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OPTIMAL INTERIOR AND LOCAL ERROR ESTIMATES OF A RECOVERED GRADIENT OF LINEAR ELEMENTS ON NONUNIFORM TRIANGULATIONS*

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Abstract

We examine a simple averaging formula for the gradient of linear finite elements in R^d whose interpolation order in the L^q-norm is $\mathcal{O}(h^2)$ for $d < 2q$ and nonuniform triangulations. For elliptic problems in R^2 we derive an interior superconvergence for the averaged gradient over quasiuniform triangulations. Local error estimates up to a regular part of the boundary and the effect of numerical integration are also investigated.

1. Introduction

Consider a model elliptic boundary value problem

$$-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) = f, \quad \text{in } \Omega, u = 0, \quad \text{on } \partial\Omega, 1.1$$

where $\Omega \subset R^d$, $d = 1, 2, 3$, is a bounded polyhedral domain with a Lipschitz boundary, $f \in L^2(\Omega)$, a_{ij} are Lipschitz-continuous functions and the matrix $\mathcal{A} = (a_{ij})$ is symmetric and uniformly positive definite with respect to $x \in \Omega$.

It is known that the finite element method applied to (1.1) may produce some superconvergence phenomena even if the used meshes are nonuniform[5,8,9,10,12]. In a recent paper [6], an interior error estimate for the recovered gradient of Galerkin piecewise linear approximations has been proposed in the case $d = 2$. This result, however, has required a high global regularity of the solution of the boundary value problem. In the present paper we derive other error estimates over some subdomains for problems of low regularity. We employ some results of [9, 13] together with a series of modified lemmas of our recent paper [6].

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In Section 2 we establish an optimal interior error estimate on subdomains in L^2-norm, whereas Section 3 is devoted to an error estimate in the so-called discrete interior L^2-norm. The effect of numerical integration is treated in Section 4 on a simple example. We present local error estimates up to a regular part of the boundary in Section 5, 6 and 7, in the continuous and discrete L^2-norm, respectively.

Throughout the paper $C, C', \ldots$ are generic positive constants and $\|\cdot\|$ is the Euclidean norm. The symbol $W_q^k(\Omega)$ stands for the Sobolev space equipped with the standard norm $\|\cdot\|_{k,q,\Omega}$ and seminorm $|\cdot|_{k,q,\Omega}$. In particular, we write

$$\|\cdot\|_{k,\Omega} = \|\cdot\|_{k,2,\Omega}, \qquad |\cdot|_{k,\Omega} = |\cdot|_{k,2,\Omega},$$

and $(.,.)_{0,\Omega}$ is the scalar product in $L^2(\Omega)$. The subspace of $W_2^1(\Omega)$ whose functions have vanishing traces is denoted by $\circ \rightarrow W_2^1(\Omega)$. The weak solution $u \in \circ \rightarrow W_2^1(\Omega)$ of (1.1) is defined by the relation

$$a(u,v) = (f,v)_{0,\Omega} \quad \forall v \in \circ \rightarrow W_2^1(\Omega), 1.2$$

where

$$a(u,v) = \int_\Omega \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx.$$

Let

$$V_h = \{v_h \in C(\overline{\Omega}) \mid v_h|_K \in P_1(K) \quad \forall K \in T_h\},$$

where T_h is a triangulation (decomposition) of $\overline{\Omega}$ into closed simplexes in the standard sense and $P_1(K)$ is the space of linear polynomials over K. Let $V_h^0 = V_h \cap \circ \rightarrow W_2^1(\Omega)$. A finite element approximation of (1.1) reads: Find $u_h \in V_h^0$ such that

$$a(u_h, v_h) = (f, v_h)_{0,\Omega}, \quad \forall v_h \in V_h^0 .1.3$$

Moreover, let $\pi_h : C(\overline{\Omega}) \rightarrow V_h$ be the usual linear interpolation operator such that

$$\pi_h v(Z) = v(Z), \quad \forall Z \in N_h,$$

where N_h is the set of nodes of T_h.

Recall that a family of triangulations $\mathcal{F} = \{T_h\}_{h\rightarrow 0}$ is said to be *regular* (*strongly regular*) if there exists a constant $\varkappa > 0$ such that for any $K \in T_h$ and any $T_h \in \mathcal{F}$ there exists a ball $\mathcal{B} \subset K$ with radius ρ_K such that $\varkappa h_K \leq \rho_K$ ($\varkappa h \leq \rho_K$), where $h_K = \operatorname{diam} K$ and $h = \max_K h_K$.

We briefly recall the definition of the weighted averaged gradient introduced in details in [6]. For $Z \in N_h$ denote by $\ell_i = \ell_i(Z)$ that straight line which passes through Z and is parallel with the axis x_i.

Now if $v \in C(\overline{\Omega})$ and $i \in \{1, ..., d\}$ we set

$$(G_h v(Z))_i = \alpha_i v(A_i) - (\alpha_i + \beta_i)v(Z) + \beta_i v(B_i), \quad \text{for } Z \in N_h,$$

where

$$\alpha_i = \frac{b_i}{a_i(b_i - a_i)}, \qquad \beta_i = \frac{a_i}{b_i(a_i - b_i)}, a_i = (A_i - Z)_i, \qquad b_i = (B_i - Z)_i, \quad i = 1, ..., d,$$

and where $(.)_i$ stands for the i-th component. Define

$$U = U(Z) = \bigcup K \in T_h K \cap Z \neq \emptyset K.$$

Let $A_i, B_i \in \ell_i \cap U$ for $Z \in N_h \cap \Omega$, $A_i \in \partial U$ for $Z \in \partial\Omega$ and let

$$C|a_i| \leq |b_i| \leq \overline{C}|a_i|$$

with $\overline{C} > C > 1$. In the case that such points A_i and B_i do not exist, we refer to [6] for the definition of $(G_h v(Z))_i$.

We introduce a continuous piecewise linear function (still denoted by $G_h v$) which is uniquely determined by the values at nodes. Hence, from now on

$$G_h v \in (V_h)^d, \quad \forall v \in C(\overline{\Omega}).$$

2. Optimal Interior Error Estimates in the L^2-Norm

Definition 2.1. *The Dirichlet problem* (1.1) *is said to be γ-regular* $(0 < \gamma \leq 1)$, *if its weak solution u belongs to $W_2^{1+\gamma}(\Omega)$ for any $f \in L^2(\Omega)$ and if*

$$\|u\|_{1+\gamma,\Omega} \leq C\|f\|_{0,\Omega}.2.1$$

Example 2.2. Let $\Omega \subset R^2$ be a bounded domain with a Lipschitz boundary $\partial\Omega$, which consists of a finite number of smooth arcs Γ_j of class W_∞^2. In corner points, let the tangents generate interior angles $\omega_j \in (0, \pi]$. If the coefficients a_{ij} belong to $W_\infty^1(\Omega)$ then the problem (1.1) is 1-regular (see [7] also for $d \geq 2$).

Example 2.3. Let $a_{ij} = \delta_{ij}$ (Kronecker's delta) and let Ω be a nonconvex polygonal domain. Then the problem (1.1) is γ-regular with some $\gamma \in (0, 1)$[4].

Lemma 2.4. *Assume that the Dirichlet problem* (1.1) *is γ-regular and the family $\mathcal{F}$ of triangulations is regular. Then*

$$\|u - u_h\|_{0,\Omega} \leq Ch^{2\gamma}|u|_{1+\gamma,\Omega}.2.2$$

Proof. Let us consider the following adjoint problem: Find $w \in \circ \to W_2^1(\Omega)$, such that

$$a(w, v) = (u - u_h, v)_{0,\Omega}, \quad \forall v \in \circ \to W_2^1(\Omega).2.3$$

By assumption and (2.1), we have $w \in W_2^{1+\gamma}(\Omega)$ and

$$\|w\|_{1+\gamma,\Omega} \le c\|u - u_h\|_{0,\Omega}.$$

Inserting $v = u - u_h$ into (2.3), we obtain by (1.2) and (1.3)

$$\|u - u_h\|_{0,\Omega}^2 = a(u - u_h, w) = a(u - u_h, w - \pi_h w)2.4 \le C|u - u_h|_{1,\Omega}|w - \pi_h w|_{1,\Omega}.$$

By interpolation theory in Sobolev spaces with fractional derivatives[2,3] and the well-known Céa's lemma we have

$$|w-\pi_h w|_{1,\Omega} \le Ch^\gamma|w|_{1+\gamma,\Omega} \le Ch^\gamma\|u-u_h\|_{0,\Omega}, 2.5|u-u_h|_{1,\Omega} \le C|u-\pi_h u|_{1,\Omega} \le Ch^\gamma|u|_{1+\gamma,\Omega}.2.6$$

Substituting (2.5) and (2.6) into (2.4), we arrive at

$$\|u - u_h\|_{0,\Omega}^2 \le Ch^{2\gamma}|u|_{1+\gamma,\Omega}\|u - u_h\|_{0,\Omega}$$

and the lemma is proved.

Definition 2.5. *Let $d = 2$. We say that the family $\mathcal{F}$ of triangulations satisfies the assumption* (A1) *in Ω^*, if*

it is generated in a subdomain $\Omega^* \subset \Omega$ by a smooth distortion (W_∞^2-diffeomorphism) of the reference uniform or

Recall that[9] in chevron triangulations the slope (+1) of diagonals of one row is followed by the slope (−1) in the next row.

Lemma 2.6. *Assume that the family $\mathcal{F}$ of triangulations satisfies* (A1) *in Ω^*, $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega^* \subset \Omega$ and $u|_{\Omega_2} \in W_2^3(\Omega_2)$. Then for h small enough there exists $C(\Omega_1) > 0$ such that*

$$|u_h - \pi_h u|_{1,\Omega_1}^2 \le C(\Omega_1)(h^4\|u\|_{3,\Omega_2}^2 + h^2|u_h - \pi_h u|_{1,\Omega_2}^2 + \|u_h - \pi_h u\|_{0,\Omega_2}^2).$$

The proof is essentially the same as that of [13, Lemmas 3.5 and 3.6] combined with the proof of [9, Lemma 5.3].

Lemma 2.7. *Let the assumptions of Lemma* 2.6 *be fulfilled. Moreover, let $u \in W_2^{1+\gamma}(\Omega)$ for some $\gamma \in (0, 1]$. Then*

$$|u_h - \pi_h u|_{1,\Omega_1}^2 \le C(\Omega_1)(h^4\|u\|_{3,\Omega_2}^2 + h^{2+2\gamma}|u|_{1+\gamma,\Omega}^2 + \|u - u_h\|_{0,\Omega_2}^2).$$

Proof. By Céa's lemma we have

$$|u - u_h|_{1,\Omega} \le C|u - \pi_h u|_{1,\Omega}.$$

Using the interpolation theory, we may write

$$|u_h - \pi_h u|_{1,\Omega_2} \le |u - u_h|_{1,\Omega} + |u - \pi_h u|_{1,\Omega} \le 2.7 \le (1+C)|u - \pi_h u|_{1,\Omega} \le \tilde{C} h^{\gamma} |u|_{1+\gamma,\Omega}.$$

Next, we have

$$\|u_h - \pi_h u\|_{0,\Omega_2} \le \|u - u_h\|_{0,\Omega_2} + \|u - \pi_h u\|_{0,\Omega} \le 2.8 \le \|u - u_h\|_{0,\Omega_2} + C h^{1+\gamma} |u|_{1+\gamma,\Omega}.$$

Combining Lemma 2.6 with (2.7) and (2.8), we obtain Lemma 2.7.

Proposition 2.8. *Let $\Omega_0 \subset\subset \Omega_1 \subset \Omega$ be d-dimensional subdomains, $q \in (d/2, \infty)$, $q \ge 1$ and let $\mathcal{F}$ be a strongly regular family of decompositions of $\overline{\Omega}$. Then there exists a constant $C > 0$ such that*

$$\| \operatorname{grad} v - G_h v\|_{0,q,\Omega_0} \le C h^2 |v|_{3,q,\Omega_1}$$

holds for any $v \in W_q^3(\Omega_1)$ and sufficiently small h.

The proof is a slight modification of that of Theorem 3.8 in [6], where the norms and seminorms over the domain Ω are replaced by those over subdomains Ω_0 and Ω_1, respectively.

Lemma 2.9. *Let $\mathcal{F}$ be a strongly regular family of decompositions of $\overline{\Omega} \subset R^d$ and let $\Omega_0 \subset\subset \Omega_1 \subset \Omega$. Then there exists a constant C such that*

$$\|G_h v_h\|_{0,\Omega_0} \le C \| \operatorname{grad} v_h\|_{0,\Omega_1} \quad \forall v_h \in V_h$$

holds for sufficiently small h.

The proof is a slight modification of that of Lemma 5.1 in [6].

Lemma 2.10. *Let $d = 2$, $q \in (2, \infty)$ and let the family $\mathcal{F}$ satisfy* (A1) *in Ω^*. Let $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega^* \subset \Omega$ and $v \in W_q^3(\Omega_1)$. Then*

$$\|G_h v - G_h \pi_h v\|_{0,q,\Omega_0} \le C h^2 \|v\|_{3,q,\Omega_1}$$

holds for sufficiently small h.

The proof is based on the same argument as that of Proposition 2.8 and on Lemma 5.2 from [6].

Theorem 2.11. *Let $d = 2$, $q \in (2, \infty)$ and let the family $\mathcal{F}$ satisfy* (A1) in Ω^*. *Let $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega_2 \subset\subset \Omega^* \subset \Omega$, $u|_{\Omega_1} \in W_q^3(\Omega_1)$, $u|_{\Omega_2} \in W_2^3(\Omega_2)$ and let the Dirichlet problem* (1.1) *be γ-regular. Then*

$$\| \operatorname{grad} u - G_h u_h\|_{0,\Omega_0} \le C(\Omega_1) h^{2\gamma} (\|u\|_{3,q,\Omega_1} + \|u\|_{3,\Omega_2} + |u|_{1+\gamma,\Omega})$$

holds for sufficiently small h.

Proof. We have

$$\| \operatorname{grad} u - G_h u_h\|_{0,\Omega_0} \leq \| \operatorname{grad} u - G_h u\|_{0,\Omega_0} + \|G_h u - G_h \pi_h u\|_{0,\Omega_0} + 2.9 + \|G_h(\pi_h u - u_h)\|_{0,\Omega_0} \equiv J_1 + J_2 + J_3.$$

Proposition 2.8 yields that

$$J_1 \leq C_1 h^2 |u|_{3,q,\Omega_1}.2.10$$

From Lemma 2.10 we obtain that

$$J_2 \leq C_2 h^2 \|u\|_{3,q,\Omega_1}.2.11$$

Since the family $\mathcal{F}$ of triangulations is strongly regular (see [6, (5.13)]), we may apply Lemma 2.9 to estimate J_3 so that

$$J_3 \leq C_3 |u_h - \pi_h u|_{1,\Omega_1}.2.12$$

Using Lemmas 2.7 and 2.4, we may write

$$|u_h - \pi_h u|_{1,\Omega_1} \leq C(\Omega_1)(h^2\|u\|_{3,\Omega_2} + h^{1+\gamma}|u|_{1+\gamma,\Omega} + \|u - u_h\|_{0,\Omega_2})2.13 \leq C_4(\Omega_1)(h^2\|u\|_{3,\Omega_2} + h^{1+\gamma}|u|_{1+\gamma,\Omega} + h^{2\gamma}|u|_{1+\gamma,\Omega})$$

Inserting the estimates (2.10)-(2.13) into (2.9), we obtain the estimate as required.

Theorem 2.12. *Let $\Omega_0 \subset\subset \Omega_1 \subset \Omega$ be d-dimensional domains, $q \in (\frac{d}{2}, \infty)$, $q \geq 1$ and let $\mathcal{F}$ be strongly regular family of decompositions of $\overline{\Omega}$. Then*

$$\| \operatorname{grad} v - G_h v\|_{1,q,\Omega_0} \leq Ch|v|_{3,q,\Omega_1}$$

holds for sufficiently small h.

Proof. Recall that

$$\|w - \pi_h w\|_{1,q,K} \leq Ch_K |w|_{2,q,K} \quad \forall K \in T_h \quad \forall w \in W_q^2(K).$$

From here we deduce that

$$\| \operatorname{grad} v - L_h v\|_{1,q,\Omega_0} \leq Ch|v|_{3,q,\Omega_1}, 2.14$$

where $(L_h v)_i = \pi_h(\partial_i v)$, $i = 1, ..., d$. Using the standard inverse inequality (see [1, Theorem 17.2]) and [6, (3.26)], we obtain for h sufficiently small

$$\|L_h v - G_h v\|_{1,q,\Omega_0} \leq Ch|v|_{3,q,\Omega_1}.2.15$$

The theorem then follows from the triangle inequality, (2.14) and (2.15).

3. Optimal Interior Error Estimates in a Discrete L^2-Norm

In the present section we consider two-dimensional problems ($d = 2$) only. Let us define a discrete L^q-norm of vectors $F(Z)$ defined at nodes Z. For

$$F = \{F(Z)\}_{Z \in N_h \cap \Omega_0}, \quad \Omega_0 \subset \Omega, \quad q \geq 1,$$

we set

$$\|F\|_{q,\Omega_0} = h^{2/q}\Big(\sum_{Z\in N_h\cap\Omega_0} \|F(Z)\|^q\Big)^{1/q}.$$

One can easily verify the triangle inequality

$$\|F+G\|_{q,\Omega_0} \le \|F\|_{q,\Omega_0} + \|G\|_{q,\Omega_0},$$

using the Minkowski inequality.

Remark 3.1. From the Hölder inequality, we obtain that

$$\|F\|_{2,\Omega_0} \le C(\Omega_0)\|F\|_{q,\Omega_0}, \quad \forall q > 2,$$

if the family $\mathcal{F}$ is strongly regular.

Lemma 3.2. *Let $q \in (2,\infty)$, $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega^* \subset \Omega$, $v \in W^3_q(\Omega_1)$ and let the family $\mathcal{F}$ satisfy* (A1) *in Ω^*. Then*

$$\|\ \mathrm{grad}\ v - G_h(\pi_h v)\|_{q,\Omega_0} \le Ch^2\|v\|_{3,q,\Omega_1} 3.1$$

holds for sufficiently small h.

Proof. Let us consider an arbitrary $Z \in N_h \cap \Omega_0$. By [6, Lemmas 3.6 and 5.2] we have

$$\|(\ \mathrm{grad}\ v - G_h(\pi_h v))(Z)\| \le \|(\ \mathrm{grad}\ v - G_h v)(Z)\| + \|G_h(v-\pi_h v)(Z)\| \le Ch^{2-2/q}\|v\|_{3,q,U(Z)}.$$

Then for sufficiently small h we may write

$$\|\ \mathrm{grad}\ v - G_h(\pi_h v)\|^q_{q,\Omega_0} = h^2 \sum_{Z\in N_h\cap\Omega_0} \|(\ \mathrm{grad}\ v - G_h(\pi_h v))(Z)\|^q \le \ \le Ch^2h^{2q-2} \sum_{Z\in N_h\cap\Omega_0} \|v\|^q_{3,q,U(Z)} \le 3Ch^{2q}\|v\|^q_{3,q}$$

and the estimate (3.1) follows.

Lemma 3.3. *Assume that the family $\mathcal{F}$ satisfies* (A1) *in Ω^*, $a_{ij} \in W^1_\infty(\Omega)$, $u \in W^3_2(\Omega_2)$ and that the Dirichlet problem* (1.1) *is γ-regular. Let $\Omega_0 \subset\subset \Omega_2 \subset\subset \Omega^* \subset \Omega$ and $q \in (2,\infty)$. Then*

$$\|G_h(u_h - \pi_h u)\|_{2,\Omega_0} \le C(\Omega_1)h^{2\gamma}(\|u\|_{3,\Omega_2} + |u|_{1+\gamma,\Omega}) 3.2$$

holds for sufficiently small h.

Proof. In the proof of Lemma 5.1 of [6] (see (5.9) there), we established the following estimate

$$|(G_h v_h(Z))_i| \le |v_h|_{1,\infty,\delta_i(Z)}, \quad \forall v_h \in V_h, \quad \forall Z \in N_h \cap \Omega,$$

where $\delta_i(Z) = \{K \in T_h \mid \mathrm{meas}_1((\overline{ZA_i} \cup \overline{ZB_i}) \cap K) > 0\}$. Since the family of triangulations is strongly regular,

$$|v_h|_{1,\infty,K} \le Ch^{-1}|v_h|_{1,K} \quad \forall K \in T_h.$$

If $E(Z,i)$ denotes that triangle, where the seminorm attains its maximum over $\delta_i(Z)$, we may write

$$|(G_h v_h(Z))_i| \le Ch^{-1}|v_h|_{1,E(Z,i)}.$$

There exists a subdomain Ω_1 such that $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega_2$. Then we have

$$\|G_h v_h\|^2_{2,\Omega_0} = h^2 \sum_{Z\in N_h\cap\Omega_0} \sum_{i=1}^{2} |(G_h v_h(Z))_i|^2 \le C \sum_{Z,i} |v_h|^2_{1,E(Z,i)} \le 6C|v_h|^2_{1,\Omega_1}, 3.3$$

for sufficiently small h. Substituting $v_h = u_h - \pi_h u$, we obtain

$$\|G_h(u_h - \pi_h u)\|_{2,\Omega_0} \le C|u_h - \pi_h u|_{1,\Omega_1}.$$

By Lemmas 2.7 and 2.4, we have

$$|u_h-\pi_h u|_{1,\Omega_1} \le C(\Omega_1)(h^2\|u\|_{3,\Omega_2}+h^{1+\gamma}|u|_{1+\gamma,\Omega}+h^{2\gamma}|u|_{1+\gamma,\Omega}) \le \ \le C(\Omega_1)h^{2\gamma}(\|u\|_{3,\Omega_2}+|u|_{1+\gamma,\Omega}).$$

Consequently,

$$\|G_h(u_h - \pi_h u)\|_{2,\Omega_0} \le C(\Omega_1)h^{2\gamma}(\|u\|_{3,\Omega_2} + |u|_{1+\gamma,\Omega}).$$

Theorem 3.4. *Let the assumptions of Theorem* 2.11 *be fulfilled. Then*

$$\|\operatorname{grad} u - G_h u_h\|_{2,\Omega_0} \le C(\Omega_0,\Omega_1)h^{2\gamma}(\|u\|_{3,q,\Omega_1} + \|u\|_{3,\Omega_2} + |u|_{1+\gamma,\Omega})$$

holds for sufficiently small h.

Proof. Since the family $\mathcal{F}$ is strongly regular, we can use Remark 3.1 and Lemma 3.2 to get the estimate

$$\|\operatorname{grad} u - G_h(\pi_h u)\|_{2,\Omega_0} \le C(\Omega_0)h^2\|u\|_{3,q,\Omega_1}.$$

Then using the triangle inequality and Lemma 3.3, we arrive at the result, as required.

4. Effect of Numerical Integration

The Galerkin approximation u_h has been defined under the assumption that all integrations in the inner products are performed exactly. In practice they will in fact be performed numerically using quadrature formulae.

We present here an example of numerical integration and derive its effect to error estimates of the averaged gradient.

Lemma 4.1. *Let $d = 2$, $\mathcal{A} = (a_{ij})$, $a_{ij} \in W^2_\infty(\Omega)$, $f \in W^2_2(\Omega)$ and define*

$$a_h(w,v) = \sum_{i,j=1}^{2} \int_\Omega (\pi_h a_{ij}) \frac{\partial w}{\partial x_i} \frac{\partial v}{\partial x_j} dx.4.1$$

Let $u_h^* \in V_h^0$ *satisfy*

$$a_h(u_h^*, v_h) = (\pi_h f, v_h)_{0,\Omega} \quad \forall v_h \in V_h^0. 4.2$$

Then

$$|u_h - u_h^*|_{1,\Omega} \leq Ch^2(|f|_{2,\Omega} + |\mathcal{A}|_{2,\infty,\Omega}\|u\|_{1,\Omega}). 4.3$$

For the proof see [13, Corollary of Lemma 3.7].

Theorem 4.2. *Let the assumptions of Theorem* 2.11 *be fulfilled. Moreover, let* $a_{ij} \in W_\infty^2(\Omega)$*,* $f \in W_2^2(\Omega)$ *and define* u_h^* *by means of* (4.1) and (4.2). *Then*

$$\|\text{ grad } u - G_h u_h^*\|_{0,\Omega_0} \leq C(\Omega_0, \Omega_1)h^{2\gamma}(|f|_{2,\Omega} + |\mathcal{A}|_{2,\infty,\Omega}\|u\|_{1,\Omega} + 4.4 + \|u\|_{3,q,\Omega_1} + \|u\|_{3,\Omega_2} + |u|_{1+\gamma,\Omega})$$

holds for sufficiently small h*.*

Proof. By the triangle inequality

$$\|\text{ grad } u - G_h u_h^*\|_{0,\Omega_0} \leq \|\text{ grad } u - G_h u_h\|_{0,\Omega_0} + \|G_h u_h - G_h u_h^*\|_{0,\Omega_0} \equiv I_1 + I_2.$$

For the first term we use the estimate of Theorem 2.11. The second term can be estimated on the basis of Lemma 2.9 and Lemma 4.1 as follows

$$\|G_h(u_h - u_h^*)\|_{0,\Omega_0} \leq C|u_h - u_h^*|_{1,\Omega_1} \leq Ch^2(|f|_{2,\Omega} + |\mathcal{A}|_{2,\infty,\Omega}\|u\|_{1,\Omega}).$$

Combining these two estimates, we arrive at (4.4), as required.

Theorem 4.3. *Let the assumptions of Theorem* 4.2 *be satisfied. Then*

$$|\!|\!| \text{ grad } u - G_h u_h^* |\!|\!|_{2,\Omega_0} \leq C(\Omega_0, \Omega_1)h^{2\gamma}(|f|_{2,\Omega} + |\mathcal{A}|_{2,\infty,\Omega}\|u\|_{1,\Omega} + \|u\|_{3,q,\Omega_1} + \|u\|_{3,\Omega_2} + |u|_{1+\gamma,\Omega})$$

holds for sufficiently small h*.*

Proof. We have

$$|\!|\!| \text{ grad } u - G_h u_h^* |\!|\!|_{2,\Omega_0} \leq |\!|\!| \text{ grad } u - G_h u_h |\!|\!|_{2,\Omega_0} + |\!|\!| G_h(u - u_h^*) |\!|\!|_{2,\Omega_0}.$$

Using (3.3), we may write

$$|\!|\!| G_h(u_h - u_h^*) |\!|\!|_{2,\Omega_0} \leq C|u_h - u_h^*|_{1,\Omega}$$

and the rest of the proof is the same as previously, i.e., it follows from Lemma 4.1 and Theorem 3.4.

5. Local Error Estimates up to the Boundary

We shall study the local behaviour of the error grad $u - G_h u_h$ in subdomains Ω_0 adjacent to the boundary $\partial\Omega$, assuming that the solution u belongs to $W_q^3(\Omega_1)$ for a subdomain Ω_1 containing Ω_0. Such a case may occur, if the part $\partial\Omega_1 \cap \partial\Omega$, the coefficients a_{ij} and the right-hand side f are smooth enough (see Remark 5.4 below).

Definition 5.1. *Let $d \geq 2$. For integer j and k we write $\Omega_j < \Omega_k$, if there exist two open domains $\mathcal{G}_j$ and $\mathcal{G}_k$ which are not contained in Ω (see Figure 1), $\mathcal{G}_j \subset\subset \mathcal{G}_k$, $\Omega_j = \mathcal{G}_j \cap \Omega$ and $\Omega_k = \mathcal{G}_k \cap \Omega$.*

Figure 1

Lemma 5.2. *Assume that $d = 2$ and $\Omega_1 < \Omega_2 \subset \Omega^*$. Let the family $\mathcal{F}$ satisfy* (A1) *in Ω^* and*

$\partial\Omega_2 \cap \partial\Omega$ corresponds to a part of a coordinate line $X = const$ or$Y = const$ in the reference uniform triangulation

Moreover, let $a_{ij} \in W^1_\infty$, $i, j = 1, 2$, and the solution $u \in W^3_2(\Omega_2)$. Then there exists a constant $C(\Omega_1) > 0$ such that

$$|u_h - \pi_h u|^2_{1,\Omega_1} \leq C(\Omega_1)(h^4 \|u\|^2_{3,\Omega_2} + h^2 |u_h - \pi_h u|^2_{1,\Omega_2} + \|u_h - \pi_h u\|^2_{0,\Omega_2})$$

holds for sufficiently small h.

The proof is essentially the same as that of [13, Lemmas 3.5 and 3.6] in combination with the proof of [9, Lemma 5.3].

Remark 5.3. The assumption (A2) can be weakened by assuming that $\partial\Omega_2 \cap \partial\Omega$ may correspond to a part of the "diagonal" straight line $X \pm Y = \text{const}$ in the reference square grid of (A1).

Remark 5.4. Some sufficient conditions for the local regularity of the solution in a neighbourhood of the boundary (i.e., for $u \in W^3_2(\Omega_k)$) are given in [11, p. 221]. They include, for instance, $f \in W^1_2(\Omega_k)$, $a_{ij} \in C^{(1),1}(\overline{\Omega}_k)$, $\partial\Omega \cap \mathcal{G}_k$ is described by a function from $C^{(4),1}$.

Lemma 5.5. *Let the assumptions of Lemma* 5.2 *be satisfied and let $u \in W^{1+\gamma}_2(\Omega)$ for some $\gamma \in (0, 1]$. Then*

$$|u_h - \pi_h u|^2_{1,\Omega_1} \leq C(\Omega_1)(h^4 \|u\|^2_{3,\Omega_2} + h^{2+2\gamma} |u|^2_{1+\gamma,\Omega} + \|u - u_h\|^2_{0,\Omega_2}).$$

The proof is the same as that of Lemma 2.7.

Proposition 5.6. *Let* $\Omega_0 < \Omega_1$*,* $d \geq 2$*,* $q \in (\frac{d}{2}, \infty)$*,* $v \in W_q^3(\Omega_1)$ *and let* $\mathcal{F}$ *be a strongly regular family of triangulations. Then*

$$\| \operatorname{grad} v - G_h v\|_{0,q,\Omega_0} \leq Ch^2 |v|_{3,q,\Omega_1}$$

holds for h *small enough.*

The proof is a slight modification of that of [6, Theorem 3.8], where the norms and seminorms over Ω are replaced by those over subdomains Ω_0 and Ω_1, respectively.

Lemma 5.7. *Let* $\Omega_0 < \Omega_1$*,* $d \geq 2$ *and let* $\mathcal{F}$ *be a strongly regular family of triangulations. Then*

$$\|G_h v_h\|_{0,\Omega_0} \leq C\| \operatorname{grad} v_h\|_{0,\Omega_1} \quad \forall v_h \in V_h$$

holds for h *small enough.*

The proof is a slight modification of that of [6, Lemma 5.1].

Lemma 5.8. *Assume that* $\Omega_0 < \Omega_1 < \Omega^*$*,* $d = 2$*,* $q \in (2, \infty)$*,* $v \in W_q^3(\Omega_1)$ *and let the family* $\mathcal{F}$ *satisfy* (A1) *in* Ω^* *and* (A2)*. Moreover, let*

for $Z \in N_h \cap \partial\Omega$ *the points* B_i *be chosen on the boundary*∂K_C *of the triangle* K_C *(of Figure* 2*), such that* 5.1$\{K \in$

Then

$$\|G_h v - G_h \pi_h v\|_{0,\Omega_0} \leq Ch^{3/2} \|v\|_{3,q,\Omega_1}$$

holds for h *small enough.*

Figure 2

Proof. For any $Z \in N_h \cap \partial\Omega \cap \partial\Omega_1$ and any $i = 1, 2$ one can prove the following estimate

$$|(G_h(v-\pi_h v))_i(Z)| = |\alpha_i(v(A_i)-\pi_h v(A_i))+\beta_i(v(B_i)-\pi_h v(B_i))| \leq Ch(|v|_{2,\infty,U}+|v|_{3,q,U}), 5.2$$

where $U = U_i(Z)$ denotes the union of the triangles $K \in T_h$ for which $\operatorname{int}(\overline{ZB_i}) \cap K \neq \emptyset$ (the proof can be found in [5], for an analogous proof see [6, Lemma 5.2]).

Let us define the following (closed) strips

$$\Omega_b = \bigcup_{K\cap\partial\Omega\cap\partial\Omega_0\neq\emptyset} K, \qquad \Omega_c = \bigcup_{K\cap\Omega_b\neq\emptyset} K.$$

Then we obviously have

$$\text{meas } \Omega_b \leq C(\Omega_0)h.$$

From (5.2) and [6, Lemma 5.2] we deduce

$$\|(G_h(v-\pi_h v))_i\|_{0,\infty,\Omega_b} \leq \max_{Z\in N_h\cap\Omega_b} |(G_h(v-\pi_h))_i(Z)| = \ = |(G_h(v-\pi_h v))_i(Z^*)| \leq Ch(|v|_{2,\infty,\Omega_c}+\|v\|_{3,q,\Omega_c}),$$

where Z^* is the node at which the minimum is attained. Consequently,

$$\|(G_h(v-\pi_h))_i\|^2_{0,\Omega_b} \leq \ \text{meas } \Omega_b \|(G_h(v-\pi_h v))_i\|^2_{0,\infty,\Omega_b} 5.3 \leq C(\Omega_0)h^3(|v|_{2,\infty,\Omega_c}+\|v\|_{3,q,\Omega_c})^2.$$

Arguing as in the proof of Lemma 2.10, we obtain

$$\|(G_h(v-\pi_h v))_i\|^2_{0,\Omega_0\setminus\Omega_b} \leq Ch^4\|v\|^2_{3,q,\Omega_1} 5.4$$

for h small enough. Thus we arrive at the estimate

$$\|(G_h(v-\pi_h v)_i\|^2_{0,\Omega_0} \leq \tilde{C}h^3(|v|^2_{2,\infty,\Omega_c} + \|v\|^2_{3,q,\Omega_1}) \leq Ch^3\|v\|^2_{3,q,\Omega_1},$$

using also the continuous embedding $W^3_q(\Omega_1) \hookrightarrow W^2_\infty(\Omega_1)$.

Theorem 5.9. *Assume that* $d = 2$, $\Omega_0 < \Omega_1 < \Omega_2 < \Omega^*$, $q \in (2, \infty)$, *the family* $\mathcal{F}$ *of triangulations satisfies* (A1) *in* Ω^* *and* (A2), *the solution* $u|_{\Omega_1} \in W^3_q(\Omega_1)$, $u|_{\Omega_2} \in W^3_2(\Omega_2)$, *the Dirichlet problem* (1.1) *is* γ*-regular and that* (5.1) *holds. Then*

$$\| \operatorname{grad} u - G_h u_h\|_{0,\Omega_0} \leq C(\Omega_1)h^\beta(\|u\|_{3,q,\Omega_1} + \|u\|_{3,\Omega_2} + |u|_{1+\gamma,\Omega})$$

where $\beta = \min\{2\gamma, \frac{3}{2}\}$, *holds for sufficiently small* h.

The proof is the same as that of Theorem 2.11, with the only exception: instead of (2.11) we use the estimate

$$\|G_h u - G_h\pi_h u\|_{0,\Omega_0} \leq Ch^{3/2}\|u\|_{3,q,\Omega_1},$$

based on Lemma 5.8. We employ the triangle inequality, Proposition 5.6, Lemmas 5.8, 5.7, 5.5 and 2.4.

Theorem 5.10. *Let the assumptions of Theorem* 5.9 *be fulfilled. Moreover, let* $a_{ij} \in W^2_\infty(\Omega)$, $f \in W^2_2(\Omega)$ *and let* u^*_h *be defined by means of* (4.1) *and* (4.2). *Then*

$$\| \operatorname{grad} u - G_h u^*_h\|_{0,\Omega_0} \le C(\Omega_0, \Omega_1) h^\beta (\|u\|_{3,q,\Omega_1} + \|u\|_{3,\Omega_2} + |u|_{1+\gamma,\Omega} + + |f|_{2,\Omega} + |\mathcal{A}|_{2,\infty,\Omega} \|u\|_{1,\Omega})$$

holds for h *small enough.*

The proof follows from the triangle inequality, Theorem 5.9, Lemma 5.7 and Lemma 4.1.

6. Local Error Estimates up to the Boundary in Some Particular Cases

We easily realize that the loss of accuracy (if $\gamma > 3/4$) in the previous section is caused by the estimate (5.2) for the boundary nodes $Z \in N_h \cap \partial\Omega$. There are situations, however, when the loss of accuracy can be removed, at least for some component of the recovered gradient. We present one such a favourable case in what follows.

Lemma 6.1. *Let* $d \ge 2$, $q \in (\frac{d}{2}, \infty)$, $\Omega_0 < \Omega_1$, $v \in W^3_q(\Omega_1)$, *let* $\mathcal{F}$ *be regular and let the points* A_i, B_i *coincide for some* $i \in \{1, ..., d\}$ *with the nodes of* T_h *for all* $Z \in N_h \cap \Omega_{0h}$, *where*

$$\Omega_{0h} = \bigcup K \in T_h K \cap \Omega_0 \ne \emptyset K.$$

Then for all $Z \in N_h \cap \Omega_{0h}$ *and* h *sufficiently small, we have*

$$|\partial_i v(Z) - (G_h \pi_h v(Z))_i| \le C(h(Z))^{2-d/q} |v|_{3,q,\delta_i(Z)},$$

where

$$h(Z) = \max_{Z\cap K \ne \emptyset, K \in T_h} h_K.$$

Proof. We slightly modify the proof of [6, Lemma 3.6]. Namely, in (3.13) and (3.14) of [6] we replace $p(Z)$, $p(A_i)$, $p(B_i)$ by $\pi_h p(Z)$, $\pi_h p(A_i)$, $\pi_h p(B_i)$, so that

$$(G_h \pi_h p(Z))_i = \partial_i p(Z) \quad \forall p \in P_2.$$

In the rest of the proof we replace $(G_h v(Z))_i$ by $(G_h \pi_h v(Z))_i$ and realize that

$$\delta_i(Z) = \bigcup \{K \in T_h \mid \operatorname{meas}_1((\overline{ZA_i} \cup \overline{ZB_i}) \cap K) > 0\} \subset \Omega_1$$

for all $Z \in \Omega_{0h}$, if h is small enough.

Lemma 6.2. *Let $d = 2$, $q \in (2, \infty)$, $\Omega_0 < \Omega_1 < \Omega^*$, $v \in W_q^3(\Omega_1)$, let $\mathcal{F}$ satisfy* (A1) *in* Ω^*, (A2) *and let the points A_i, B_i coincide for some $i \in \{1, 2\}$ with the nodes of T_h for all $Z \in N_h \cap \partial\Omega_{0h} \cap \partial\Omega$. Then*

$$|\partial_i v(Z) - (G_h \pi_h v(Z)))_i| \leq C(h(Z))^{2-2/q} \|v\|_{3,q,\delta_i(Z)}$$

holds for $Z \in N_h \cap \Omega_{0h}$.

Proof. If $Z \in N_h \cap \partial\Omega_{0h} \cap \partial\Omega$, we use the same argument as that in the proof of Lemma 6.1.

If $Z \in N_h \cap (\Omega_{0h} \setminus (\partial\Omega_{0h} \cap \partial\Omega))$, we may use [6, Lemmas 3.6 and 5.2] to obtain

$$|\partial_i v(Z) - (G_h \pi_h v(Z))_i| \leq |\partial_i v(Z) - (G_h v(Z))_i| + |(G_h v(Z) - G_h \pi_h v(Z))_i| \leq \leq C(h(Z))^{2-2/q}(|v|_{3,q,\delta_i(Z)} + \|v\|_{3,q,\delta_i(Z)})$$

Proposition 6.3. (i) *Let the assumptions of Lemma* 6.1 *be fulfilled and let $\mathcal{F}$ be strongly regular. Then for sufficiently small h*

$$\|\partial_i v - (G_h \pi_h v)_i\|_{0,q,\Omega_0} \leq Ch^2 |v|_{3,q,\Omega_1}.$$

(ii) *If the assumptions of Lemma* 6.2 *are satisfied then*

$$\|\partial_i v - (G_h \pi_h v)_i\|_{0,q,\Omega_0} \leq Ch^2 \|v\|_{3,q,\Omega_1}$$

holds for h small enough.

The proof is parallel to that of [6, Theorem 3.8]. Instead of Lemma 3.6 there we use Lemma 6.1 or Lemma 6.2 and replace grad by ∂_i and $(G_h v)_i$ by $(G_h \pi_h v)_i$.

Theorem 6.4. *Assume that $d = 2$, $q \in (2, \infty)$, $\Omega_0 < \Omega_1 < \Omega_2 < \Omega^*$, the family $\mathcal{F}$ satisfies* (A1) *in* Ω^*, (A2) *and A_i, B_i concide for some $i \in \{1, 2\}$ with the nodes of T_h for all $Z \in N_h \cap \partial\Omega_1 \cap \partial\Omega$, the solution $u|_{\Omega_1} \in W_q^3(\Omega_1)$, $u|_{\Omega_2} \in W_2^3(\Omega_2)$ and the Dirichlet problem* (1.1) *is γ-regular. Then*

$$\|\partial_i u - (G_h u_h)_i\|_{0,\Omega_0} \leq C(\Omega_0, \Omega_1) h^{2\gamma} (\|u\|_{3,\Omega_2} + \|u\|_{3,q,\Omega_1} + \|f\|_{0,\Omega})$$

holds for h sufficiently small.

Proof. We have

$$\|\partial_i u - (G_h u_h)_i\|_{0,\Omega_0} \leq \|\partial_i u - (G_h \pi_h u)_i\|_{0,\Omega_0} + \|(G_h \pi_h u - G_h u_h)_i\|_{0,\Omega_0} \equiv I_1 + I_2 .6.1$$

Proposition 6.3 (ii) yields that

$$I_1 \leq Ch^2 \|v\|_{3,q,\Omega_1} .6.2$$

Since $\mathcal{F}$ is strongly regular, we deduce from Lemma 5.7 that

$$I_2 \leq C|u_h - \pi_h u|_{1,\Omega_1} 6.3$$

if h is small. Making use of Lemma 5.5 and Lemma 2.4, we may write

$$|u_h-\pi_h u|_{1,\Omega_1} \le \tilde{C}(\Omega_1)(h^2\|u\|_{3,\Omega_2}+h^{1+\gamma}|u|_{1+\gamma,\Omega}+\|u-u_h\|_{0,\Omega_2}) \le 6.4 \le C(\Omega_1)h^{2\gamma}(\|u\|_{3,\Omega_2}+|u|_{1+\gamma,\Omega}).$$

Combining (6.1)-(6.4), we arrive the required estimate.

7. Local Error Estimates up to the Boundary in a Discrete L^2-Norm

We shall consider domains $\Omega_0 < \Omega_1$ and the set

$$\Omega_0^* = \Omega_0 \cup (\partial\Omega_0 \cap \partial\Omega).$$

For $F = \{F(Z)\}_{Z\in N_h\cap\Omega_0^*}$ define the discrete norm (cf. Section 3)

$$|\!|\!|F|\!|\!|_{q,\Omega_0^*} = h^{2/q}\Big(\sum_{Z\in N_h\cap\Omega_0^*} \|F(Z)\|^q\Big)^{1/q}, \quad q \ge 1.$$

Any such vector can be decomposed as follows

$$F = F^b + F^{int}, 7.1$$

where

$$F^b(Z) = \{\, 0 \text{ for } Z \in \Omega_0 \qquad F(Z) \text{for } Z \in \partial\Omega_0 \cap \partial\Omega\ , \quad F^{int}(Z) = \{\, F\,(Z) \text{for } Z \in \Omega_0 \qquad 0 \text{for } Z \in \partial\Omega_0 \cap \partial\Omega\ .$$

Lemma 7.1. *Let $d = 2$, $\Omega_0 < \Omega_1 \subset \Omega^*$, $q \in (2,\infty)$, $v \in W_q^3(\Omega_1)$, let the family $\mathcal{F}$ satisfy* (A1) *in* Ω^*, (A2) *and let* (5.1) *hold. Then*

$$|\!|\!| \text{ grad } v - G_h\pi_h v|\!|\!|_{2,\Omega_0^*} \le C(\Omega_0)h^{3/2}\|v\|_{3,q,\Omega_1}$$

holds for sufficiently small h.

Proof. If $Z \in N_h \cap \partial\Omega_0 \cap \partial\Omega$, we make use of [6, Lemma 3.6] and (5.2) to obtain

$$\| \text{ grad } v(Z)-G_h\pi_h v(Z)\| \le \| \text{ grad } v(Z)-G_h v(Z)\|+\|G_h v(Z)-G_h\pi_h v(Z)\| \le \ \le \tilde{C}h(|v|_{2,\infty,U(Z)}+\|v\|_{3,q,U(Z)}) \le \tilde{C}h$$

Then

$$|\!|\!|(\text{ grad } v-G_h\pi_h v)^b|\!|\!|_{2,\Omega_0^*} \le Ch\Big(\sum_{Z\in N_h\cap\partial\Omega\cap\partial\Omega_0} h^2\|v\|^2_{3,q,\Omega_1}\Big)^{1/2} \le 7.2 \le Ch(Ch^{-1})^{1/2}h\|v\|_{3,q,\Omega_1} = C'h^{3/2}\|v\|_{3,q,\Omega_1},$$

since the number of nodes in $\partial\Omega_0 \cap \partial\Omega$ is bounded by Ch^{-1}.

If $Z \in N_h \cap \Omega_0$, we argue as in the proof of Lemma 3.2 using [6, Lemmas 3.6 and 5.2], to get

$$|\!|\!|(\text{ grad } v - G_h\pi_h v)^{int}|\!|\!|_{2,\Omega_0^*} \le Ch^2\|v\|_{3,q,\Omega_1}.7.3$$

By (7.1), the triangle inequality, (7.2) and (7.3), we deduce the estimate, as required.

Lemma 7.2. *Assume that* $d = 2$, $\Omega_0 < \Omega_2 < \Omega^*$, $q \in (2, \infty)$, $u \in W_2^3(\Omega_2)$, $a_{ij} \in W_\infty^1(\Omega_2)$, *the Dirichlet problem* (1.1) *is regular and the family* $\mathcal{F}$ *satisfies* (A1) *in* Ω^* *and* (A2). *Then*

$$\|G_h(u_h - \pi_h u)\|_{2,\Omega_0^*} \leq Ch^{2\gamma}(\|u\|_{3,\Omega_2} + |u|_{1+\gamma,\Omega})$$

holds for sufficiently small h.

Proof. Arguing as in the proof of Lemma 3.3 (cf. (3.3)), we derive the estimate

$$\|G_h v_h\|_{2,\Omega_0^*} \leq C|v_h|_{1,\Omega_1}$$

for sufficiently small h, where Ω_1 is any subdomain such that $\Omega_0 < \Omega_1 < \Omega_2$.

Substituting $v_h = u_h - \pi_h u$ and using Lemmas 5.5 and 2.4, we obtain

$$\|G_h v_h\|_{2,\Omega_0^*} \leq C|u_h - \pi_h u|_{1,\Omega_1} \leq C(h^2\|u\|_{3,\Omega_2} + h^{2\gamma}|u|_{1+\gamma,\Omega}) \leq \leq Ch^{2\gamma}(\|u\|_{3,\Omega_2} + |u|_{1+\gamma,\Omega}).$$

Theorem 7.3. *Assume that* $d = 2$, $\Omega_0 < \Omega_1 < \Omega_2 < \Omega^*$, $q \in (2, \infty)$, *the solution* $u|_{\Omega_2} \in W_2^3(\Omega_2)$, $u|_{\Omega_1} \in W_q^3(\Omega_1)$, $a_{ij} \in W_\infty^1(\Omega)$, *the family* $\mathcal{F}$ *satisfies* (A1) *in* Ω^*, (A2) *and the Dirichlet problem* (1.1) *is* γ*-regular. Moreover, let* (5.1) *be fulfilled. Then*

$$\|\operatorname{grad} u - G_h u_h\|_{2,\Omega_0^*} \leq Ch^\beta(\|u\|_{3,\Omega_2} + \|u\|_{3,q,\Omega_1} + |u|_{1+\gamma,\Omega})$$

holds for sufficiently small h, *where* $\beta = \min\{\frac{3}{2}, 2\gamma\}$.

The proof follows immediately from the triangle inequality, Lemmas 7.1 and 7.2.

Finally let us consider some particular cases, enabling us to remove the loss of accuracy, as in Section 6.

Lemma 7.4. (i) *If* $d = 2$ *and the assumptions of Lemma* 6.1 *are satisfied then for sufficiently small* h

$$\|\partial_i v - (G_h \pi_h v)_i\|_{q,\Omega_0^*} \leq Ch^2|v|_{3,q,\Omega_1}.$$

(ii) *If the assumptions of Lemma* 6.2 *are satisfied then for sufficiently small* h

$$\|\partial_i v - (G_h \pi_h v)_i\|_{q,\Omega_0^*} \leq Ch^2\|v\|_{3,q,\Omega_1}.$$

The proof follows from Lemmas 6.1 and 6.2, respectively.

Theorem 7.5. *Let the assumptions of Theorem* 6.4 *be fulfilled. Then*

$$\|\partial_i u - (G_h u_h)_i\|_{2,\Omega_0^*} \leq Ch^{2\gamma}(\|u\|_{3,\Omega_2} + \|u\|_{3,q,\Omega_1} + |u|_{1+\gamma,\Omega})$$

holds for sufficiently small h.

The proof follows from the triangle inequality, Lemma 7.4 (ii) and Lemma 7.2.

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