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A MULTIGRID METHOD FOR NONLINEAR PARABOLIC PROBLEMS*1)

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Abstract

The multigrid algorithm in [13] is developed for solving nonlinear parabolic equations arising from the finite element discretization. The computational cost of the algorithm is approximate $O(N_kN)$ where N_k is the dimension of the finite element space and N is the number of time steps.

1. Introduction

The finite element methods for solving nonlinear parabolic problems are studied by many authors, such as Douglas and Dupont[5], Wheeler[4], Luskin[3], etc. They proposed various ways of computing the problems and proved the optimal order convergence rates of the methods, such as the linearized methods, the predictor-corrector methods, the extrapolation methods, the alternating direction methods and the iterative methods[2], etc. The multigrid methods for solving parabolic problems are studied by some authors, such as Hachbusch[14,15], Bank and Dupont[12], Brandt and Greenwald[16] as well as Yu[13]. But these methods are given mainly for linear parabolic equations. For nonlinear parabolic problems Hachbusch and Brandt in [14], [15], [16] gave the multigrid methods by using the integral differential equation and the frozen-τ technique.

In this paper we present a multigrid procedure for two-dimension nonlinear parabolic problems. The method is an extension of our earlier algorithm in [13] for linear parabolic problems. The iterative methods for solving the system of nonlinear algebraic equations are avoided because the unknown function $U_k^{n+\theta}$ in the nonlinear coefficient $a(x, U_k^{n+\theta})$ and the right term $f(x,t,U_k^{n+\theta})$ in the system of nonlinear algebraic equations is replaced by $I_kU_{k-1}^{n+\theta}$ in the multigrid procedure, where I_k denotes an intergrid transfer operator, θ a weighted function and $U_{k-1}^{n+\theta}$ the solutions of the equation in the (k-1)th level. We analyze the convergence of our algorithm and the computational cost of N

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time steps. The asymptotically computational cost is $O(NN_k)$ where N_k is the dimension of the discrete finite element space and N is the number of time steps. In addition, the methods can be applied to more general nonlinear parabolic problems.

The paper is organized as follows. In Section 2, we give the basic assumptions and properties by using of the finite element discretizing a nonlinear parabolic equation. In Section 3 we extend the time-dependent fully multigrid algorithm in [13] to the nonlinear parabolic equation. In Section 4 we analyze the convergence of the algorithm and in Section 5 we consider the computational cost and the development.

2. Notations and Preliminaries

We consider nonlinear parabolic initial value problems as follows:

$$\{ \ \partial u \partial t = \nabla(a(x,u)\nabla u)+f(x,t,u), \ (x,t) \in \Omega \times [0,T], u(x,t) = 0, \ (x,t) \in \partial\Omega \times [0,T], u(x,0) = u_0(x), \ x \in \Omega, 2.1$$

where $\Omega \subset R^2$ is a convex polygonal domain, ∇ is a gradient operator on $x = (x_1, x_2)$ directions. Assume that the nonlinear coefficient $a(x,p)$ satisfies the condition: there are constants $K_0, \ K_1 > 0$ such that

$$0 < K_0 \le a(x,u) \le K_1, \ \forall (x,p) \in \bar{\Omega} \times R^1 .2.2$$

$a(x,p)$ and $f(x,t,p)$ hold uniformly Lipschitz condition with respect to p, i.e., there is a constant $L > 0$ such that

$$|a(x,p_1){-}a(x,p_2)| \le L|p_1{-}p_2|, \ \forall (x,p) \in \bar{\Omega}{\times}R^1, |f(x,t,p_1){-}f(x,t,p_2)| \le L|p_1{-}p_2|, \ \forall (x,t,p) \in \bar{\Omega}{\times}[0,T]{\times}R^1 .2.3$$

Further assume that for any $t \in [0,T], \ f(x,t,0) \in L^2(\Omega)$. Thus by (2.3), we have

$$|f(x,t,v(x,t))| \le |f(x,t,0)| + L|v(x,t)| \in L^2(\Omega), \ \forall v(x,t) \in L^2(\Omega).$$

The variational form of problem (2.1) is : Find a continuously differentiable mapping $u(t) = u(x,t) : [0,T] \to H_0^1(\Omega)$ such that

$$\{ (\ \partial u \partial t, v) + a(u;u,v) = (f(u),v), (u(x,0),v) = (u_0(x),v), \ \forall v \in H_0^1(\Omega) .2.4$$

where $a(u;u,v) = \int_\Omega a(x,u)\nabla u \nabla v dx, \ (f(u),v) = \int_\Omega f(x,t,u) v dx.$

Under the assumptions (2.3) and (2.4), a solution of the variational problem (2.4) such that $\|\nabla u\|_{L^\infty(L^\infty)} < +\infty$, if it exists, must be unique where $\|\nabla u\|_{L^\infty(L^\infty)}$ is defined by

$$\|\nabla u\|_{L^\infty(L^\infty)} = \|\|\nabla u\|_{L^\infty(\Omega)}\|_{L^\infty[0,T]}.$$

In the following we assume that a solution of the problem (2.4) exists and is unique. And the solution is smooth enough for the finite element analysis.

Let Γ be a mesh partition of the domain Ω (the triangulation or quadrilateral partition) which satisfies the partition quasi-uniformity conditions [17]. Since Ω is a convex polygonal domain, we can make the partition satisfy that $\Omega = \cup_{\tau\in\Gamma}\tau$. Let $\mathcal{M} \subset H_0^1(\Omega)$ be the finite element space of the piecewise linear interpolation or the quadratic interpolation corresponding to the mesh partition. Then the inverse inequality in $\mathcal{M}$ holds, i.e., there exists a constant $c_0 > 0$ such that

$$\|\varphi\|_{H_0^1} \le c_0 h^{-1}\|\varphi\|_{L^2}, \forall\varphi \in \mathcal{M}, 2.5$$

where h denotes the maximum value in the element edge sizes of the mesh partition Γ of the domain Ω.

Let Π be an interpolation operator from $H_0^1 \cap H^2(\Omega)$ onto $\mathcal{M}$. Then Π satisfies the approximation property: for $\forall u \in H^2(\Omega)$,

$$\|u - \Pi u\|_{L^2} + h\|u - \Pi u\|_{H^1} \le ch^2\|u\|_{H^2}, 2.6$$

where $H^p(\Omega)$ denotes the Sobolev space of p order whose norm is defined by $\|\varphi\|_{H^p}$. $p = 0,\ H^p = L^2(\Omega)$.

Let $\triangle t > 0$ be a time step size, $t_n = n\triangle t$, $\bar{J} = \{0, 1, 2, \cdots N\}$, $N = [\frac{T}{\triangle t}]$. Assume that the solution u of (2.4) be smooth enough with respect to t so that the differential quotient $\frac{\partial u}{\partial t}$ may be replaced by the difference quotient. Set $t_{n+\theta} = \frac{1}{2}(1+\theta)t_{n+1} + \frac{1}{2}(1-\theta)t_n$, $U^n = U(x, t_n)$, $U^{n+\theta} = \frac{1}{2}(1+\theta)U^{n+1} + \frac{1}{2}(1-\theta)U^n$, $f(U^{n+\theta}) = f(x, t_{n+\theta}, U^{n+\theta})$, $\theta \in [0, 1]$. Then we have the finite element method for solving the variational problem (2.4): Find $\{U^j\}_{j=1}^N : \bar{J} \to \mathcal{M}_k$ such that

$$\{ \quad (U^{n+1} - U^n\triangle t, v) + a(U^{n+\theta}; U^{n+\theta}, v) = (f(U^{n+\theta}), v),\ \forall v \in \mathcal{M}_k, (u(x,0), v) = (u_0(x), v). 2.7$$

(2.7) is the Crank-Nicolson scheme when $\theta = 0$. (2.7) is the fully implicit scheme when $\theta = 1$. Obviously, (2.7) for any $\theta \in [0, 1]$ is a system of nonlinear algebraic equations at each time step $t_j = j\triangle t$. By using of the Brower's fixed point theorem, we can prove that a solution of (2.7) exists. By using of the prior error estimate of the approximate solution, we can prove that the solution of (2.7) is unique[15].

3. Time-Dependent Fully Multigrid Method

We now give the mesh partitions of the domain Ω (the triangulation or quadrilateral partition) level after level. Let Γ_1 be an initial mesh partition of the domain Ω which satisfies the quasi-uniformity conditions and $\Omega = \cup_{\tau\in\Gamma_1}\tau$. And Γ_k $(k \ge 1)$ is a partition obtained by connecting the midpoints of edges of elements in Γ_{k-1}. Then Γ_k satisfies the quasi-uniformity condition, $\Omega = \cup_{\tau\in\Gamma_k}\tau$ and $h_k = \frac{1}{2}h_{k-1}$ where $h_k = \max_{\tau\in\Gamma_k} h_\tau$.

Let $\mathcal{M}_k (k \ge 1)$ be a finite element space of the piecewise linear interpolation or the quadratic interpolation associated with the partitions Γ_k $(k \ge 1)$. Then $\mathcal{M}_{k-1} \subset \mathcal{M}_k \subset H_0^1(\Omega)$.

Let I_k be an intergrid transfer operator, $I_k : \mathcal{M}_{k-1} \to \mathcal{M}_k$. I_k is defined as the piecewise linear interpolation or the average of values of the neighboring nodal points.

Since $\mathcal{M}_{k-1} \subset \mathcal{M}_k$, I_k is a natural inclusion operator, i.e., $I_k v = v$ for $\forall v \in \mathcal{M}_{k-1}$. Let I_k^t be the conjugate operator of I_k or the restriction operator, $I_k^t : \mathcal{M}_k \to \mathcal{M}_{k-1}$, which satisfies

$$(I_k^t u_k, v_{k-1}) = (u_k, I_k v_{k-1}), \quad \forall u_k \in \mathcal{M}_k,\ v_{k-1} \in \mathcal{M}_{k-1}.3.1$$

By the nested property of the finite element space, there exists a matrix $B_k = [b_{ij}]_{N_{k-1}\times N_k}$ represented by the basis functions of the space $\mathcal{M}_{k-1}$ under the basis function of the space $\mathcal{M}_k$ such that $I_k = B_k^T$, $I_k^t = B_k$[13].

The intergrid transfer operator in the above definition has the properties as follows:

$$i) \qquad |I_k v\|_{L^2} = \|v\|_{L^2}, \forall v \in \mathcal{M}_{k-1}, ii) \qquad |\nabla(I_k v)\|_{L^2} \le \|\nabla v\|_{L^2},\ \forall v \in \mathcal{M}_{k-1}, 3.2$$

where (3.2) holds by the definition of I_k.

The multigrid method for solving the system of nonlinear algebraic equations (2.7) first makes the nonlinear terms in (2.7) linearization, i.e., $a(x,\ U^{n+\theta})$ is replaced by $a(x, I_k U_{k-1}^{n+\theta})$ and $f(x,t,U^{n+\theta})$ is replaced by $f(x,t,\ I_k U_{k-1}^{n+\theta})$. If the solutions U_{k-1}^{n+1} and U_{k-1}^n on the (k-1)th level as well as U_k^n on the k'th level are known, then we obtain a system of linearized algebraic equations:

$$\{ \quad (U_k^{n+1} - U_k^n \triangle t, v) + a(I_k U_{k-1}^{n+\theta}; U_k^{n+\theta}, v) = (f(I_k U_{k-1}^{n+\theta}), v), (u(x,0), v) = (u_0(x), v),\ \forall v \in \mathcal{M}_k.3.3$$

By (2.2)and (2.3) assumptions, we can prove that a solution of (3.3) exists. In Section 4, we will prove that the solution is unique. And the error order is $O(\triangle t + h_k^2)$ when $\theta \ne 0$ and $O(\triangle t^2 + h_k^2)$ when $\theta = 0$.

Let $\{\psi_i^k\}_{i=1}^{N_k}$ and $\{\psi_i^{k-1}\}_{i=1}^{N_{k-1}}$ be the basis functions of $\mathcal{M}_k$ and $\mathcal{M}_{k-1}$, respectively. Then $U_{k-1}^{n+1} = \sum_{i=1}^{N_{k-1}} \alpha_i^{n+1}\psi_i^{k-1}$ and $U_k^{n+1} = \sum_{i=1}^{N_k} \alpha_i^{n+1}\psi_i^k$. By the definition of I_k, we know that $I_k U_{k-1}^{n+1} = \sum_{i=1}^{N_k} \beta_i^{n+1}\psi_i^k$ where

$$\beta_k^{n+1} = \{\beta_1^{n+1}, \beta_2^{n+1}, \cdots, \beta_{N_k}^{n+1}\}^T = B_k^T \alpha_{k-1}^{n+1}, \alpha_{k-1}^{n+1} = \{\alpha_1^{n+1}, \alpha_2^{n+1}, \cdots, \alpha_{N_{k-1}}^{n+1}\}^T.$$

Set

$$a)\ C_k = [(\psi_i^k, \psi_j^k)]_{N_k\times N_k}, b) A_k^n(\alpha) = [(a(I_k(\sum_{i=1}^{N_{k-1}} \alpha_i^{n+1}\psi_i^{k-1}))\nabla\psi_i^k, \nabla\psi_j^k)]_{N_k\times N_k} \qquad = [a(\sum_{i=1}^{N_k} \beta_i^{n+1}\psi_i^k; \psi_i^k, \psi_i^k)]_{N_k}$$

then (3.3) can be written in the vector-matrix form as:

$$(C_k + 12(1+\theta)\triangle t A_k^n(\alpha))\alpha_k^{n+1} = \triangle t F_k^n(\alpha) + C_k\alpha_k^n - 12(1-\theta)\triangle t A_k^n(\alpha)\alpha_k^n.3.5$$

In the following we will give the time-dependent k'th level algorithm for solving the system of linear algebraic equations (3.5). Assume that the solutions U_{k-1}^{n+1} and U_{k-1}^n on the (k-1)th level and U_k^n on the k'th level are known. Then an initial approximate value of the solution at (n+1)th step time on the k'th level is taken as:

$$U_{k,0}^{n+1} = U_k^n + I_k(U_{k-1}^{n+1} - U_{k-1}^n)\ (\alpha_{k,0}^{n+1} = \alpha_k^n + B_k^T(\alpha_{k-1}^{n+1} - \alpha_{k-1}^n)).3.6$$

1) Pre-smoothing: performing ν_1 time smoothing iterations on the k level:

$$U_{k,\nu_1}^{n+1} = S_k^{\nu_1} U_{k,0}^{n+1} \ (\alpha_{k,\nu_1}^{n+1} = S_k^{\nu_1} \alpha_{k,0}^{n+1})3.7$$

where S_k is a smoothing iterative operator, such as the Jacobi iteration, the Gauss-Seidel iteration and the preconditioned conjugate gradient iteration. The iterative methods are discussed later.

2) Coarse grid correction: the coarse grid equation is that $\forall v \in \mathcal{M}_{k-1}$,

$$(\hat{U}_{k-1}^{n+1} - U_{k-1}^{n} \triangle t, v) + a(U_{k-1}^{n+\theta}; \hat{U}_{k-1}^{n+\theta}, v) = (f(U_{k-1}^{n+\theta}), v) + [(f(I_k U_{k-1}^{n+\theta}), I_k v) - (U_{k,\nu_1}^{n+1} - U_k^n \triangle t, I_k v) - a(I_k U_{k-1}^{n+1}; 12(1+$$

where $\hat{U}_{k-1}^{n+\theta} = 12(1+\theta)\hat{U}_{k-1}^{n+1} + 12(1-\theta)U_{k-1}^{n}$. (3.8) is written in the vector-matrix form as:

$$(C_{k-1} + 12(1+\theta)\triangle t \hat{A}_{k-1}^{n}(\alpha))\hat{\alpha}_{k-1}^{n+1} = \triangle t \hat{F}_{k-1}^{n}(\alpha) + C_{k-1}\alpha_{k-1}^{n} - 12(1-\theta)\triangle t \hat{A}_{k-1}^{n}(\alpha)\alpha_{k-1}^{n} + B_k^T[\triangle t F_k^n(\alpha) + C_k(\alpha_{k,\nu_1}^{n+1} - \alpha_k^n$$

where

$$\hat{U}_{k-1}^{n+1} = \sum_{i=1}^{N_{k-1}} \hat{\alpha}_i^{n+1}\psi_i^{k-1}, \hat{A}_{k-1}^{n}(\alpha) = [a(\sum_{i=1}^{N_{k-1}} \alpha_i^{n+\theta}\psi_i^{k-1}; \psi_i^{k-1}, \psi_j^{k-1})], \hat{F}_{k-1}^{n}(\alpha) = \{(f(\sum_{i=1}^{N_{k-1}} \alpha_i^{n+\theta}\psi_i^{k-1}), \psi_i^{k-1})\}_{N_{k-1}}^{T}.$$

Let $\hat{U}_{k-1,p}^{n+1}$ be a solution of (3.8) obtained by using p time iterations and $\hat{U}_{k-1,0}^{n+1} = U_{k-1}^{n+1}$ as the initial approximate value. Then the corrective value U_{k,ν_1+1}^{n+1} of the iterative solution of (3.7) on the (k-1)th level is defined as

$$U_{k,\nu_1+1}^{n+1} = U_{k,\nu_1}^{n+1} + I_k(\hat{U}_{k-1,p}^{n+1} - U_{k-1}^{n+1})(\alpha_{k,\nu_1+1}^{n+1} = \alpha_{k,\nu_1}^{n+1} + B_k^T(\hat{\alpha}_{k-1,p}^{n+1} - \alpha_{k-1}^{n+1})).3.9$$

3) Post-smoothing: performing ν_2 time smoothing iterations on the k'th level:

$$U_{k,\nu_1+\nu_2+1}^{n+1} = S_k^{\nu_2} U_{k,\nu_1+1}^{n+1} \ (\alpha_{k,\nu_1+\nu_2+1}^{n+1} = S_k^{\nu_2} \alpha_{k,\nu_1+1}^{n+1}).3.10$$

Thus we obtain a approximate solution value of the equation (3.3) at (n+1)th step time on the k level as follows:

$$U_k^{n+1} = U_{k,\nu_1+\nu_2+1}^{n+1} \ (\alpha_k^{n+1} = \alpha_{k,\nu_1+\nu_2+1}^{n+1}).$$

The multigrid scheme is defined as a recursive process for the level k. If we carry out the multigrid operation for each time step n, we get a time-dependent fully multigrid method. Obviously, the above multigrid procedure for solving the nonlinear parabolic equation (2.1) can be extended to the circumstances of the variable time step size.

We now consider to determine the initial approximate values of solutions in the above multigrid procedure. Because the k'th level algorithm depends on the solution values U_{k-1}^{n+1}, U_{k-1}^{n} and U_k^n, therefore the fully multigrid iterative procedure depends on the solution values U_k^0 for $k = 1, 2, \cdots$ and U_1^n for $n = 1, 2, \cdots, N$.

The approximate solutions $U_k^0 (k = 1, 2, \cdots)$ are determined by the following scheme. $U_1^0 = \bar{U}_1^0$ is obtained by exactly solving the following equation (3.11). U_k^0 for $k > 1$

is obtained by using $I_kU_{k-1}^0$ as an initial approximate value to carry out the multigrid iterations for the equation (3.11). The exact solution $\bar{U}_k^0 (k = 1, 2, \cdots)$ satisfies the equation:

$$(\bar{U}_k^0, v) + a(u_0; \bar{U}_k^0, v) = (f(u_0(x)), v), \ \forall v \in \mathcal{M}_k.3.11$$

(3.11) is written in the vector-matrix form as

$$(C_k + A_k(\alpha))\alpha_k^0 = F_k 3.12$$

where C_k definition is same as the above. $A_k(\alpha) = [a(u_0(x); \psi_i^k, \psi_j^k)]$ and $F_k = \{(f(u_0(x)), \psi_j^k)\}_{N_k}^T$.

Note that (3.11) or (3.12) is a discrete elliptic equation, therefore the convergence of the multigrid algorithm can be found in the Bank and Dupont [12].

The solution values U_1^n $(n = 1, 2, \cdots, N)$ according to the different θ values will be considered in the following two situations in order to preserve the accuracy of values of the approximate solutions.

1) When $\theta \neq 0$, U_1^{n+1} is obtained by solving the following linear equation:

$$(U_1^{n+1} - U_1^n \triangle t, v) + a(U_1^n; U_1^{n+\theta}, v) = (f(U_1^n), v), \ \forall v \in \mathcal{M}_1, 3.13$$

for $n = 0, 1, 2, \cdots, N-1$.

2) When $\theta = 0$, U_1^1 is obtained by applying the predictor and twice corrector methods. Let U_1^* be a solution of the following predictor equation,

$$(U_1^* - U_1^0 \triangle t, v) + a(U_1^0; (U_1^* + U_1^0)/2, v) = (f(U_1^0), v), \ \forall v \in \mathcal{M}_1.3.14$$

Set $U_1^{*\frac{1}{2}} = (U_1^* + U_1^0)/2$. Let U_1^{**} be a solution of the following corrector equation,

$$(U_1^{**} - U_1^0 \triangle t, v) + a(U_1^{*\frac{1}{2}}; (U_1^{**} + U_1^0)/2, v) = (f(U_1^{*\frac{1}{2}}), v), \ \forall v \in \mathcal{M}_1.3.15$$

Set $U_1^{**\frac{1}{2}} = (U_1^{**} + U_1^0)/2$. Then U_1^1 is obtained by the equation:

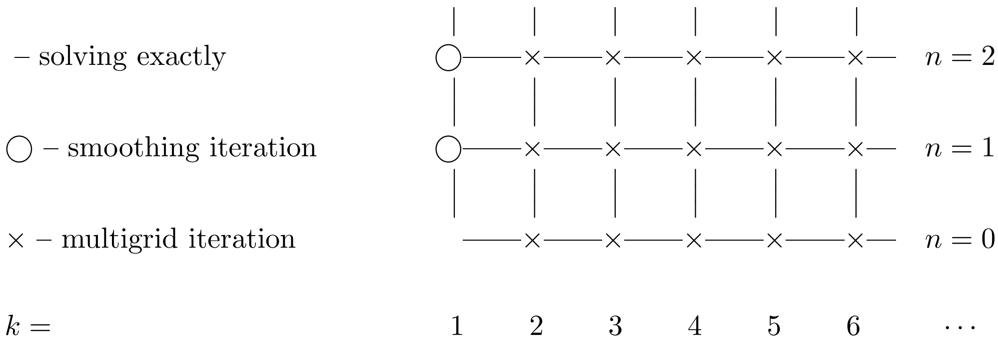
$$(U_1^1 - U_1^0 \triangle t, v) + a(U_1^{**\frac{1}{2}}; U_1^{\frac{1}{2}}, v) = (f(U_1^{**\frac{1}{2}}), v), \ \forall v \in \mathcal{M}_1.3.16$$

The solution $U_1^{n+1} (n = 1, 2, \cdots, N-1)$ is obtained by applying the modified Crank-Nicolson method.

$$(U_1^{n+1} - U_1^n \triangle t, v) + a(EU_1^n; U_1^{n+\frac{1}{2}}, v) = (f(EU_1^n), v), \ \forall v \in \mathcal{M}_1, 3.17$$

where $EU_1^n = \frac{3}{2}U_1^n - \frac{1}{2}U_1^{n-1}$.

The above exact solutions of the equations (3.13)-(3.17) can by replaced by the approximate solutions, which are obtained by the smoothing iterative method defined in (3.7). The convergence, see [1], [2]. Therefore, the multigrid scheme is described by the diagram as:



Now we will give some considerations for the smoothing iterative scheme (3.7).

Set $\tilde{A}_k^n(\alpha) = C_k + \frac{1}{2}(1+\theta)A_k^n(\alpha) = \tilde{D}_k^n(\alpha) - \tilde{L}_k^n(\alpha) - \tilde{U}_k^n(\alpha)$ where $\tilde{D}_k^n(\alpha) = diag(\tilde{A}_k^n(\alpha))$, $\tilde{L}_k^n(\alpha)$ and $\tilde{U}_k^n(\alpha)$ are the strictly upper triangle and lower triangle matrix, respectively. $\tilde{F}_k^n(\alpha) = \triangle t F_k^n(\alpha) + C_k\alpha_k^n - \frac{1}{2}(1-\theta)\triangle t A_k^n(\alpha)\alpha_k^n$. Then the equation (3.4) can be written in the form as:

$$\tilde{A}_k^n(\alpha)\alpha_k^{n+1} = \tilde{F}_k^n(\alpha).3.18$$

Set $\tilde{A}_k(\alpha) = C_k + A_k(\alpha)$ where $A_k(\alpha) = [(\nabla\psi_i^k, \nabla\psi_i^k)]$. Then $\tilde{A}_k(\alpha)$ and $A_k(\alpha)$ are independent of time t. The smoothing iteration (3.7) can be chosen as:

1) The Jacobi iterative method: for $i = 1, 2, \cdots, \nu$

$$\alpha_{k,i}^{n+1} = \tilde{D}_k^{n\ -1}(\alpha)(\tilde{L}_k^n(\alpha)+\tilde{U}_k^n(\alpha))\alpha_{k,i-1}^{n+1}+\tilde{D}_k^{n\ -1}(\alpha)\tilde{F}_k^n(\alpha) = (I-\tilde{D}_k^{n\ -1}(\alpha)\tilde{A}_k^n(\alpha))\alpha_{k,i-1}^{n+1}+\tilde{D}_k^{n\ -1}(\alpha)\tilde{F}_k^n(\alpha).3.19$$

Usually, we do not use (3.19) to perform the smoothing iterative computation in the multigrid scheme. We use the modified form of (3.19). Let $0 < \lambda_1 \le \lambda_2 \le \cdots \le \Lambda_{N_k}$ be the eigenvalues of $A_k(\alpha)$ and $\{\chi_i\}_{i=1}^{N_k} \subset \mathcal{M}_k$ be orthogonal eigenfunctions of $A_k(\alpha)$. By the assumption (2.2), we have

$$\lambda_i K_0(\chi_i, \chi_i) = K_0(A_k(\alpha)\chi_i, \chi_i) \le (A_k^n(\alpha)\chi_i, \chi_i) \le K_1(A_k(\alpha)\chi_i, \chi_i) = K_1\lambda_i(\chi_i, \chi_i).$$

The modified Jacobi iterative method is that

$$\alpha_{k,i}^{n+1} = (I - 11 + \frac{1}{2}(1+\theta)\triangle t\Lambda_{N_k}K_1\tilde{A}_k^n(\alpha))\alpha_{k,i-1}^{n+1} + 11 + \frac{1}{2}(1+\theta)\triangle t\Lambda_{N_k}K_1\tilde{F}_k^n(\alpha).3.20$$

The smoothing iterative matrix is that $S_k = I - 11 + \frac{1}{2}(1+\theta)\triangle t\Lambda_{N_k}K_1\tilde{A}_k^n(\alpha)$ and the convergence radius satisfies that

$$\rho(S_k) \le 1 - 1 + \frac{1}{2}(1+\theta)\triangle t\Lambda_{N_k}K_0 1 + \frac{1}{2}(1+\theta)\triangle t\Lambda_{N_k}K_1 = (1-K_0K_1)(1+1\frac{1}{2}(1+\theta)\triangle t\Lambda_{N_k}K_1)^{-1}.$$

Note that $\Lambda_{N_k} \le ch_k^{-2}$. Hence if we choose $\triangle t \sim O(h_k^2)$, the convergence radius of the modified Jacobi iterative method has

$$\rho(S_k^\nu) \le (1 - K_0K_1).3.21$$

2) The Gauss-Seidel iterative method: for $i = 1, 2, \cdots, \nu$

$$\alpha_{k,i}^{n+1} = \alpha_{k,i}^{n+1} + (I - \tilde{L}_k^n(\alpha))^{-1} \tilde{D}_k^{n\ -1}(\alpha)(\tilde{F}_k^n(\alpha) - \tilde{A}_k^n(\alpha)\alpha_{k,i-1}^{n+1}) = (I - (I - \tilde{L}_k^n(\alpha))^{-1} \tilde{D}_k^{n\ -1}(\alpha)\tilde{A}_k^n(\alpha))\alpha_{k,i-1}^{n+1} \quad + (I - \tilde{L}_k^n(

Let u be the solution of (2.1) which satisfies

$$u \in L^\infty(H^3),\ \partial u \partial t \in L^2(H^1) \cap L^\infty(H^2), \partial^2 u \partial t^2 \in L^\infty(H^1),\ \partial^3 u \partial t^3 \in L^2(L^2) \cap L^1(H^1).4.1$$

Then under the assumptive conditions (2.2) and (2.3), the finite element solution of (2.7) has the following error estimation of the convergence[3−5].

Lemma 1. *Let u be the solution of* (2.4). $\tilde{U}_k^n (n \geq 1)$ *and* $\tilde{U}_k^0$ *are the solutions* (2.7) *and* (3.12), *respectively. Then for* $\theta \in [0,1]$, *there are the constants* $c^*, \tau_0 > 0$ *independent of* $h_k, \{\tilde{U}_k^n\}$ *and* $\triangle t$ *such that* $\triangle t \leq \tau_0$, *we have*

$$\|u(t_n) - \tilde{U}_k^n\|_{L^2} + h_k \|u(t_n) - \tilde{U}_k^n\|_{H_0^1} \leq \{ \quad c^*(h_k^2 + \triangle t^2),\ \theta = 0, c^*(h_k^2 + \triangle t),\ \theta \neq 0.4.2$$

In the following we will prove that the finite element solution of the discrete equation (3.3) still has the error estimation (4.2).

Lemma 2. *Assume that we have obtained the finite element solutions* $\bar{U}_{k-1}^{n+1}$, $\bar{U}_{k-1}^n$ *on the k-1 level and* $\bar{U}_k^n$ *on the k level. And* $\bar{U}_k^{n+1}$ *is the finite element solution of* (3.3). $\tilde{U}_k^{n+1}$ *is the finite element solution of* (2.7) *on the k level. Then for* $\theta \in [0,1]$, $\triangle t \sim O(h_k^2)$, *there are the constants* $c^*, \tau_0 > 0$ *independent of* $h_k, \{\tilde{U}_k^n\}, \{\bar{U}_k^n\}$ *and* $\triangle t$ *such that* $\triangle t \leq \tau_0$, *we have*

$$\|\tilde{U}_k^n - \bar{U}_k^n\|_{L^2} + h_k \|\tilde{U}_k^n - \bar{U}_k^n\|_{H_0^1} \leq \{ \quad c^*(h_k^2 + \triangle t^2),\ \theta = 0, c^*(h_k^2 + \triangle t),\ \theta \neq 0.4.3$$

Proof. Set $\xi_k^n = \tilde{U}_k^n - \bar{U}_k^n$. Then (2.7) subtracting (3.3), we have the equality:

$$(\xi_k^{n+1} - \xi_k^n \triangle t, v) + a(I_k \bar{U}_{k-1}^{n+\theta}, \xi_k^{n+\theta}, v) = a(I_k \bar{U}_{k-1}^{n+\theta} - \tilde{U}_k^{n+\theta}, \tilde{U}_k^{n+\theta}, v) + (f(\tilde{U}_k^{n+\theta}) - f(I_k \bar{U}_k^{n+\theta}), v),\ \forall v \in \mathcal{M}_k.4.4$$

Since

$$12\triangle t(\xi_k^{n+1} - \xi_k^n, \xi_k^{n+1} + \xi_k^n) = 1\triangle t(\xi_k^{n+1} - \xi_k^n, \xi_k^{n+\theta}) - \theta 2\triangle t(\xi_k^{n+1} - \xi_k^n, \xi_k^{n+1} - \xi_k^n),$$

hence by assumptions (2.2), (2.3) and (4.1), taking $v = \xi_k^{n+\theta}$ in (4.4), we obtain

$$12\triangle t(\|\xi_k^{n+1}\|_{L^2}^2 - \|\xi_k^n\|_{L^2}^2) + K_0 \|\nabla \xi_k^{n+\theta}\|_{L^2} = 12\triangle t(\xi_k^{n+1} - \xi_k^n, \xi_k^{n+1} + \xi_k^n) + K_0 \|\nabla \xi_k^{n+\theta}\|_{L^2} \leq 1\triangle t(\xi_k^{n+1} - \xi_k^n, \xi_k^{n+\theta}) + a(I_k$$

where the function u is the solution of (2.4) and the constant c depends on L and $\|\nabla u\|_{L^\infty(L^\infty)}$. Taking $\epsilon = \frac{K_0}{c+K_1}$ and adding $\frac{K_0}{2}\|\xi_k^{n+\theta}\|_{L^2}^2$ in two sides of the inequality (4.5), we have

$$1\triangle t(\|\xi_k^{n+1}\|_{L^2}^2 - \|\xi_k^n\|_{L^2}^2) + K_0 \|\xi_k^{n+\theta}\|_{H_0^1} \leq c\{\|\xi_k^{n+\theta}\|_{L^2}^2 + \|u - \tilde{U}_k^{n+\theta}\|_{L^2}^2 + \|I_k \bar{U}_{k-1}^{n+\theta} - \tilde{U}_k^{n+\theta}\|_{L^2}^2\}$$

where the constant c depends on K_0, K_1, L and $\|\nabla u\|_{L^\infty(L^\infty)}$. The above inequality sums up for n. Since $\bar{U}_k^0$ and $\tilde{U}_k^0$ are the solution of (3.12), hence $\xi_k^0 = 0$. By the discrete Gronwall inequality, we get

$$(1 - c\triangle t)\|\xi_k^n\|_{L^2}^2 + K_0 \triangle t \sum_{i=0}^{n-1} \|\xi_k^{i+1}\|_{H_0^1}^2 \leq c\triangle t\{\sum_{i=0}^{n-1} \|u - \tilde{U}_k^{i+\theta}\|_{L^2}^2 + \sum_{i=0}^{n-1} \|I_k \bar{U}_{k-1}^{i+\theta} - \tilde{U}_k^{i+\theta}\|_{L^2}^2\}.$$

Thus when $1 - c\triangle t \geq \nu_0 > 0$, i.e., $\triangle t \leq \frac{1-\nu_0}{c} = \tau_0$, we have

$$\|\xi_k^n\|_{L^2}^2+\triangle t\sum_{i=0}^{n-1}\|\xi_k^{i+1}\|_{H_0^1}^2 \leq c\triangle t\{\sum_{i=0}^{n-1}\|u-\tilde{U}_k^{i+\theta}\|_{L^2}^2+\sum_{i=0}^{n-1}\|I_k\bar{U}_{k-1}^{i+\theta}-\tilde{U}_k^{i+\theta}\|_{L^2}^2\} \leq c\triangle t\{\sum_{i=0}^{n-1}\|u-\tilde{U}_k^{i+\theta}\|_{L^2}^2+\sum_{i=0}^{n-1}\|u-\Pi_{k-1}u$$

here Π_{k-1} is a interpolation operator from $u \in H_0^1 \cap H^2(\Omega)$ onto $\mathcal{M}_{k-1}$. Therefore by (4.2), (2.6), i) of (3.2) and $h_k = \frac{1}{2}h_{k-1}$, we obtain

$$\|\xi_k^n\|_{L^2}^2+\triangle t\sum_{i=0}^{n-1}\|\xi_k^{i+1}\|_{H_0^1}^2 \leq R_k^2+c\triangle t\sum_{i=0}^{n-1}\|\Pi_{k-1}u-\bar{U}_{k-1}^{i+\theta}\|_{L^2}^2 \leq R_k^2+c\triangle t\sum_{i=0}^{n-1}\|u-\bar{U}_{k-1}^{i+\theta}\|_{L^2}^2 4.6$$

where $R_k = \{\, c^*\,(h_k^2 + \triangle t^2),\ \theta = 0,$
$c^*(h_k^2 + \triangle t),\ \theta \neq 0$. The finite element solution $U_1^n(n = 0, 1, 2, \cdots, N-1)$ defined in (3.13)-(3.17) satisfy that

$$\|u - \bar{U}_1^n\|_{L^2} \leq R_1,$$

(see [1], [3], [4]). Hence by (4.6), we can prove that for $j \leq k-1$,

$$\|u - \bar{U}_j^n\|_{L^2} \leq R_j.$$

Thus by $h_k = \frac{1}{2}h_{k-1}$ and (4.6), we obtain

$$\|\xi_k^n\|_{L^2}^2 + \triangle t\sum_{i=0}^{n-1}\|\xi_k^{i+1}\|_{H_0^1}^2 \leq R_k^2 + R_{k-1}^2 \leq R_k^2 . 4.7$$

Note that the assumption $\triangle t \sim O(h_k^2)$, we know that (4.3) holds.

Applying Lemma 1, Lemma 2 and the triangle inequality, we obtain the convergence of the finite element solution of the equation (3.3) as follows.

Theorem 1. *Let u be the solution of (2.4) and satisfy the assumptive conditions (2.2), (2.3) and (4.1). Let $\bar{U}_k^n(n \geq 2)$ be the solution of (3.3) and $\bar{U}_1^n, \bar{U}_k^0$ be the solutions of (3.13)-(3.17) and (3.12), respectively. Then for $\theta \in [0,1]$ and $\triangle t \sim O(h_k^2)$, there are the constants $c^*, \tau_0 > 0$ independent of $h_k, \{\bar{U}_k^n\}$ and $\triangle t$ such that $\triangle t \leq \tau_0$, we have*

$$\|u(t_n) - \bar{U}_k^n\|_{L^2} + h_k\|u(t_n) - \bar{U}_k^n\|_{H_0^1} \leq \{ \quad c^*(h_k^2 + \triangle t^2),\ \theta = 0, c^*(h_k^2 + \triangle t),\ \theta \neq 0.4.8$$

In the following we consider the convergence of the k level iterative solutions defined by (3.6)-(3.10). We first consider the error arising from the k-1 level correction. Let $\bar{U}_{k-1}^{n+1}$ be the exact solution of the equation (3.3) on the k-1 level and $\hat{U}_{k-1}^{n+1}$ be the solution of the equation (3.8). Then

$$(\hat{U}_{k-1}^{n+1} - \bar{U}_{k-1}^n\triangle t, v)+a(\bar{U}_{k-1}^{n+\theta};\hat{U}_{k-1}^{n+\theta}-\bar{U}_{k-1}^{n+\theta}, v) = (f(\bar{U}_{k-1}^{n+\theta})-f(I_{k-1}\bar{U}_{k-2}^{n+\theta}), v) \quad +a(\bar{U}_{k-1}^{n+\theta}-I_{k-1}\bar{U}_{k-2}^{n+\theta};\bar{U}_{k-1}^{n+\theta}, v)+(\bar{U}_{k-}$$

Taking $v = \hat{U}_{k-1}^{n+1} - \bar{U}_{k-1}^{n+1}$, by (2.2), (2.3), (4.1) and ϵ inequality, we obtain

$$\|\hat{U}_{k-1}^{n+1}-\bar{U}_{k-1}^{n+1}\|_{L^2}^2+12(1+\theta)K_0\triangle t\|\nabla(\hat{U}_{k-1}^{n+1}-\bar{U}_{k-1}^{n+1})\|_{L^2}^2 \leq (\hat{U}_{k-1}^{n+1}-\bar{U}_{k-1}^{n+1}, \hat{U}_{k-1}^{n+1}-\bar{U}_{k-1}^{n+1}) \quad +\triangle ta(\bar{U}_{k-1}^{n+\theta};\hat{U}_{k-1}^{n+1}-\bar{U}_{k-1}^{n+1}, \hat{U}$$

where the constant c in (4.9) depends on K_1, L and $\|\nabla u\|_{L^\infty(L^\infty)}$. Applying i.), ii.) of (3.2) and taking $\varepsilon = \frac{1}{2}$, we have

$$\|\hat{U}^{n+1}_{k-1} - \bar{U}^{n+1}_{k-1}\|^2_{L^2} + 12(1+\theta)K_0\triangle t\|\nabla(\hat{U}^{n+1}_{k-1} - \bar{U}^{n+1}_{k-1})\|^2_{L^2} \le c\triangle t[\|\bar{U}^{n+\theta}_{k-1} - I_{k-1}\bar{U}^{n+\theta}_{k-2}\|^2_{L^2} + \|\nabla(u - \bar{U}^{n+\theta}_{k-1})\|^2_{L^2}] + c[\|\bar{U}^{n+1}_{k} - U$$

where the second term is the error of the smoothing iterative solution. By i.) of (3.2) and theorem 1, we have

$$\|\bar{U}^{n+\theta}_{k-1} - I_{k-1}\bar{U}^{n+\theta}_{k-2}\|^2_{L^2}\triangle t + \|\nabla(u - \bar{U}^{n+\theta}_{k-1})\|^2_{L^2}\triangle t \le \|u - I_{k-1}\bar{U}^{n+\theta}_{k-2}\|^2_{L^2}\triangle t + \|u - \bar{U}^{n+\theta}_{k-1}\|^2_{H^1_0}\triangle t \le \|u - \Pi_{k-1}u\|^2_{L^2}\triangle t + \|\Pi_{k-1}$$

where Π_{k-1} is an interpolation operator from $H^1_0 \cap H^2(\Omega)$ onto $\mathcal{M}_{k-1}$. Note that $\triangle t \sim h^2_k$, we obtain

$$\|\hat{U}^{n+1}_{k-1} - \bar{U}^{n+1}_{k-1}\|^2_{L^2} + \triangle t\|\nabla(\hat{U}^{n+1}_{k-1} - \bar{U}^{n+1}_{k-1})\|^2_{L^2} \le R^2_{k-1} + c[\|\bar{U}^{n+1}_{k} - U^{n+1}_{k,\nu_1}\|^2_{L^2} + \triangle t\|\nabla(\bar{U}^{n+1}_{k} - U^{n+1}_{k,\nu_1})\|^2_{L^2}].$$

Thus we have

Lemma 3. *Assume that u satisfy the assumptive conditions* (2.2), (2.3) *and* (4.1). *Let* $\bar{U}^{n+1}_{k-1}$ *be the solution of* (3.3) *on the (k-1)th level and* $\hat{U}^{n+1}_{k-1}$ *be the solutions of* (3.8). *Then when* $\theta \in [0,1]$, $\triangle t \sim O(h^2_k)$, *we have*

$$\|\hat{U}^{n+1}_{k-1} - \bar{U}^{n+1}_{k-1}\|_{L^2} + h_k\|\hat{U}^{n+1}_{k-1} - \bar{U}^{n+1}_{k-1}\|_{H^1_0} \le R_{k-1} + c[\|\bar{U}^{n+1}_{k} - U^{n+1}_{k,\nu_1}\|_{L^2} + \triangle t\|\nabla(\bar{U}^{n+1}_{k} - U^{n+1}_{k,\nu_1}\|_{L^2}]^{\frac{1}{2}}, 4.10$$

where $R_{k-1} = \{\, c^*\,(h^2_{k-1} + \triangle t^2),\ \theta = 0,$
$c^*(h^2_{k-1} + \triangle t),\ \theta \neq 0$. *The constants* c^*, c *depend on* $K_0,\ K_1,\ L,\ \|\nabla u\|_{L^\infty(L^\infty)}$.

Let $\hat{U}^{n+1}_{k-1,p}$ is an approximate solution of the equation (3.8) obtained by p time smoothing iterations. By using of the Euclidean norm, there exists a constant $0 < \gamma < 1$ such that

$$\|\tilde{\hat{A}}^{n,\frac{1}{2}}_{k}(\alpha)(\hat{\alpha}^{n+1}_{k-1} - \alpha^{n+1}_{k-1,p})\|_e \le \gamma^p\|\tilde{\hat{A}}^{n,\frac{1}{2}}_{k-1}(\alpha)(\hat{\alpha}^{n+1}_{k-1} - \alpha^{n+1}_{k-1})\|_e 4.11$$

where $\tilde{\hat{A}}^{n,\frac{1}{2}}_{k}(\alpha) = C_{k-1} + \frac{1}{2}(1+\theta)\triangle t\hat{A}^n_{k-1}(\alpha)$. By (2.2), (4.11) can be written equivalently in the form as:

$$\|\hat{U}^{n+1}_{k-1} - \hat{U}^{n+1}_{k-1,p}\|^2_{L^2} + 12(1+\theta)\triangle tK_0\|\nabla(\hat{U}^{n+1}_{k-1} - \hat{U}^{n+1}_{k-1,p})\|^2_{L^2} \le \gamma^{2p}[\|\hat{U}^{n+1}_{k-1} - \bar{U}^{n+1}_{k-1}\|^2_{L^2} + 12(1+\theta)\triangle tK_1\|\nabla(\hat{U}^{n+1}_{k-1} - \bar{U}^{n+1}_{k-1})\|^2_{L^2}]. 4.12$$

Therefore, the error of the coarse corrective solution of (3.7) satisfies the inequality that

$$\|\bar{U}^{n+1}_{k} - U^{n+1}_{k,\nu_1+1}\|^2_{L^2} + 12(1+\theta)\triangle tK_0\|\nabla(\bar{U}^{n+1}_{k} - U^{n+1}_{k,\nu_1+1})\|^2_{L^2} \le \|\bar{U}^{n+1}_{k} - U^{n+1}_{k,\nu_1}\|^2_{L^2} + \|I_k(\hat{U}^{n+1}_{k-1,p} - \bar{U}^{n+1}_{k-1})\|^2_{L^2} \quad + 12(1+\theta)\triangle$$

where $R_{k-1} = \{\, c^*\,(h^2_{k-1} + \triangle t^2),\ \theta = 0,$
$c^*(h^2_{k-1} + \triangle t),\ \theta \neq 0,\ I_1 = \|\bar{U}^{n+1}_{k} - U^{n+1}_{k,\nu_1}\|^2_{L^2} + 12(1+\theta)\triangle tK_0\ \|\nabla(\bar{U}^{n+1}_{k} - U^{n+1}_{k,\nu_1})\|^2_{L^2}$.

The inequality (4.13) shows that the error of the coarse corrective solution is bounded by the error of the smoothing iterative solution of (3.3) adding the error of the finite element solution of (3.8).

We now consider the solution error of the smoothing iterative scheme (3.7). The smoothing iterative methods (3.20), (3.23) and (3.26) by using of the Euclidean norm have the error estimation:

$$\|\tilde{A}_k^{n,\frac{1}{2}}(\alpha)(\bar{\alpha}_k^{n+1} - \alpha_{k,\nu_1}^{n+1})\|_e \leq \rho(S_k^{\nu_1})\|\tilde{A}_k^{n,\frac{1}{2}}(\alpha)(\bar{\alpha}_k^{n+1} - \alpha_{k,0}^{n+1})\|_e 4.14$$

where $\rho(S_k^{\nu_1})$ satisfies inequalities (3.21), (2.24) and (3.27). Similar to (4.12), (4.14) can be written in the form:

$$\|\bar{U}_k^{n+1} - U_{k,\nu_1}^{n+1}\|_{L^2}^2 + 12(1+\theta)\triangle t K_0 \|\nabla(\bar{U}_k^{n+1} - U_{k,\nu_1}^{n+1})\|_{L^2}^2 \leq \rho(S_k^{\nu_1})[\|\bar{U}_k^{n+1} - U_{k,0}^{n+1}\|_{L^2}^2 + 12(1+\theta)\triangle t K_1 \|\nabla(\bar{U}_k^{n+1} - U_{k,0}^{n+1})\|$$

Thus by (4.13) and (4.15), the k'th level algorithm defined in (3.6)-(3.10) has the result:

Theorem 2. *Let $\bar{U}_k^{n+1}$ be the exact solution of* (3.3) *and $\bar{U}_{k,\nu_1+\nu_2+1}^{n+1}$ be the iterative solution of the k'th level algorithm for* (3.3)*. If there exists a constant $0 < \gamma < 1$ such that* (4.11) *or* (4.12) *holds for the (k-1)th level, then when $\nu_1 + \nu_2$ is large enough, we have*

$$\|\bar{U}_k^{n+1} - U_{k,\nu_1+\nu_2+1}^{n+1}\|_{L^2}^2 + 12(1+\theta)K_0\triangle t\|\nabla(\bar{U}_k^{n+1} - U_{k,\nu_1+\nu_2+1}^{n+1})\|_{L^2}^2 \leq \gamma R_{k-1}^2 + \gamma[\|\bar{U}_k^{n+1} - U_{k,0}^{n+1}\|_{L^2}^2 + 12(1+\theta)K_0\triangle t\|\nabla(\bar{U}_k^{n+1} - U_{k,0}^{n+1}$$

Proof. by (4.15), we have

$$\|\bar{U}_k^{n+1} - U_{k,\nu_1+\nu_2+1}^{n+1}\|_{L^2}^2 + 12(1+\theta)K_0\triangle t\|\nabla(\bar{U}_k^{n+1} - U_{k,\nu_1+\nu_2+1}^{n+1})\|_{L^2}^2 \leq \rho(S_k^{\nu_2})[\|\bar{U}_k^{n+1} - U_{k,\nu_1+1}^{n+1}\|_{L^2}^2 \quad + 12(1+\theta)K_1\triangle t\|\nabla($$

By (4.13), we get

$$\|\bar{U}_k^{n+1} - U_{k,\nu_1+\nu_2+1}^{n+1}\|_{L^2}^2 + 12(1+\theta)K_0\triangle t\|\nabla(\bar{U}_k^{n+1} - U_{k,\nu_1+\nu_2+1}^{n+1})\|_{L^2}^2 \leq \rho(S_k^{\nu_2})[cI_1 + (1+\gamma^p)R_{k-1}^2] \leq c\rho(S_k^{\nu_1+\nu_2})[\|\bar{U}_k^{n+1} -$$

here R_{k-1} is the error of the finite element solution of the equation (3.8). Hence if $\gamma \leq max\{c\rho(S_k^{\nu_1+\nu_2}), (1+\gamma^p)\rho(S_k^{\nu_2})\} < 1$, then (4.16) holds.

Theorem 3. *Let u be the solution of* (2.4) *and satisfy the assumptive conditions* (2.2)*,* (2.3) *and* (4.1)*. Let $U_{k,\nu_1+\nu_2+1}^n$ be the k'th level iterative solution of* (3.6)-(3.10)*. Then there are the constants $c^*, \tau_0 > 0$ independent of h_k and $\triangle t$ such that if $\triangle t \sim O(h_k^2)$ and $\triangle t \leq \tau_0$, we have*

$$\|u(t_{n+1}) - U_{k,\nu_1+\nu_2+1}^{n+1}\|_{L^2} + h_k\|u(t_{n+1}) - U_{k,\nu_1+\nu_2+1}^{n+1}\|_{H_0^1} \leq R_k.4.17$$

Proof. By (3.2), (2.5) and the triangle inequality, we have

$$\|\bar{U}_k^{n+1} - U_{k,0}^{n+1}\|_{L^2}^2 + 12(1+\theta)K_0\triangle t\|\nabla(\bar{U}_k^{n+1} - U_{k,0}^{n+1})\|_{L^2}^2 = \|\bar{U}_k^{n+1} - \bar{U}_k^n - I_k(\bar{U}_{k-1}^{n+1} - \bar{U}_{k-1}^n)\|_{L^2}^2 + 12(1+\theta)K_0\triangle t \quad \cdot\|\nabla(\bar{U}_k^n$$

By theorem 1, we obtain

$$\|\bar{U}_k^{n+1} - U_{k,0}^{n+1}\|_{L^2}^2 + 12(1+\theta)K_0\triangle t\|\nabla(\bar{U}_k^{n+1} - U_{k,0}^{n+1})\|_{L^2}^2 \leq R_k^2.4.19$$

In virtue of (4.16), (4.19) and theorem 1 as well as $h_k = \frac{1}{2}h_{k-1}$, we get

$$\|u(t_{n+1})-U^{n+1}_{k,\nu_1+\nu_2+1}\|_{L^2}+h_k\|u(t_{n+1})-U^{n+1}_{k,\nu_1+\nu_2+1}\|_{H^1_0} \le \|u-\bar{U}^{n+1}_k\|_{L^2}+h_k\|u-\bar{U}^{n+1}_k\|_{H^1_0} \quad +\|\bar{U}^{n+1}_k-U^{n+1}_{k,\nu_1+\nu_2+1}\|_{L^2}+$$

In the following we analyze the convergence of the multigrid scheme. By the assumptive conditions (2.2), (2.3) and (4.1), the solutions $U^n_1 (n = 1, 2, \cdots, N)$ of the equations (3.13)-(3.17) satisfy the error estimation[2−3]:

$$\|u(t_n) - U^n_1\|_{L^2} + h_1\|u(t_n) - U^n_1\|_{H^1_0} \le R_1.4.20$$

Hence by (4.18), (4.19), (2.3) and (2.5), we have

$$\|\bar{U}^{n+1}_k-U^{n+1}_{k,0}\|_{L^2} = \|\bar{U}^{n+1}_k-U^n_k-I_k(U^{n+1}_{k-1}-U^n_{k-1})\|_{L^2} \le \|\bar{U}^{n+1}_k-\bar{U}^n_k-I_k(\bar{U}^{n+1}_{k-1}-\bar{U}^n_{k-1})\|_{L^2}+\|I_k(\bar{U}^{n+1}_{k-1}-U^{n+1}_{k-1})\|_{L^2}+\|\bar{U}$$

By theorem 2 and (4.20), the inequality (4.21) is recurred about the level k. We have

$$\|\bar{U}^{n+1}_k-U^{n+1}_{k,0}\|_{L^2} \le R_k+\gamma R_{k-1}+\gamma\|\bar{U}^{n+1}_{k-1}-U^{n+1}_{k-1}\|_{L^2} \quad +c[\|\bar{U}^n_k-U^n_k\|_{L^2}+\gamma\|\bar{U}^n_{k-1}-U^n_{k-1}\|_{L^2}] \le \sum_{i=2}^{k}\gamma^{k-i}R_i+\gamma^{k-1}\|\bar{U}^{n+}_1$$

Applying $h_k = \frac{1}{2}h_{k-1}$, we have

$$\|\bar{U}^{n+1}_k-U^{n+1}_{k,0}\|_{L^2} \le R_k\sum_{i=1}^{k}(2\gamma)^{k-i}+c\sum_{i=2}^{k}\gamma^{k-i}\|\bar{U}^n_i-U^n_i\|_{L^2} \le \epsilon_0 R_k+c\sum_{i=2}^{k}\gamma^{k-i}\|\bar{U}^n_i-U^n_i\|_{L^2}$$

where $\epsilon_0 = \frac{1-(2\gamma)^{k+1}}{1-2\gamma}$. Therefore, we obtain

$$\|\bar{U}^{n+1}_k - U^{n+1}_k\|_{L^2} \le \gamma\epsilon_0 R_k + c\gamma\sum_{i=2}^{k}\gamma^{k-i}\|\bar{U}^n_i - U^n_i\|_{L^2}4.22$$

here we used (4.16). (4.22) is recurred about n. We obtain

$$\|\bar{U}^{n+1}_k - U^{n+1}_k\|_{L^2} \le R_k.4.23$$

Similar to (4.23), we can prove that

$$h_k\|\bar{U}^{n+1}_k - U^{n+1}_k\|_{H^1_0} \le R_k.4.24$$

Therefore, we obtain the result of convergence of the multigrid algorithm.

Theorem 4. *Assume that conditions* (2.2), (2.3) *and* (4.1) *hold. Then the approximate solution defined by multigrid algorithm satisfies the inequality:*

$$\|u(t_{n+1}) - U^{n+1}_k\|_{L^2} + h_k\|u(t_{n+1}) - U^{n+1}_k\|_{H^1_0} \le R_k 4.25$$

where the constant c^* is independent of $h_k, \triangle t$ and $\{U^n_k\}$.

The proof of the theorem 4 can be obtained by the triangle inequality, theorem 1 and (4.23), (4.24).

5. Computational Cost and Development

Because the coefficient $a(x,u)$ and right term $f(x,t,u)$ in the nonlinear parabolic equation (2.1) associate with the known function u, much computational time is costed in forming the algebraic system (3.5) in the time-dependent fully multigrid algorithm. If N_k denotes the dimension of the finite element space $\mathcal{M}_k$ on the k'th level, the computational cost for forming the algebraic systems (3.5) can be bounded by c_1N_k. In addition, the computational cost of $\nu_1+\nu_2$ time smoothing iterations (3.20), (3.23) or (3.26) is bounded by $c_2(\nu_1+\nu_2)N_k$. The computational cost of p time coarse corrective iterations is bounded by $c_3(N_{k-1}+pN_{k-1})$. Thus the computational cost of the k'th level algorithm is that

$$c_1N_k + c_2(\nu_1+\nu_2)N_k + c_3(N_{k-1}+pN_{k-1}).$$

Usually, the iterative frequencies ν_1, ν_2, p all are not larger than 4. By the relation $N_k \sim 4N_{k-1}$, we obtain that the computational cost of the k level algorithm is bounded by c_4N_k. Therefore, the computational cost of the multigrid algorithm satisfies that

$$\sum_{j\le K} c_4N_j \le c_4N_k(1+14+14^2+\cdots+) \le 43c_4N_k.$$

If N denotes the number of time steps, then the computational cost of the time-dependent fully multigrid algorithm is bounded by $O(NN_k)$.

Note that the multigrid algorithm defined in Section 3 has a few restrictions for the equation (2.1). Hence the multigrid scheme can be extended to more general nonlinear parabolic equation, such as the equation

$$\{\, c\ (x,u)\partial u\partial t = \nabla(a(x,u)\nabla u)+\vec{b}(x,u)\nabla u+f(x,t,u), (x,t)\in\Omega\times[0,T], a(x,u)\partial u\partial t+\vec{n}\vec{b}(x,u) = g(x,t), (x,t)\in\partial\Omega$$

here $\vec{n}$ denotes the unit outer normal direction of the domain boundary, $\vec{b}(x,u) = (b_1(x,u), b_2(x,u))$.

By using finite element discretizing the equation (5.1), we obtain that a system of linearized algebraic equations is simillar to (3.3)

$$(c(I_kU_{k-1}^{n+\theta})U_k^{n+1} - U_k^n\triangle t, v)+a(I_kU_{k-1}^{n+\theta}; U_{k-1}^{n+\theta}, v) \quad -(\vec{b}(I_kU_{k-1}^{n+\theta})\nabla U_{k-1}^{n+\theta}, v) = (f(I_kU_{k-1}^{n+\theta}), v) \quad +< g(t_{n+\theta}), v >, \forall$$

where $< \cdot,\cdot >$ denotes the inner product on the boundary $\partial\Omega$.

Similar to (3.6)-(3.10), we can define the time-dependent fully multigrid algorithm for solving the equation (5.2). The convergence proof of the algorithm needs for the nonlinear coefficient $c(x,u)$ and $\vec{b}(x,u)$ some constrain conditions, here it is omitted.

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