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SUBCONVERGENCE OF FINITE ELEMENTS AND A SELF–ADAPTIVE ALGORITHM*1)2)

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1. Introduction

So far, there have been many papers concerning with the superconvergence of finite elements. Under some conditions, a higher convergence accuracy is obtained at some specific points. This phenomenon is called superconvergence. The theory of supercovergence[1] tells us that, in order to gain superconvergence, two conditions must be satisfied. One is the good subdivision and the second is the existence of locally sufficient smooth solution. If these conditions are not satisfied, e.g., low smoothness of the solution, occurs a contrary phenomenon that some local errors is greater than that of average errors. This phenomenon, we call *subconvergence.*

Subconvergence is conflicting with superconvergence. It often occurs in the neighourhood of singular points, where higher accuracy algorithm is followed with great interest by engineers but it is very difficult. So it is necessary to study the theory of subconvergence.

Bubuska[2] proposed the h–p version that locally refining subdivision or locally increasing the degree of piecewise polynomials gets locally higher accuracy. To implement this method, it is necessary to use the posterior data to find the elements at which the subconvergence occurs. Otherwise, refining all elements and not distinguishing the 'good' ones with the 'bad' ones, the algorithm is very inefficiency and it is not a high accuracy one.

This paper uses the idea of subconvergence to obtain a simple criterion Δ_e, which only depends on the approximate solution u^h and the function f. Comparing the sizes of Δ_e, the element e_0 with the worst convergence can be found. Then, e_0 and the neighbouring elements are refined by the h-version or p-version. Repeating this process many times, the high accuracy results may be obtained for the problem with singularity.

The outline of this paper is as follows. Section 1 presents some definitions about the subconvergence. Section 2 gives a method of locally refinement or local increasing the degree of polynomials and some main results. In section 3 several examples are provided.

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2. Subconvergence Point, Subconvergence Element and A Self–Adaptive Method

Consider the model problem: to find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = (f,v), \quad v \in H_0^1(\Omega) \tag{2.1}$$

where $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$, $f \in L^2(\Omega)$, and $a(\cdot,\cdot)$ an elliptic bilinear form with some singularity, Ω a bounded domain in $R^N (N = 1,2,3)$. We call

$$\|u\| = \sqrt{a(u,u)}$$

the energy norm.

Let T^h be a regular subdivision on Ω, $S^h(\Omega)$ a piecewise linear finite element space and

$$S_0^h(\Omega) = \{v \in S^h(\Omega) : v|_{\partial\Omega} = 0\}$$

Suppose $u^h \in S_0^h(\Omega)$ be the Galerkin approximation of the solution u of problem (2.1). Then the energy

$$\left\|u - u^h\right\|^2 = \sum_{e \in T^h} \left\|u - u^h\right\|_e^{\ 2}$$

Generally speaking, the element $e \in T^h$ with large $\left\|u - u^h\right\|_e^{\ 2}$ is a *subconvergence element* or bad element.

Now, we want to find a criterion to say $\left\|u - u^h\right\|_e^2$ large or small. To do this, we denote, by $S_0^{h/2}(\Omega)$, $p-$degree finite element space on T^h or the linear finite element space after many times refinement of T^h, and denote, by $\bar{G}$, the set of base functions of finite element space $S_0^{h/2}(\Omega)$. It is obvious that there exists a $\delta_0 \in G_e \equiv \text{span}\{\phi \in \bar{G} : (\mathit{supp}\ \phi) \cap e \neq \emptyset\}$ such that

$$\Delta_e \equiv \left| a(u - u^h, \delta_0) \,/\, \|\delta_0\| \right| = \max_{\delta \in G_e} \left| a(u - u^h, \delta) \,/\, \|\delta\| \right| \tag{2.2}$$

If $\Delta_{e_0} = \max_{e \in T^h}\{\Delta_e\}$, we call e_0 the bad element of T^h.

If we refine T^h by mid-point locally again and again and the point z_0 belongs to the bad element infinitly, we call z_0 *subconvergence point.* Since

$$\Delta_e \equiv \left| a(u - u^h, \delta_0) \,/\, \|\delta_0\| \right| \tag{2.3}$$

it is a posterior datum, which can be determined by the computer. It can be seen below that replacing the error energy $\left\|u - u^h\right\|_e$ on element e with Δ_e is reasonable (see Th.1 & Th. 2).

Let $e \in T^h$ be a bad element. Refine e by mid-point and then denote all new nodal points by N'_e. For each point $z \in N'_e$, we have a basis function $\phi_z \in \bar{G}$. Introduce the following notations

$$S_0^{h/2}(D_e) = \text{span}\{\phi_z : z \in N'_e\} \tag{2.4}$$

where

$$D_e = \cup\{\text{supp}(\phi_z) : z \in N'_e\} \tag{2.5}$$

Therefore, for $e \in T^h$, a new finite element space by local refinement or locally increasing the degree of polynomials is obtained:

$$V_e^h(\Omega) = S^h(\Omega) \oplus S^{h/2}(D_e) = \{v + \phi : v \in S^h(\Omega), \phi \in S^{h/2}(D_e)\}$$

Let $u_e^h(\Omega) \in V_e^h$ be the Galerkin approximation of u. In the next section, we will prove

$$\left\| u - u_e^h \right\|^2 - \left\| u - u^h \right\|^2 \geq {\Delta_e}^2$$

It follows that the criterion Δ_e indicates to some extent the change of the error on the element e.

For convenience, we also use $u^I \in S_0^h(\Omega)$ and $u_e^I \in V_e^h$ to denote the interpolations of the exact solution u on the finite element space S_0^h and V_e^h respectively.

In section 3, a concrete method of subconvergence local refinement is presented and a high accuracy result can be obtained automatically in the neighbourhood of the singular point (subconvergence point).

3. Some Methods of Local Refinement and Locally Increasing the Degree of Polynomials

At first, we give some methods of local refinement for finite element.

Example 1. One dimensional element

Let $e = [\alpha, \beta]$ be a one dimensional linear element. Thus

$$D_e = e, \qquad S_0^{h/2}(D_e) = \{\alpha\phi_0 : \alpha \in R\}$$

where ϕ_0 is the basis function at the middle point c of e :

$$\phi_0(x) = \begin{cases} 1, & \text{when} \quad x = c \\ 0, & \text{when} \quad x \leq c - h/2 \quad \text{or} \quad x \geq c + h/2 \ . \\ \text{linear}, & \text{other} \end{cases}$$

If

$$S_0^h = \text{span}\{\phi_1, \phi_2, \cdots, \phi_n\}$$

then

$$V_e = \text{span}\{\phi_1, \phi_2, \cdots, \phi_n, \phi_0\}$$

and the criterion

$$\Delta_e = \left| (f, \phi_0) - a(u^h, \phi_0) \right| / ||\phi_0||. \tag{3.1}$$

Example 2. Two dimensional triangular linear element

Let $e = \Delta z_1 z_2 z_3$ (Figure 1) and let z_4, z_5, z_6 the three middel points of the sides of e with the basis function ϕ_4, ϕ_5, ϕ_6 respectively:

$$D_e = \cup\{\text{supp}(\phi_i), i = 4, 5, 6\} \cup e$$

$$S_0^{h/2}(D_e) = \text{span}\{\phi_4, \phi_5, \phi_6\}.$$

The supports of these basis functions do not belong to e. However,

$$\Delta_e = |(f, \delta_0) - a(u, \delta_0)| \,/||\delta_0|| \equiv \|\delta_0\| \tag{3.2}$$

can be used as a criterion, where $\delta_0 \in S_0^{h/2}(D_e)$ is the solution of the simple equation

$$a(\delta_0, v) = (f, v) - a(u^h, v), \quad v \in S_0^{h/2}(D_e) \tag{3.3}$$

where $u^h \in S_0^h(\Omega)$ is the finite element solution before refinement.

Example 3. Two dimensional bilinear element

Let $e =$ rectangular $z_1z_2z_3z_4$ (Figure 2). Add on e five nodal points z_0, z_5, z_6, z_7, z_8 and denote by $\phi_i \in \bar{G}$ the basis function at each point z_i, $i = 0, 5, 6, 7, 8$. Let

$$S_0^{h/2}(D_e) = \text{span}\{\phi_0, \phi_5, \phi_6, \phi_7, \phi_8\}$$

$$D_e = \cup\{\text{supp}\phi_i, \quad i = 0, 5, 6, 7, 8\}.$$

The criterion

$$\Delta_e = \Big|(f, \delta_0) - a(u^h, \delta_0)\Big| \,/\,||\delta_0|| = ||\delta_0|| \tag{3.4}$$

where $\delta_0 \in S_0^{h/2}(D_e)$ is the solution of the simple equation

$$a(\delta_0, v) = (f, v) - a(u^h, v), \quad v \in S_0^{h/2}(D_e). \tag{3.5}$$

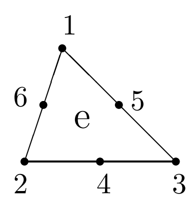

Figure 1.

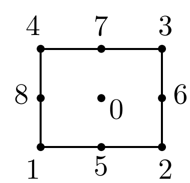

Figure 2.

Those similar to (3.1) are called the first kind criterion and those similar to (3.2) and (3.4) are called the second kind criterion.

Δ_e in Example 1 – Example 3 is defined by changing the sizes of h. We call it the $h - v$ criterion, which was studied in [4] and in [5]. If we define Δ_e by increasing the degree of polynomials, we call it the $p - v$ criterion, which will be demonstrated as follows:

Denote the linear finite element on T^h by $S_0^h(\Omega)$ again.

Let

$$S_0^{h/2}(\Omega) = \{v \in C_0(\Omega) : v|_e \in P_2(e),\, e \in T^h\}, \quad \text{for Example1 \& Example2}$$

$$S_0^{h/2}(\Omega) = \{v \in C_0(\Omega) : v|_e \in Q_2(e),\, e \in T^h\}, \quad \text{for} \quad \text{Example3}$$

where $P_2(e)$ is the set of all quadratic polynomials on e and $Q_2(e)$ the set of all biquadratic polynomials on e. And let

$$D_e = \begin{cases} \cup\{e' \in T^h : \bar{e}' \cap \bar{e} \neq \emptyset\}, & \text{for multi} - \text{dimensional space} \\ e \quad \text{for one} - \text{dimensional space} \end{cases}$$

$$S_0^{h/2}(D_e) = \{v \in S_0^{h/2} : \text{supp}\,(v) \subset D_e\}$$
$$V_e = S_0^h(\Omega) \oplus S_0^{h/2}(D_e)$$

the $p - v$ criterion Δ_e can be defined by (3.1) (3.2) and (3.4) similarly.

The proofs of the following two theorems are valid not only for $h - v$ criterion but also for the $p - v$ criterion.

Theorem 1. *Let $S_0^h(\Omega)$ be the finite element space on the coarse subdivision T^h, $e \in T^h$ (generally, e being the subconvergence element), $u^h \in S_0^h$, $u_e^h \in V_e^h \equiv S_0^h(\Omega) \oplus S_0^{h/2}(D_e)$ be the corresphonding Galerkin solution. Then, there exists the decomposition*

$$\left\| u - u_e^h \right\|^2 = \left\| u - u^h \right\|^2 - \|w\|^2 \tag{3.6}$$

where $w = u_e^h - u^h$ satisfies

$$\|w\|^2 \geq {\Delta_e}^2 \tag{3.7}$$

Proof. It follows from the definition that $u - u^h = u - u_e^h + w$ and

$$a(u - u_e^h, w) = 0$$

So, (3.6) is valid. Next,

$$\|w\|^2 = a(w, w) = a(w, w - w^I) \equiv a(w, \delta) \tag{3.8}$$

Since $w = (u_e^h - u) - (u^h - u)$ and $\delta \equiv w - w^I \in S^{h/2}(D_e)$, we have $a(u_e^h - u, \delta) = 0$. Therefore,

$$\|w\|^2 = a(u^h - u, \delta) \tag{3.9}$$

If Δ_e is the first kind criterion, $S_0^{h/2}(D_e) = \text{span}\{\phi_0\}$. Set

$$\delta = \delta(z_0)\phi_0 \tag{3.10}$$

where ϕ_0 is the basis function at the center z_0 of e. It follows from (3.8) that

$$\|w\| \leq \|\delta\| = |\delta(z_0)| \cdot \|\phi_0\|$$

Then (3.8) and (3.10) give the relations:

$$|\delta(z_0)| = ||w||^2 \,/\, a(u - u^h, \phi_0) \leq |\delta(z_0)|^2 \cdot \, ||\phi_0||^2 \,/\, a(u - u^h, \phi_0)$$
$$|\delta(z_0)| \geq |a(u - u^h, \phi_0)| \,/\, ||\phi_0||^2$$
$$||w||^2 = |\delta(z_0)| \cdot \, |a(u - u^h, \phi_0)| \geq |a(u - u^h, \phi_0)|^2 \,/\, ||\phi_0||^2 = {\Delta_e}^2$$

so (3.7) is valid.

If Δ_e is the second kind criterion, there exists a $\delta_0 \in S_0^{h/2}(D_e)$ such that

$$a(\delta_0, \phi) = (f, \phi) - a(u^h, \phi) = a(u - u^h, \phi), \quad \text{for any} \quad \phi \in S_0^{h/2}(D_e).$$

According to the energy product, we have

$$u - u^h - \delta_0 \perp S_0^{h/2}(D_e)$$

and then

$$||u - u^h - \delta_0 + \phi||^2 = ||u - u^h - \delta_0||^2 + ||\phi||^2, \quad \text{for any} \quad \phi \in S_0^{h/2}(D_e)$$

Letting $\phi = \delta_0$, we have

$$\left\|u - u^h\right\|^2 = \left\|u - u^h - \delta_0\right\|^2 + \|\delta_0\|^2 . \tag{3.11}$$

Noticing $||u - u^h - w||^2 = ||u - u_e^h||^2 \le ||u - u^h - \delta_0||^2$ and comparing (3.11) and (3.6), we get

$$\|w\|^2 = \left\|u - u^h\right\|^2 - \left\|u - u_e^h\right\|^2 \ge \left\|u - u^h\right\|^2 - \left\|u - u^h - \delta_0\right\|^2 = ||\delta_0||^2 = {\Delta_e}^2$$

which again gives (3.7).

Theorem 2. *Suppose the conditions of Theorem* 1 *be satisfied. Let* $u^h \in S_0^h(\Omega)$ *and* $u^{h/2} \in S_0^{h/2}(\Omega)$ *be the Galerkin solution on* $S_0^h(\Omega)$ *and* $S_0^{h/2}(\Omega)$ *respectively. Then, we have*

$$\Delta_e \le ||u^{h/2} - u^h||_{D_e} \tag{3.12}$$

Conversly, for the second kind criterion, there exists two positive constants c' *and* c *such that*

$$c' \sqrt{\sum_e {\Delta_e}^2} \le ||u^{h/2} - u^h|| \le c \sqrt{\sum_e {\Delta_e}^2} \tag{3.13}$$

where the summation is for all elements.

Proof. According to the definition, we have

$$\Delta_e = |a(u - u_h, \delta)| \,/\, \|\delta_0\| = a(u^{h/2} - u^h, \delta_0)| \,/\, \|\delta_0\| \le \|u^{h/2} - u^h\|_{D_e}.$$

If Δ_e is the first kind criterion, (3.12) can be proved similarly. It follows easily from (3.12) that the left part of (3.13) is valid.

On the contrary, since $u^{h/2} - u^h \in S_0^{h/2}(\Omega)$, denoting $v = u^{h/2} - u^h$, we have

$$||v||^2 = a(v, u^{h/2} - u^h) = a(v, u - u^h) = a(v - v^I, u - u^h)$$

where $v^I \in S_0^h(\Omega)$ is an interpolation on the coarse mesh.

Without loss of generality, suppose T^h be a triangular subdivision and, for each element $e \in T^h$, let $\phi_i (i = 1, 2, 3)$ be the basis functions at the mid-points of three sides of e with respect to $S_0^{h/2}(\Omega)$. Set

$$\delta_e = \frac{1}{2} \sum_{i=1}^{3} [v(z_i) - v^I(z_i)] \phi_i \in S_0^{h/2}(D_e). \tag{3.14}$$

Then,

$$v - v^I = \sum_e \delta_e.$$

It is easily seen that

$$||v||^2 = \sum_e a(\delta_e, u - u^h) = \sum_e a(\delta_e, \delta_0)$$

where δ_0 is defined by (3.2) or (3.5) and depends on e. Therefore,

$$\|v\|^2 \le c \sqrt{\sum_e \|\delta_e\|^2} \sqrt{\sum_e {\Delta_e}^2} \tag{3.15}$$

where $\Delta_e = ||\delta_0||$.

Next, it follows from (3.14) that

$$\begin{aligned} &||\delta_e|| \le \tfrac{1}{2} ||v - v^I||_{0,\infty,e} \max_{1 \le i \le 3} \|\phi_i\|^2 \le c h_e |v|_{1,\infty,e} \\ &\le c\, |v|_{1,e} \le c\, \|v\|_e \end{aligned}$$

Substituting it into (3.15) gets

$$\|v\|^2 \le c \sqrt{\sum_e \|v\|_e^2} \sqrt{\sum_e {\Delta_e}^2} = c\, \|v\| \sqrt{\sum_e {\Delta_e}^2}.$$

Therefore,

$$\|u^{h/2} - u^h\| = \|v\| \le c \sqrt{\sum_e {\Delta_e}^2}.$$

This completes the proof.

Remark. The theory of the local extrapolation[3] shows that $u - (4u^{h/2} - u^h)/3$ has a higher approximate accuracy than $u - u^h$. Thus, not strictly speaking,

$$\frac{4}{3}(u^{h/2} - u^h) = \frac{1}{3}(4u^{h/2} - u^h) - u^h \simeq u - u^h.$$

It follows from above theorem that Δ_e is a reasonable and computable for testing $\|u - u^h\|_{D_e}$ being 'good' or 'bad'.

3. Some Numerical Examples

Consider the two-point boundary value problem with singular points $x = 0$:

$$\begin{cases} -\frac{d^2 u(x)}{dx^2} + \frac{1}{x} \frac{du(x)}{dx} + \frac{1}{x^2} u(x) = \frac{7}{4} \frac{1}{x^{3/2}} - \frac{2}{x} \\ u(0) = u(1) = 0 \end{cases}$$

its exact solution is $\sqrt{x} - x$. Since $u'(x) = \frac{1}{2}x^{-1/2} - 1$ is not square integrable, $u \notin H_0^1([0,1])$.

We use the piecewise linear finite element method to solve the approximate solution u^h. Two methods are applied. One is the uniform subdivision and the other is the subconvergence local refinement. The method of the subconvergence local refinement proceeds as follows:

Step 1 (n=2): Divide $[0,1]$ into $n = 2$ elements and compute. Utilize (2.2) to find the number i_0 of the bad element.

Step 2 (n=3): Refine the $i_0 -$ th element by midpoint, then $[0,1]$ is divided into 3 elements. Use (2.2) again to find the number i_0 of new bad element.

If $\Delta_{e_{i_0}} = \max_e \Delta_e \leq \delta$, we stop. Otherwise, we refine the bad element by mid-point to get $n+1$ elements on $[0,1]$, then compute with the linear finite element and find the number of new bad element.

The following is several group results.

(I) The distribution of bad elements (subconvergence local refinement)

subdivision element number $n =$	2	3	4	5	6	7	8	9	10	11
bad element number $i =$	1	1	1	1	6	1	1	1	1	8

It may be seen from above results that the singular point $x = 0$ is in the majority of bad elements in the cycling process of the subconvergence local refinement. This shows that the criterion (2.2) is a effective method for finding singular points.

(II) The displacement error $|u(x) - u^h(x)|$ in the neighbourhood of the singular point:

x	uniform subdivision	subconvergence local refinement
1/2	$1.80 \times E - 1(n = 2)$	$1.80 \times E - 1(n = 2)$
1/4	$1.46 \times E - 1(n = 4)$	$5.76 \times E - 2(n = 3)$
1/8	$1.04 \times E - 2(n = 8)$	$5.52 \times E - 2(n = 4)$
1/16	$7.37 \times E - 2(n = 16)$	$4.61 \times E - 2(n = 5)$
1/32	$5.21 \times E - 2(n = 32)$	$3.60 \times E - 2(n = 6)$

(III) The derivative error $|u'(x) - u^{h\prime}(x)|$ in the neighbourhood of the sigular point:

x	uniform subdivision	subconvergence local refinement
1/4	$6.6 \times E - 1(n = 2)$	$6.6 \times E - 1(n = 2)$
1/8	$1.67 \times E - 1(n = 4)$	$3.6 \times E - 1(n = 3)$
1/16	$1.67(n = 8)$	$3.8 \times E - 1(n = 4)$
1/32	$1.48(n = 16)$	$4.3 \times E - 1(n = 5)$
1/64	$3.33(n = 32)$	$5.0 \times E - 1(n = 6)$

Practical computations show that

(1) the error decreases by exponential function as $h = 1/n$ does.

(2) the accuracy of the subconvergence local refinement is better than that of uniform subdivision but the number of the elements of the subconvergence local refinement is $lnn\,/\,nln2$ times of that of the latter.

(3) Apart from the singular point, the derivative has superconvergence. This is in accord with the theory of the local superconvergence.

x	$\|u'(x) - u^{h\prime}(x)\|$			
3/8	$8.59E-2(n=7)$	$8.82E-2(n=8)$	$8.86E-2(n=9)$	
1/8	$8.72E-2(n=6)$	$8.92E-2(n=7)$	$7.29E-2(n=8)$	$6.97E-2(n=9)$

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