a convex set of R^*. Hence $\Phi : \Omega \to R^*$ maps Ω into R^*.

7. Making the scalar product of the vector $\delta^2 w_j^{n+\alpha} h_j^{(2)} \tau^{n+\frac{1}{2}}$ with the vector finite difference equation (13) and summing up the resulting products for $j = 1,2,\cdots,J-1$, we have

$$\begin{aligned} &\sum_{j=1}^{J-1}\left(\delta^2 w_j^{n+\alpha}, w_j^{n+1}-w_j^n\right)h_j^{(2)} \\ =\;& \tau^{n+\frac{1}{2}}\sum_{j=1}^{J-1}\left(\delta^2 w_j^{n+\alpha}, A(u-z)_j^{n+\alpha}\delta^2 w_j^{n+\alpha}\right)h_j^{(2)} \\ &+\tau^{n+\frac{1}{2}}\sum_{j=1}^{J-1}\Big(\delta^2 w_j^{n+\alpha}, B(u,u-z)_j^{n+\alpha}\overline{\delta}^0 w_j^{n+\alpha} \\ &\quad +C(u,u-z)_j^{n+\alpha}\overline{\delta}^1 w_j^{n+\alpha} + D(u,u-z)_j^{n+\alpha}\tilde{\delta}^0 w_j^{n+\alpha} \\ &\quad +E(u,u-z)_j^{n+\alpha}\tilde{\delta}^1 w_j^{n+\alpha} + R_j^{n+\alpha}\Big)h_j^{(2)}. \end{aligned} \tag{15}$$

For the first term on the right hand part of the equality (15), there is

$$\begin{aligned} &\tau^{n+\frac{1}{2}}\sum_{j=1}^{J-1}\left(\delta^2 w_j^{n+\alpha}, A(u-z)_j^{n+\alpha}\delta^2 w_j^{n+\alpha}\right)h_j^{(2)} \\ \geq\ & \sigma_0\tau^{n+\frac{1}{2}}\left\|\delta^2 w_h^{n+\alpha}\right\|_2^2\ , \end{aligned} \tag{16}$$

since by the assumption (II), the $m\times m$ matrix A and then $A(u-z)_j^{n+\alpha}$ are positively definite with constant $\sigma_0>0$ given in (4).

For the left part of the equality (15), we have

$$\begin{aligned} &-\sum_{j=1}^{J-1}\left(\delta^2 w_j^{n+\alpha}, w_j^{n+1}-w_j^n\right)h_j^{(2)} \\ =\ & \tfrac{1}{2}\left\|\delta w_h^{n+1}\right\|_2^2-\tfrac{1}{2}\left\|\delta w_h^n\right\|_2^2-\left(\tfrac{1}{2}-\alpha\right)\left\|\delta\left(w_h^{n+1}-w_h^n\right)\right\|_2^2\ . \end{aligned} \tag{17}$$

For the last term of the above equality (17), we have

$$\begin{aligned} &\left\|\delta\left(w_h^{n+1}-w_h^n\right)\right\|_2^2 \\ =\ & \sum_{j=1}^{J-1}\left|\frac{\left(w_{j+1}^{n+1}-w_{j+1}^n\right)-\left(w_j^{n+1}-w_j^n\right)}{h_{j+\frac{1}{2}}}\right|^2 h_{j+\frac{1}{2}} \\ \leq\ & \left(\frac{2\tau^{n+\frac{1}{2}}}{h_*}\right)^2\sum_{j=1}^{J-1}\left|\frac{w_j^{n+1}-w_j^n}{\tau^{n+\frac{1}{2}}}\right|^2 h_j^{(2)}\ . \end{aligned}$$

Substituting (13) into the right sum of thr above inequality, we get

$$\begin{aligned} &\left\|\delta\left(w_h^{n+1}-w_h^n\right)\right\|_2^2 \\ \leq\ & \left(\frac{2\tau^{n+\frac{1}{2}}}{h_*}\right)^2\sum_{j=1}^{J-1}\Big|A(u-z)_j^{n+\alpha}\delta^2 w_j^{n+\alpha} \\ &+B_j^{n+\alpha}\overline{\delta}^0 w_j^{n+\alpha}+C_j^{n+\alpha}\overline{\delta}^1 w_j^{n+\alpha} \\ &+D_j^{n+\alpha}\tilde{\delta}^0 w_j^{n+\alpha}+E_j^{n+\alpha}\tilde{\delta}^1 w_j^{n+\alpha}+R_j^{n+\alpha}\Big|^2 h_j^{(2)} \\ \leq\ & \left(\frac{2\tau^{n+\frac{1}{2}}}{h_*}\right)^2\Bigg\{(1+\epsilon_1)\sum_{j=1}^{J-1}\left|A(u-z)_j^{n+\alpha}\delta^2 w_j^{n+\alpha}\right|^2 h_j^{(2)} \\ &+\left(1+\frac{1}{\epsilon_1}\right)\sum_{j=1}^{J-1}\Big|B_j^{n+\alpha}\overline{\delta}^0 w_j^{n+\alpha}+C_j^{n+\alpha}\overline{\delta}^1 w_j^{n+\alpha} \\ &+D_j^{n+\alpha}\tilde{\delta}^0 w_j^{n+\alpha}+E_j^{n+\alpha}\tilde{\delta}^1 w_j^{n+\alpha}+R_j^{n+\alpha}\Big|^2 h_j^{(2)}\Bigg\}\ , \end{aligned}$$

where $\epsilon_1>0$ is a small constant to be determined.

Hence we have

$$\begin{aligned} &\sum_{j=1}^{J-1}\left|A(u-z)_j^{n+\alpha}\delta^2 w_j^{n+\alpha}\right|^2 h_j^{(2)} \\ \leq\ & \sum_{j=1}^{J-1}\frac{\rho^2\left(A(u-z)_j^{n+\alpha}\right)}{\sigma\left(A(u-z)_j^{n+\alpha}\right)}\left(\delta^2 w_j^{n+\alpha}, A(u-z)_j^{n+\alpha}\delta^2 w_j^{n+\alpha}\right)h_j^{(2)}\ , \end{aligned}$$

where the symbols $\rho\left(A(u-z)_j^{n+\alpha}\right)$ and $\sigma\left(A(u-z)_j^{n+\alpha}\right)$ are defined by

$$\begin{aligned}\rho(A) = & \sup_{\xi\in R^m} \tfrac{|A\xi|}{|\xi|} , \\ \sigma(A) = & \inf_{\xi\in R^m} \tfrac{(\xi,A\xi)}{|\xi|^2}\end{aligned} \tag{18}$$

respectively, that is, $\rho(A)$ is the radius of spectrum or the norm of the matrix A with respect to Euclidean metric and $\sigma(A)$ measures the positive definitness of the matrix A. Here we have

$$\rho(A) \geq \sigma(A) \geq \sigma_0 > 0 .$$

(V) Suppose that the unequal meshsteps

$$h = \{h_{j+\frac{1}{2}} \big| j = 0,1,\cdots,J-1\}$$

and

$$\tau = \{\tau^{n+\frac{1}{2}} \big| n = 0,1,\cdots,N-1\}$$

are so chosen that there is the restriction

$$\left(\frac{1}{2}-\alpha\right)\frac{4\tau}{h_*^2} \max_{\substack{(x,t)\in Q_T, \\ |u|\leq \bar{\delta}_0 G, \\ |p|\leq \bar{\delta}_1 G}} \frac{\rho^2(A(x,t,u,p))}{\tau(A(x,t,u,p))} \leq 1-\epsilon , \tag{19}$$

where $\epsilon = 1$ for $\frac{1}{2} \leq \alpha \leq 1$ and $0 < \epsilon < 1$ for $0 \leq \alpha < \frac{1}{2}$.

It is clear that when $\frac{1}{2} \leq \alpha \leq 1$, there are no any restriction on the choice of the meshsteps $h = \{h_{j+\frac{1}{2}} \big| j = 0,1,\cdots,J-1\}$ and $\tau = \{\tau^{n+\frac{1}{2}} \big| n = 0,1,\cdots,N-1\}$.

Under the restriction (V), we have

$$\begin{aligned} & (\tfrac{1}{2}-\alpha)\|\delta(w_h^{n+1}-w_h^n)\|_2^2 \\ \leq\ & (1+\epsilon_1)(1-\epsilon\tau^{n+\frac{1}{2}}\sum_{j=1}^{J-1}(\delta^2 w_j^{n+\alpha}, A(u-z)_j^{n+\alpha}\delta^2 w_j^{n+\alpha}h_j^{(2)} \\ & +(1+\tfrac{1}{\epsilon_1})(1-\epsilon)\tfrac{\tau^{n+\frac{1}{2}}}{\sigma_0}\sum_{j=1}^{J-1}|B_j^{n+\alpha}\bar{\delta}^0 w_j^{n+\alpha} \\ & +C_j^{n+\alpha}\bar{\delta}^1 w_j^{n+\alpha} + D_j^{n+\alpha}\tilde{\delta}^0 w_j^{n+\alpha} \\ & +E_j^{n+\alpha}\tilde{\delta}^1 w_j^{n+\alpha} + R_j^{n+\alpha}|^2 h_j^{(2)} \end{aligned} \tag{20}$$

and

$$\begin{aligned} & \tau^{n+\frac{1}{2}}\sum_{j=1}^{J-1}\Big(\delta^2 w_j^{n+\alpha}, B_j^{n+\alpha}\bar{\delta}^0 w_j^{n+\alpha} + C_j^{n+\alpha}\bar{\delta}^1 w_j^{n+\alpha} \\ & +D_j^{n+\alpha}\tilde{\delta}^0 w_j^{n+\alpha} + E_j^{n+\alpha}\tilde{\delta}^1 w_j^{n+\alpha} + R_j^{n+\alpha}\Big) h_j^{(2)} \\ \leq\ & \tfrac{\epsilon^2\sigma_0}{2}\tau^{n+\frac{1}{2}}\|\delta^2 w_h^{n+\alpha}\|_2^2 + \tfrac{2}{\epsilon^2\sigma_0}\tau^{n+\frac{1}{2}}\sum_{j=1}^{J-1}\Big|B_j^{n+\alpha}\bar{\delta}^0 w_j^{n+\alpha} \\ & +C_j^{n+\alpha}\bar{\delta}^1 w_j^{n+\alpha} + D_j^{n+\alpha}\tilde{\delta}^0 w_j^{n+\alpha} \\ & +E_j^{n+\alpha}\tilde{\delta}^1 w_j^{n+\alpha} + R_j^{n+\alpha}\Big|^2 h_j^{(2)} . \end{aligned}$$

Let $\epsilon_1 = \epsilon$, then the equality (15) can be replaced by the following inequality:

$$\begin{aligned} &\|\delta w_h^{n+1}\|_2^2 - \|\delta w_h^n\|_2^2 + \epsilon^2\sigma_0\tau^{n+\frac{1}{2}}\|\delta^2 w_h^{n+\alpha}\|_2^2 \\ \leq\ & 10\left[(1-\epsilon)\left(1+\frac{1}{\epsilon}\right)+\frac{2}{\epsilon^2}\right]\frac{\tau^{n+\frac{1}{2}}}{\sigma_0}\left\{\sum_{j=1}^{J-1}\left|B_j^{n+\alpha}\bar{\delta}^0 w_j^{n+\alpha}\right|^2 h_j^{(2)}\right. \\ & + \sum_{j=1}^{J-1}\left|C_j^{n+\alpha}\bar{\delta}^1 w_j^{n+\alpha}\right|^2 h_j^{(2)} + \sum_{j=1}^{J-1}\left|D_j^{n+\alpha}\tilde{\delta}^0 w_j^{n+\alpha}\right|^2 h_j^{(2)} \\ & \left.+ \sum_{j=1}^{J-1}\left|E_j^{n+\alpha}\tilde{\delta}^1 w_j^{n+\alpha}\right|^2 h_j^{(2)} + \sum_{j=1}^{J-1}\left|R_j^{n+\alpha}\right|^2 h_j^{(2)}\right\} \ . \end{aligned} \tag{21}$$

Since $z_\Delta \in \Omega$, there are

$$\begin{aligned} & \sum_{j=1}^{J-1}\left|B(u,u-z)_j^{n+\alpha}\bar{\delta}^0 w_j^{n+\alpha}\right|^2 h_j^{(2)} \\ \leq\ & C_1\left\{\|w_h^{n+1}\|_2^2 + \|w_h^n\|_2^2\right\} \ , \\ & \sum_{j=1}^{J-1}\left|C(u,u-z)_j^{n+\alpha}\bar{\delta}^1 w_j^{n+\alpha}\right|^2 h_j^{(2)} \\ \leq\ & C_1\left\{\|\delta w_h^{n+1}\|_2^2 + \|\delta w_h^n\|_2^2\right\} \ , \\ & \sum_{j=1}^{J-1}\left|D(u,u-z)_j^{n+\alpha}\tilde{\delta}^0 w_j^{n+\alpha}\right|^2 h_j^{(2)} \\ \leq\ & C_1\left\{\|w_h^{n+1}\|_2^2 + \|w_h^n\|_2^2\right\} \ , \\ & \sum_{j=1}^{J-1}\left|E(u,u-z)_j^{n+\alpha}\tilde{\delta}^1 w_j^{n+\alpha}\right|^2 h_j^{(2)} \\ \leq\ & C_1\left\{\|\delta w_h^{n+1}\|_2^2 + \|\delta w_h^n\|_2^2\right\} \ , \\ & \sum_{j=1}^{J-1}\left|R_j^{n+\alpha}\right|^2 h_j^{(2)} \leq C_1(\tau+h)^2 \ , \end{aligned} \tag{22}$$

where C_1 is a constant dependent on G and on the ratio constant M_h^* of meshsteps. Then we have

$$\begin{aligned} &\|\delta w_h^{n+1}\|_2^2 - \|\delta w_h^n\|_2^2 + \epsilon^2\sigma_0\tau^{n+\frac{1}{2}}\|\delta^2 w_h^{n+\alpha}\|_2^2 \\ \leq\ & C_2\tau^{n+\frac{1}{2}}\left\{\|\delta w_h^{n+1}\|_2^2 + \|\delta w_h^n\|_2^2 + \|w_h^{n+1}\|_2^2 + \|w_h^n\|_2^2 + (\tau+h)^2\right\} \ . \end{aligned}$$

By means of the Lemma 3, we then have

$$\|w_h^n\|_2 \leq \sqrt{2}l\|\delta w_h^n\|_2 \ ,$$

$$\|w_h^{n+1}\|_2 \leq \sqrt{2}l\|\delta w_h^{n+1}\|_2 \ .$$

Hence we get

$$\|\delta w_h^{n+1}\|_2^2 - \|\delta w_h^n\|_2^2 + \epsilon^2\sigma_0\tau^{n+\frac{1}{2}}\|\delta^2 w_h^{n+\alpha}\|_2^2 \le C_3\tau^{n+\frac{1}{2}}\left\{\|\delta w_h^{n+1}\|_2^2 + \|\delta w_h^n\|_2^2 + (\tau+h)^2\right\} \ . \tag{23}$$

Using the Lemma 4, we get

$$\max_{n=0,1,\cdots,N} \|\delta w_h^n\|_2 \le C_4(\tau+h) \ , \tag{24}$$

where C_4 is a constant dependent on G and also on the ratio constant M_h^* of meshsteps. Then we have also the estimates

$$\max_{0\le n\le N} \|w_h^n\|_2 \ , \qquad \max_{0\le n\le N} \|w_h^n\|_\infty \ ,$$
$$\left(\sum_{n=0}^{N-1} \tau^{n+\frac{1}{2}} \left\|\delta^2 w_h^{n+\alpha}\right\|_2^2\right)^{\frac{1}{2}} \ , \qquad \left(\sum_{n=0}^{N-1} \tau^{n+\frac{1}{2}} \left\|\frac{w_h^{n+1}-w_h^n}{\tau^{n+\frac{1}{2}}}\right\|_2^2\right)^{\frac{1}{2}} \tag{25}$$
$$\le C_5(\tau+h) \ ,$$

where C_5 depends on G and M_h^*.

8. By the Lemma 2, there are estimates

$$\max_{\substack{0\le j\le J \\ 0\le n\le N}} |w_j^n| \le C_5(\tau+h) \ ,$$
$$\max_{\substack{0\le j\le J-1 \\ 0\le n\le N}} |\delta w_{j+\frac{1}{2}}^n| \le C_5\left(\frac{\tau}{h_*^{\frac{1}{2}}} + h^{\frac{1}{2}}\sqrt{M_h^*}\right) \ . \tag{26}$$

Taking

$$\tau,\ h \le \min\left(\frac{G}{2C_5} \ , \quad \left(\frac{G}{2C_5}\right)^2 \frac{1}{M_h^*}\right) ,\tau \le \frac{G}{2c_3} h_*^{\frac{1}{2}} \ , \tag{27}$$

we have the estimate

$$\max_{\substack{0\le j\le J \\ 0\le n\le N}} |w_j^n| \ , \qquad \max_{\substack{0\le j\le J-1 \\ 0\le n\le N}} |\delta w_{j+\frac{1}{2}}^n| \le G \ . \tag{28}$$

This shows that $w_\Delta = \Phi(z_\Delta) \in \Omega$. Therefore $\Phi(\Omega) \subset \Omega$. The mapping $\Phi : \Omega \to R^*$ maps Ω into $\Omega \subset R^*$ itself. By means of Brouwer theorem on fixed point, the mapping has at least one fixed point w_Δ, such that $\Phi(w_\Delta) = w_\Delta$, that is w_Δ is a solution of the system

$$\begin{aligned}\frac{w_j^{n+1}-w_j^n}{\tau^{n+\frac{1}{2}}} = \; & A(u-w)_j^{n+\alpha}\delta^2 w_j^{n+\alpha} \\ & +B(u,u-w)_j^{n+\alpha}\overline{\delta}^0 w_j^{n+\alpha} \\ & +C(u,u-w)_j^{n+\alpha}\overline{\delta}^1 w_j^{n+\alpha} \\ & +D(u,u-w)_j^{n+\alpha}\tilde{\delta}^0 w_j^{n+\alpha} \\ & +E(u,u-w)_j^{n+\alpha}\tilde{\delta}^1 w_j^{n+\alpha} + R_j^{n+\alpha}\end{aligned} \qquad (13)_0$$

$$(j = 1, 1, \cdots, J-1; \;\; n = 0, 1, \cdots, N-1)$$

and the homogeneoous discrete boundary and discrete initial conditions (11). Then $v_\Delta = u_\Delta - w_\Delta$ is the solution of the finite difference system $(1)_\Delta$, $(2)_\Delta$ and $(3)_\Delta$.

Theorem 1. *Suppose that the conditions* (I), (II), (III), (IV) *and* (V) *are fulfilled and the unique smooth vector solution* $u(x,t)$ *of the boundary problem* (2) *and* (3) *for the nonlinear parabolic system* (1) *of partial differential equations has the estimates*

$$\|u\|_{C(Q_T)}\,, \;\; \|u_x\|_{C(Q_T)}\,, \;\; \|u_{xx}\|_{C(Q_T)}\,, \;\; \|u_t\|_{C(Q_T)} \;\le G \;\; .$$

and also

(VI) *the meshsteps* h *and* τ *are so chosen such that* $\frac{\tau}{\sqrt{h_*}}$ *is sufficiently small.*

Then for sufficiently small unequal meshsteps h *and* τ*, the general finite difference scheme* $(1)_\Delta$, $(2)_\Delta$ *and* $(3)_\Delta$ *with nonuniform meshes corresponding to the original problem* (1)*,* (2) *and* (3) *has at least one solution* $v_\Delta = \left\{ v_j^n \middle| j = 0, 1, \cdots, J; \;\; n = 0, 1, \cdots, N \right\}$ *with estimates*

$$\left\{ \max_{\substack{0 \le j \le J \\ 0 \le n \le N}} |w_j^n| \;, \quad \max_{\substack{0 \le j \le J-1 \\ 0 \le n \le N}} |\delta w_{j+\frac{1}{2}}^n| \right\} \le 2G \;\; . \qquad (29)$$

4. Convergence

9. As the consequence of the existence theorem, we have the following theorem of absolute and relative convergence.

Theorem 2. *Under of the conditions of Theorem 1, for the difference* $w_\Delta = \left\{ w_j^n = u_j^n - v_j^n \middle| j = 0, 1, \cdots, J; \;\; n = 0, 1, \cdots, N \right\}$ *of the discrete vector function*

$u_\Delta = \left\{ u_j^n = u(x_j, t^n) \middle| j = 0, 1, \cdots, J; \quad n = 0, 1, \cdots, N \right\}$ *of the solution* $u(x,t)$ *for the boundary problem* (2) *and* (3) *of the nonlinear parabolic system* (1) *and the corresponding discrete vector solution* $v_\Delta = \left\{ v_j^n \middle| j = 0, 1, \cdots, J; \quad n = 0, 1, \cdots, N \right\}$ *for the general finite difference system* $(1)_\Delta$, $(2)_\Delta$ *and* $(3)_\Delta$, *there are estimates*

$$\max_{0 \le n \le N} \|w_h^n\|_2 \ , \quad \max_{0 \le n \le N} \|w_h^n\|_\infty \ , \quad \max_{0 \le n \le N} \|\delta w_h^n\|_2 \ ,$$

$$\left(\sum_{n=0}^{N-1} \left\| \delta^2 w_h^{n+\alpha} \right\|_2^2 \tau^{n+\frac{1}{2}} \right)^{\frac{1}{2}} , \quad \left(\sum_{n=0}^{N-1} \left\| \frac{w_h^{n+1} - w_h^n}{\tau^{n+\frac{1}{2}}} \right\|_2^2 \tau^{n+\frac{1}{2}} \right)^{\frac{1}{2}} , \tag{30}$$

$$= O(\tau + h)$$

and

$$\max_{0 \le n \le N} \|\delta w_h^n\|_\infty = O\left(\frac{\tau}{h_*^{\frac{1}{2}}} , \quad h^{\frac{1}{2}} \right) \ , \tag{31}$$

where the ratio constant M_h^* *keeps bounded.*

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