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THE LARGE TIME CONVERGENCE OF SPECTRAL METHOD FOR GENERALIZED KURAMOTO-SIVASHINSKY EQUATIONS*1)

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Abstract

In this paper we use the spectral method to analyse the generalized Kuramoto-Sivashinsky equations. We prove the existence and uniqueness of global smooth solution of the equations. Finally, we obtain the error estimation between spectral approximate solution and exact solution on large time.

1. Introduction

The following nonlinear evolution equation

$$\phi_t + \frac{1}{2}\phi_x^2 + \nu\phi + \alpha\phi_{xx} + \beta\phi_{xxx} + \gamma\phi_{xxxx} = 0, \alpha, \beta, \nu > 0, \tag{1.1}$$

arises in different problem of physics. Kuramoto[1] derived it for the study of dissipative structure of reaction-diffusion. Sivashinsky[2] derived it independently in studying the propagation of a flame front in case of mild combustion. Then it was obtained in bifurcating solution of the Navier-Stokes equation[3] and in viscous film flow[4]. In [5-8], the bifurcating solution, universal attractor was studed for (1.1). The paper [9] proposed a class of generalized KS equations. In [10-11], for some equations of the generalized KS type, the author studed the existence, the uniqness of the global smooth solution, the asymptotic behaviour at $t \to \infty$, the structure of traveling wave solution and approximate solution by Lie group and infinitesimal transformation.

In infinite dimensional dynamical system, as the Navier-Stokes equation, the Boussinesq-Newton equation produced many important physical phenomena, for example, the universal attractor, the chaos etc. It was need to study the large time new computational methods and convergence proof. Similar , it is need to study the large time convergence and error estimation of the approximate solution for KS equation.

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In this paper we use the spectral method to analyse the periodic initial value problem of the generalized KS equations of the form

$$\begin{cases} u_t + \alpha u_{xx} + \beta u_{xxx} + \gamma u_{xxxx} + f(u)_x \\ \qquad\qquad + \phi(u)_{xx} = g(u) + h(x,t), \qquad t > 0, x \in R, \alpha, \gamma > 0, \\ u \mid_{t=0} = u_0(x), x \in R, u(x+2\pi, t) = u(x,t), \qquad t \geq 0, x \in R. \end{cases} \tag{1.2}$$

We prove the convergence of the approximate solution and estimate the error for $t \geq 0$. If differentiating (1.1) with respect to x and setting $u = \phi_x$, then (1.1) is particular form of (1.2).

2. On Large Time Priori Error of Approximate Solution, the Existence and the Uniqueness of Global Solution

Let $\Omega = [0, 2\pi]$, $L^2(\Omega)$ denote the set of all square integrable functions with the inner product $(u,v) = \int_0^{2\pi} u(x)v(x)dx$ and the norm $\|u\|^2 = (u,u)$. Let $L^\infty(\Omega)$ denote the Lebesgue space with the norm $\|u\|_{L^\infty} = \text{ess sup}_{x\in\Omega} \mid u(x) \mid$ and H_p^m denote the periodic Sobolev space with the norm $\|u\|_m = (\sum_{|\alpha|\leq m} \|D^\alpha u\|^2)^{\frac{1}{2}}$. We define $L^2(R^+; H_p^m(\Omega)) = \{u(x,t) \in H_p^m(\Omega) | \int_0^{+\infty} \|u(x,t)\|_m^2 dt < +\infty\}$ and $L^\infty(R^+; H_p^m(\Omega)) = \{u(x,t) \in H_p^m(\Omega) \mid \sup_{0\leq t<+\infty} \|u(x,t)\|_m < +\infty\}$. Let $v_j = \frac{1}{\sqrt{2\pi}} e^{ijx}$ are basis functions, $S_N = \text{Span}\{v_j(x), -N \leq j \leq N\}$.

We construct an approximate solution for periodic initial value problem (1.2) as following: find $u_N \in S_N$, such that

$$\begin{aligned} (u_{Nt} + \alpha u_{Nxx} + \beta u_{Nxxx} + \gamma u_{Nxxxx} + f(u_N)_x \\ + \phi(u_N)_{xx} - g(u_N) - h(x,t), v_j) = 0, \end{aligned} \tag{2.1}$$

$$u_N \mid_{t=0} = u_{0N}(x) = F_N u_0(x), \qquad -N \leq j \leq N, \tag{2.2}$$

where F_N is orthogonal projecting operator from $L^2(\Omega)$ to S_N in the inner product of $L^2(\Omega)$.

Lemma 1. *If the following conditions are satisfied*

1. $\phi'(u) \leq \phi_0, \phi_0 > 0$;
2. $g(0) = 0,\ g'(u) \leq g_0,\ g_0 < -\frac{1}{2}(\alpha + \phi_0 + 1),\ \gamma > \frac{1}{2}(\alpha + \phi_0)$;
3. $h(x,t) \in L^2(Q_\infty),\ u_0(x) \in L^2(\Omega),\ Q_\infty = \Omega \times R^+$,

then for the solution of problem (2.1)-(2.2)*, we have*

$$\|u_N(.,t)\|^2 \leq e^{2(g_0 + \frac{1}{2}(\alpha+\phi_0+1))t} \|u_0(x)\|^2 + \|h\|^2_{L^2(Q_\infty)}, \qquad 0 \leq t < +\infty,$$

$$\int_0^{+\infty} \|u_N(.,t)\|^2 \, dt \leq \frac{1}{2|g_0 + \frac{1}{2}(\alpha + \phi_0 + 1)|} (\|h\|^2_{L^2(Q_\infty)} + \|u_0\|^2),$$

$$\int_0^{+\infty} \|u_{Nxx}(.,t)\|^2 \, dt \leq E_1,$$

where the constant E_1 is dependent on $\|u_0(x)\|, \|h\|_{L^2(Q_\infty)}$ and is independent of N.

Proof. From (2.1) we have

$$(u_{Nt}+\alpha u_{Nxx}+\beta u_{Nxxx}+\gamma u_{Nxxxx}+f(u_N)_x+\phi(u_N)_{xx}-g(u_N)-h(x,t),u_N)=0, \quad (2.3)$$

Obvioursly

$$(u_{Nt},u_N)=\frac{1}{2}\frac{d}{dt}\|u_N\|^2, \qquad (\beta u_{Nxxx},u_N)=-\beta(u_{Nxx},u_{Nx})=0,$$
$$(\alpha u_{Nxx},u_N)=-\alpha\|u_{Nx}\|^2, \qquad (\gamma u_{Nxxxx},u_N)=\gamma\|u_{Nxx}\|^2,$$

$$(\phi(u_N)_{xx},u_N)=-(\phi'(u_N)u_{Nx},u_{Nx})\geq-\phi_0\|u_{Nx}\|^2.$$

By using the assumption of Lemma

$$(f(u_N)_x,u_N)=-(f(u_N),u_{Nx})=-\int_\Omega\frac{d}{dx}\int_0^{u_N}f(\tau)d\tau=0,$$

$$(g(u_N),u_N)=(g(u_N)-g(0),u_N)=(g'(\xi)u_N,u_N)\leq g_0\|u_N\|^2,$$

$$(h(x,t),u_N)\leq\frac{1}{2}(\|u_N\|^2+\|h(.,t)\|^2),$$

$$\|u_{Nx}\|^2=\int_0^{2\pi}u_{Nx}^2dx=-\int_0^{2\pi}u_Nu_{Nxx}dx\leq\|u_N\|\|u_{Nxx}\|\leq\frac{1}{2}(\|u_N\|^2+\|u_{Nxx}\|^2).$$

According to the above estimates, we get

$$\frac{1}{2}\frac{d}{dt}\|u_N\|^2+(\gamma-\frac{1}{2}(\alpha+\phi_0))\|u_{Nxx}\|^2\leq\lambda\|u_N\|^2+\frac{1}{2}\|h(.,t)\|^2, \quad (2.4)$$

where $\lambda=g_0+\frac{1}{2}(\alpha+\phi_0+1)$. Since $\gamma-\frac{1}{2}(\alpha+\phi_0)>0$, we have

$$\frac{d}{dt}\|u_N\|^2\leq 2\lambda\|u_N\|^2+\|h(.,t)\|^2.$$

Setting $e^{-2\lambda t}\|u_N(.,t)\|^2=y(t)$, then we obtain

$$\frac{d}{dt}y(t)\leq e^{-2\lambda t}\|h(.,t)\|^2.$$

Integrating both sides of the above inequality, we get

$$y(t)\leq y(0)+\int_0^t e^{-2\lambda\tau}\|h(.,\tau)\|^2d\tau,$$

i.e.

$$\|u_N(.,t)\|^2\leq e^{2\lambda t}\int_0^t e^{-2\lambda\tau}\|h(.,\tau)\|^2d\tau+e^{2\lambda t}\|u_{0N}\|^2\leq\|h\|^2_{L^2(Q_\infty)}+e^{2\lambda t}\|u_0\|^2.$$

If integrating the both sides of the above first inequality from 0 to $+\infty$,we have

$$\int_0^{+\infty}\|u_N(.,t)\|^2dt\leq\int_0^{+\infty}e^{2\lambda t}\int_0^t e^{-2\lambda\tau}\|h(.,\tau)\|^2d\tau dt+\int_0^{+\infty}e^{2\lambda t}\|u_0\|^2\,dt.$$

Because of

$$\int_0^{+\infty} e^{2\lambda t} \int_0^t e^{-2\lambda\tau} \|h(.,\tau)\|^2\, d\tau dt = \int_0^{+\infty} (\int_\tau^{+\infty} e^{2\lambda t} dt) e^{-2\lambda\tau} \|h(.,\tau)\|^2 d\tau$$
$$= \frac{1}{2 \mid \lambda \mid} \|h\|^2_{L^2(Q_\infty)},$$

we can obtain

$$\int_0^{+\infty} \|u_N(.,t)\|^2\, dt \le \frac{1}{2|\lambda|} (\|h\|^2_{L^2(Q_\infty)} + \|u_{0N}\|^2)$$
$$\le \frac{1}{2|g_0 + \frac{1}{2}(\alpha + \phi_0 + 1)|} (\|h\|^2_{L^2(Q\infty)} + \|u_0\|^2).$$

Finally, according to (2.4), we have

$$\int_0^{+\infty} \|u_{Nxx}(.,t)\|^2\, dt \le E_1,$$

where the constant E_1 is dependent on $\|u_0\|$ and $\|h\|_{L^2(Q_\infty)}$, but is independent of N. The Lemma has been proved.

Lemma 2. *Let $u(x)$ is a differentiable periodic function and the periodic is $2D$, then we have*

$$\|u\|_{L^\infty} \le \frac{1}{\sqrt{D}} [\|u\| + \sqrt{2D} \|u\|^{\frac{5}{6}} \|u_{xxx}\|^{\frac{1}{6}}], \qquad \|u_x\|_{L^\infty} \le \sqrt{2D} \|u_{xx}\|,$$
$$\int_{-D}^{D} |u|^{2p}\, dx \le C_1(D, p, \|u\|) + C_2(D, p, \|u\|) \|u_{xxx}\|^{\frac{p-1}{3}},$$

where $p > 1, C_1(D, p, \|u\|) = \frac{2^{2(p-1)}}{D^{p-1}} \|u\|^{2p}, C_2(D, p, \|u\|) = 2^{3(p-1)} \|u\|^{\frac{5p+1}{3}}$.

Proof. Let $x_0 \in [-D, D]$, by integration by parts, we have

$$\int_{x_0}^{x_0+D} u^2(x)\, dx = (x - x_0 - D) u^2(x) \mid_{x_0}^{x_0+D} -2 \int_{x_0}^{x_0+D} (x - x_0 - D) u(x) u'(x)\, dx.$$

i.e.

$$u^2(x_0) D = \int_{x_0}^{x_0+D} u^2(x)\, dx + 2 \int_{x_0}^{x_0+D} (x - x_0 - D) u(x) u'(x) dx$$

Hence

$$\|u(x)\|_{L^\infty} \le \frac{1}{\sqrt{D}} [\|u\| + \sqrt{2D} \|u(x)\|^{\frac{1}{2}} \|u_x\|^{\frac{1}{2}}]. \tag{2.5}$$

According to $\|u_x\| \le \|u\|^{\frac{1}{2}} \|u_{xx}\|^{\frac{1}{2}}$ and$\|u_{xx}\| \le \|u_x\|^{\frac{1}{2}} \|u_{xxx}\|^{\frac{1}{2}}$, we obtain $\|u_x\| \le \|u\|^{\frac{2}{3}} \|u_{xxx}\|^{\frac{1}{3}}$. Substituting this inequality into (2.5), we have

$$\|u\|_{L^\infty} \le \frac{1}{\sqrt{D}} [\|u\| + \sqrt{2D} \|u\|^{\frac{5}{6}} \|u_{xxx}\|^{\frac{1}{6}}].$$

Because of $u(x-D)=u(x+D)$, by Lagrange mean value theorem, we have $\xi\in(-D,D)$, such that $u'(\xi)=0$. For any $x_0\in(-D,D)$, we have

$$u_x(x_0)=\int_{\xi}^{x_0}u_{xx}(x)\,dx,$$

hence $\|u_x\|_{L^\infty}\le\sqrt{2D}\|u_{xx}\|$. Finally

$$\begin{aligned}\int_{-D}^{D}|u|^{2p}\,dx &\le \|u\|_{L^\infty}^{2p-2}\|u\|^2\le[\frac{1}{\sqrt{D}}(\|u\|+\sqrt{2D}\|u\|^{\frac{5}{6}}\|u_{xxx}\|^{\frac{1}{6}})]^{2p-2}\|u\|^2\\ &\le\frac{2^{2p-2}}{D^{p-1}}[\|u\|^{2p-2}+(2D)^{p-1}\|u\|^{\frac{5}{3}(p-1)}\|u_{xxx}\|^{\frac{1}{3}(p-1)}]\|u\|^2\\ &=C_1(D,p,\|u\|)+C_2(D,p,\|u\|)\|u_{xxx}\|^{\frac{p-1}{3}},\end{aligned}$$

where $C_1(D,p,\|u\|)=\frac{2^{2p-2}}{D^{p-1}}\|u\|^{2p}, C_2(D,p,\|u\|)=2^{3(p-1)}\|u\|^{\frac{5p+1}{3}}$.

Lemma 3. *Suppose that the conditions of Lemma 1 are satisfied, and assume that $|f(u)|\le A|u|^p$, $1\le p<7$, $|\phi'(u)|\le B|u|^q$, $0<q<4$; $u_0\in H_p^1(\Omega)$, $h_x(x,t)\in L^2(Q_\infty)$, then for the approximate solution $u_N(x,t)$ of (2.1)-(2.2), there are the estimates*

$$\sup_{0\le t<+\infty}\|u_{Nx}(.,t)\|\le E_2, \tag{2.6}$$

$$\int_0^{+\infty}\|u_{Nx}(.,t)\|^2\,dt\le E_3, \tag{2.7}$$

$$\int_0^{+\infty}\|u_{Nxxx}(.,t)\|^2\,dt\le E_4, \tag{2.8}$$

where the constants E_2,E_3,E_4 are independent of N.

Proof. According to $v_j''=-j^2v_j$, by (2.1) we have

$$(u_{Nt}+\alpha u_{Nxx}+\beta u_{Nxxx}+\gamma u_{Nxxxx}+f(u_N)_x+\phi(u_N)_{xx}-g(u_N)-h(x,t),u_{Nxx})=0. \tag{2.9}$$

Since

$$(u_{Nt},u_{Nxx})=-\frac{1}{2}\frac{d}{dt}\|u_{Nx}\|^2,$$

$$(\alpha u_{Nxx},u_{Nxx})=\alpha\|u_{Nxx}\|^2\le\alpha\|u_N\|^{\frac{2}{3}}\|u_{Nxxx}\|^{\frac{4}{3}}\le\frac{\gamma}{6}\|u_{Nxxx}\|^2+(\frac{\gamma}{4})^{-2}\frac{\alpha^3}{3}\|u_N\|^2,$$

$$(u_{Nxxx},u_{Nxx})=0,\qquad(\gamma u_{Nxxxx},u_{Nxx})=-\gamma\|u_{Nxxx}\|^2,$$

by using condition (1) and Lemma 2, we have

$$\begin{aligned}|(f(u_N)_x,u_{Nxx})|&=|(f(u_N),u_{Nxxx})|\le\|f(u_N)\|\|u_{Nxxx}\|\\ &\le\frac{\gamma}{6}\|u_{Nxxx}\|^2+\frac{3}{2\gamma}A^2\|u_N\|_{L^{2p}}^{2p}\\ &\le\frac{\gamma}{6}\|u_{Nxxx}\|^2+\frac{3}{2\gamma}A^2(C_1+C_2\|u_{Nxxx}\|^{\frac{p-1}{3}})\end{aligned}$$

where C_1, C_2 are $C_1(\pi,\, p, \|u_N\|)$ and $C_2(\pi,\, p, \|u_N\|)$ of Lemma 2 respectivily. According to Young's inequality

$$ab \le \epsilon \frac{a^s}{s} + \epsilon^{1-t}\frac{b^t}{t}, \qquad s,t>1, \frac{1}{s}+\frac{1}{t}=1, \qquad \epsilon, a,\, b>0,$$

we have

$$|(f(u_N)_x, u_{Nxx})| \le \frac{\gamma}{3}\|u_{Nxxx}\|^2 + \frac{3}{2\gamma}A^2C_1 + C_3(\pi,\, p,\, \|u_N\|),$$

where $C_3(\pi,\, p,\, \|u_N\|) = (\frac{p-1}{\gamma})^{\frac{p-1}{7-p}}\frac{7-p}{6}(\frac{3}{2\gamma}A^2C_2)^{\frac{6}{7-p}} \sim \|u_N\|^{\frac{2(5p+1)}{7-p}}$. By integration by parts, we obtain

$$\begin{aligned}|(\phi(u_N)_{xx}, u_{Nxx})| = |(\phi(u_N)_x, u_{Nxxx})| &= |(\phi'(u_N)u_{Nx}, u_{Nxxx})| \\ &\le \frac{\gamma}{6}\|u_{Nxxx}\|^2 + \frac{3}{2\gamma}\|\phi'(u_N)u_{Nx}\|^2.\end{aligned}$$

According to Lemma 2 and Young's inequality, we have

$$\begin{aligned}&\frac{3}{2\gamma}\|\phi'(u_N)u_{Nx}\|^2 \le \frac{3}{2\gamma}\|\phi'(u_N)\|^2_{L^\infty}\|u_{Nx}\|^2 \le \frac{3B^2}{2\gamma}\|u_N\|^{2q}_{L^\infty}\|u_{Nx}\|^2 \\ &\le \frac{3B^2}{2\gamma}[\frac{1}{\sqrt{\pi}}(\|u_N\| + \sqrt{2\pi}\|u_N\|^{\frac12}\|u_{Nx}\|^{\frac12})]^{2q}\|u_{Nx}\|^2 \\ &\le \frac{3B^2}{2\gamma}\pi^{-q}2^{2q}\|u_N\|^{2q}\|u_N\|^{\frac43}\|u_{Nxxx}\|^{\frac23} + \frac{3}{2\gamma}B^2 2^{3q}\|u_N\|^{\frac23(q+2)+q}\|u_{Nxxx}\|^{\frac13(q+2)} \\ &\le \frac{\gamma}{6}\|u_{Nxxx}\|^2 + C_4(\pi,\, q,\, \|u_N\|)\end{aligned}$$

where

$$\begin{aligned}C_4(\pi,\, q,\, \|u_N\|) &= (\frac{4}{\gamma})^{\frac12}\frac{2}{3}(\frac{3}{2\gamma}B^2\pi^{-q}2^{2q}\|u_N\|^{2q+\frac43})^{\frac32} \\ &+(\frac{\gamma}{2(q+2)})^{\frac{2+q}{q-4}}\frac{4-q}{6}(\frac{3}{2\gamma}B^2 2^{3q}\|u_N\|^{\frac{5q+4}{3}})^{\frac{6}{4-q}} \sim \|u_N\|^{3q+2} + \|u_N\|^{\frac{2(5q+4)}{4-q}}.\end{aligned}$$

Finally

$$\begin{aligned}&-(g(u_N), u_{Nxx}) = (g'(u_N)u_{Nx}, u_{Nx}) \le g_0\|u_{Nx}\|^2, \\ &-(h(x,t), u_{Nxx}) = (h_x, u_{Nx}) \le \|h_x\|\|u_{Nx}\| \le \frac{|g_0|}{2}\|u_{Nx}\|^2 + \frac{1}{2|g_0|}\|h_x\|^2.\end{aligned}$$

Substituting the above estimates into (2.9), we have

$$\begin{aligned}&\frac12\frac{d}{dt}\|u_{Nx}\|^2 + \frac{\gamma}{6}\|u_{Nxxx}\|^2 \le g_0\|u_{Nx}\|^2 + \frac{|g_0|}{2}\|u_{Nx}\|^2 \\ &+ \frac{1}{2|g_0|}\|h_x\|^2 + \frac{3}{2\gamma}A^2C_1 + C_3 + C_4 + (\frac{4}{\gamma})^2\frac{\alpha^3}{3}\|u_N\|^2,\end{aligned}$$

i.e.

$$\frac{d}{dt}\|u_{Nx}\|^2 + \frac{\gamma}{3}\|u_{Nxxx}\|^2 \le g_0\|u_{Nx}\|^2 + \frac{1}{|g_0|}\|h_x\|^2 + C_5(\pi,\, p,\, q,\, \|u_N\|), \qquad (2.10)$$

where

$$C_5(\pi,\, p,\, q,\, \|u_N\|) = 2[(\frac{4}{\gamma})^2\frac{\alpha^3}{3}\|u_N\|^2 + \frac{3}{2\gamma}A^2C_1 + C_3 + C_4].$$

Multiplying both sides of (2.10) by $e^{-g_0 t}$ and integrating from 0 to t, we have

$$\begin{aligned}\|u_{Nx}(t)\|^2 &\le e^{g_0 t}\int_0^t e^{-g_0\tau}[\frac{1}{|g_0|}\|h_x\|^2 + C_5]\,d\tau + e^{g_0 t}\|u_{0Nx}\|^2\\ &\le \frac{1}{|g_0|}\|h_x\|^2_{L^2(Q_\infty)} + \sup_{0\le t<+\infty}|C_5|\frac{1}{|g_0|}(1-e^{g_0 t}) + e^{g_0 t}\|u_{0Nx}\|^2.\end{aligned}$$

According to Lemma 1 and Lemma 2, we get

$$\begin{aligned}&\sup_{0\le t<+\infty}\|u_{Nx}(t)\|^2 \le \frac{1}{|g_0|}\|h_x\|^2_{L^2(Q_\infty)}\\ &+\frac{2}{|g_0|}C_5(\pi,p,q,\sup_{0\le t<+\infty}\|u_N\|) + \sup_{0\le t<+\infty}e^{g_0 t}\|u_{0Nx}\|^2 \le E_2.\end{aligned}$$

Similar to Lemma 1, we can proof (2.7), (2.8).

Lemma 4. *Suppose that the conditions of Lemma 3 is satisfied and assume that* $f(u)\in C^{m-1}$, $g(u)$, $\phi(u)\in C^m$, $h(x,t)\in C^{m-2}$, $m\ge 2$, $u_0\in H^m$, $\sum_{j=0}^{m-2}\|D_x^j h\|_{L^2(Q_\infty)} < +\infty$, *then for the approximate solution* $u_N(x,t)$ *of* (2.1)-(2.2), *there are the estimates*

$$\sup_{0\le t<+\infty}\|D_x^m u_N\| \le E_m, \tag{2.11}$$

$$\int_0^{+\infty}\|D_x^m u_N\|^2\,dt \le E'_m, \tag{2.12}$$

$$\int_0^{+\infty}\|D_x^{m+2} u_N\|^2\,dt \le E'_{m+2}, \tag{2.13}$$

where the constants E_m, E'_m, E'_{m+2} *are independent of* N.

Proof. By the equation (2.1), we obtain

$$\begin{aligned}&(u_{Nt} + \alpha u_{Nxx} + \beta u_{Nxxx} + \gamma u_{Nxxxx} + f(u_N)_x\\ &\quad + \phi(u_N)_{xx} - g(u_N) - h(x,t),\, D_x^{2m}u_N) = 0,\end{aligned} \tag{2.14}$$

Similar to Lemma 3, we have

$$(u_{Nt}, D_x^{2m}u_N) = (-1)^m\frac{1}{2}\frac{d}{dt}\|D_x^m u_N\|^2,\;\; (u_{Nxx}, D_x^{2m}u_N) = (-1)^{m-1}\|D_x^{m+1}u_N\|^2,$$

$$\begin{gathered}(\beta u_{Nxxx}, D_x^{2m}u_N) = (-1)^{m-2}\beta(D_x^{m+1}u_N, D_x^{m+2}u_N) = 0,\\ (\gamma u_{Nxxxx}, D_x^{2m}u_N) = (-1)^{m-2}\gamma\|D_x^{m+2}u_N\|^2,\\ |(f(u_N)_x, D_x^{2m}u_N)| = |(D_x^{m-1}f(u_N), D_x^{m+2}u_N)| \le \|D_x^{m-1}f(u_N)\|\|D_x^{m+2}u_N\|,\\ |(\phi(u_N)_{xx}, D_x^{2m}u_N)| = |(D_x^m\phi(u_N), D_x^{m+2}u_N)| \le \|D_x^m\phi(u_N)\|\|D_x^{m+2}u_N\|,\\ (g(u_N), D_x^{2m}u_N) = (-1)^m(D_x^m g(u_N), D_x^m u_N),\end{gathered}$$

$$|(h(x,t),D_x^{2m}u_N)| = |(D_x^{m-2}h(x,t),D_x^{m+2}u_N)| \le \|D_x^{m-2}h(x,t)\|\|D_x^{m+2}u_N\|$$
$$\le \frac{\gamma}{6}\|D_x^{m+2}u_N\|^2 + \frac{3}{2\gamma}\|D_x^{m-2}h(x,t)\|^2.$$

Since

$$\|D_x^{m-1}f(u_N)\| \le C_1(m,\|u_N\|_{L^\infty},b)(1+\|D_x^{m-2}u_N\|)^{m-2}\|D_x^{m-1}u_N\|,$$
$$\|D_x^{m}\phi(u_N)\| \le C_2(m,\|u_N\|_{L^\infty},b')(1+\|D_x^{m-1}u_N\|)^{m-1}\|D_x^{m}u_N\|,$$

$$(D_x^m g(u_N),D_x^m u_N) \le g_0\|D_x^m u_N\|^2 + C_3(m,\|u_N\|_{L^\infty},b'')(1+\|D_x^{m-1}u_N\|)^{m-1}\|D_x^m u_N\|^2,$$

where $b = \max_{\substack{0\le j\le m-1\\ |u|\le\|u_N\|_{L^\infty}}} |D^j f(u)|$, $b' = \max_{\substack{0\le j\le m\\ |u|\le\|u_N\|_{L^\infty}}} |D^j \phi(u)|$, and $b'' = \max_{\substack{0\le j\le m\\ |u|\le\|u_N\|_{L^\infty}}} |D^j g(u)|$. By using Young's inequality again, we have

$$\alpha\|D_x^{m+1}u_N\|^2 \le \alpha\|D_x^m u_N\|\|D_x^{m+2}u_N\| \le \alpha\|D_x^{m+2}u_N\|^{1+\frac{m}{m+2}}\|u_N\|^{\frac{2}{m+2}}$$
$$\le \frac{\gamma}{6}\|D_x^{m+2}u_N\|^2 + (\frac{\gamma(m+2)}{6(m+1)})^{-(m+1)}\frac{\alpha^{m+2}}{m+2}\|u_N\|^2.$$

$$C_1(1+\|D_x^{m-2}u_N\|)^{m-2}\|D_x^{m-1}u_N\|\|D_x^{m+2}u_N\|$$
$$\le \frac{\gamma}{6}\|D_x^{m+2}u_N\|^2 + \frac{3}{2\gamma}[C_1(1+\|D_x^{m-2}u_N\|)^{m-2}\|D_x^{m-1}u_N\|]^2,$$

$$C_2(1+\|D_x^{m-1}u_N\|)^{m-1}\|D_x u_N\|\|D_x^{m+2}u_N\|$$
$$\le \frac{\gamma}{6}\|D_x^{m+2}u_N\|^2 + (\frac{\gamma(m+2)}{6(m+1)})^{-(m+1)}\frac{1}{m+2}[C_2(1+\|D_x^{m-1}u_N\|)^{m-1}]^{m+2}\|u_N\|^2,$$

$$C_3(1+\|D_x^{m-1}u_N\|)^{m-1}\|D_x^m u_N\| \le \frac{|g_0|}{2}\|D_x^m u_N\|^2 + \frac{1}{2|g_0|}(C_3(1+\|D_x^{m-1}u_N\|))^{2(m-1)}.$$

Substituting the above estimates into (2.14), we get

$$\frac{1}{2}\frac{d}{dt}\|D_x^m u_N\|^2 + \frac{\gamma}{3}\|D_x^{m+2}u_N\|^2$$
$$\le \frac{g_0}{2}\|D_x^m u_N\|^2 + C(m,\|u_N\|_{L^\infty},\|u_N\|,\|D_x u_N\|,...,\|D_x^{m-1}u_N\|),$$

where

$$C(m,\|u_N\|_{L^\infty},\|u_N\|,\|D_x u_N\|,...,\|D_x^{m-1}u_N\|)$$
$$= (\frac{\gamma(m+2)}{6(m+1)})^{-(m+1)}\frac{\alpha^{m+2}}{m+2}\|u_N\|^2 + \frac{1}{2|g_0|}(C_3(1+\|D_x^{m-1}u_N\|))^{2(m-1)}$$
$$+\frac{3}{2\gamma}[C_1(1+\|D_x^{m-2}u_N\|)^{m-2}\|D_x^{m-1}u_N\|]^2 + \frac{3}{2\gamma}\|D_x^{m-2}h(x,t)\|^2$$
$$+(\frac{\gamma(m+2)}{6(m+1)})^{-(m+1)}\frac{1}{m+2}[C_2(1+\|D_x^{m-1}u_N\|)^{m-1}]^{m+2}\|u_N\|^2.$$

According to Lemma 1 and Lemma 3, we assume that

$$\sup_{0\le t<+\infty}\|D^{s-2}u_N\|\le E_{m-1},\ \int_0^{+\infty}\|D^s u_N\|^2\,dt\le E'_{m-1},\ 0\le s\le m+1.$$

Noting $g_0<0$ and

$$\int_0^{+\infty}C(m,\|u_N\|_{L^\infty},\|u_N\|,\|D_x u_N\|,...,\|D_x^{m-1}u_N\|)dt\le E^*,$$

where E^* is a constant independent of N. By using Gronwall's inequality, we obtain the conclusion of Lemma 4.

By using the above estimates and compact argument, we obtain

Theorem 1. *Suppose that the following conditions are satisfied* $f(u)\in C^{m-1}$, $g(u)$, $\phi(u)\in C^m$, $h(x,y)\in C^{m-2}$, $(m\ge 2)$*;* $|f(u)|\le A|u|^p$, $1\le p<7$, $|\phi'(u)|\le B|u|^q$, $0<q<4$, $\phi'(u)\le\phi_0$, $\phi_0>0$*;* $g'(u)\le g_0<0$, $g_0<-\frac{1}{2}(\alpha+\phi_0+1)$, $\gamma>\frac{1}{2}(\alpha+\phi_0)$, $g(0)=0$*;* $D^j h(x,t)\in L^2(Q_\infty)$, $j=0,...,m-2$*;* $u_0\in H^m(\Omega)$*, then there exist a global smooth solution* $u(x,t)$ *for periodic initial value problem* (1.2) *and* $u(x,t)\in L^\infty(R^+;H_p^m(\Omega))\cap L^2(R^+;H_p^{m+2}(\Omega))$*. The approximate solution* $u_N(x,t)$ *of* (2.1)-(2.2) *not only converge to the solution of* (1.2) *uniformly for* t *but also in* $L^2(Q_\infty)$ *sense.*

Theorem 2. *The global smooth solution of problem* (1.2) *is unique. Hence the approximate solutions* $u_N(x,t)$ *of* (2.1)-(2.2) *converge to the smooth solution of* (1.2).

Proof. Assume that there are two solutions u, v of (1.2). Setting $w=u-v$, then w satisfy

$$\begin{aligned}&w_t+\alpha w_{xx}+\beta w_{xxx}+\gamma w_{xxxx}+(f(u)-f(v))_x+(\phi(u)-\phi(v))_{xx}=g(u)-g(v),\\&w\mid_{t=0}=0.\end{aligned}$$

According to the mean value theorem and method of energy, we easily obtain $w=0$, for $t\ge 0$.

3. The Error Estimation of an Approximate Solution by the Spectral Method

Suppose that the $u(x,t)$ is the solution of periodic initial value problem (1.2), $u_N(x,t)$ is the solution of (2.1)-(2.2). Setting $u-u_N=u-F_Nu-(u_N-F_Nu)=\xi-\zeta$. By (2.1) and (1.2), ζ satisfy the following system

$$\begin{aligned}&(\zeta_t+\alpha\zeta_{xx}+\beta\zeta_{xxx}+\gamma\zeta_{xxxx}+f(u_N)_x-f(u)_x+\phi(u_N)_{xx}-\phi(u)_{xx}-g(u_N)\\&+g(F_Nu),\ v_j)=(\xi_t+\alpha\xi_{xx}+\beta\xi_{xxx}+\gamma\xi_{xxxx}-g(u)+g(F_Nu),\ v_j),\\&\qquad\zeta(0)=0,\qquad j=-N,...,0,...,N.\end{aligned}$$

Setting $\zeta(x,t)=\sum_{j=-N}^N\mu_j(t)v_j$. Multiplying both sides of above equations by $\mu_j(t)$ and summing them up for $-N$ to N and according to the orthogonality relation $(\xi_t+\alpha\xi_{xx}+\beta\xi_{xxx}+\gamma\xi_{xxxx},\zeta)=0$, we have

$$\begin{aligned}&(\zeta_t+\alpha\zeta_{xx}+\beta_{xxx}+\gamma\zeta_{xxxx}+f(u_N)_x-f(u)_x+\phi(u_N)_{xx}-\phi(u)_{xx}\\&-g(u_N)+g(F_Nu),\ \zeta)=(-g(u)+g(F_Nu),\ \zeta).\end{aligned}\tag{3.1}$$

Similar to the proof of above Lemma 1, we obtain

$$(\zeta_t,\zeta)=\frac{1}{2}\frac{d}{dt}\|\zeta\|^2,\ (\alpha\zeta_{xx},\zeta)=-\alpha\|\zeta_x\|^2\geq -\frac{\alpha}{2}(\|\zeta\|^2+\|\zeta_{xx}\|^2),$$
$$(\beta\zeta_{xxx},\zeta)=0,\qquad (\gamma\zeta_{xxxx},\zeta)=\gamma\|\zeta_{xx}\|^2.$$

We now have to estimate $(f(u_N)_x-f(u)_x,\zeta)=-(f(u_N)-f(u),\zeta_x)$, we use Taylor formula

$$\begin{aligned}f(u_N)-f(u)&=f'(u)(u_N-u)+\frac{f''(\eta_1)}{2}(u_N-u)^2\\&=f'(0)(u_N-u)+f''(\rho_1)u(u_N-u)+\frac{f''(\eta_1)}{2}(u_N-u)^2\\&=f'(0)(\zeta-\xi)+f''(\rho_1)u(\zeta-\xi)+\frac{f''(\eta_1)}{2}(\zeta-\xi)^2.\end{aligned}$$

So that

$$|(f(u_N)_x-f(u)_x,\zeta)|=|f'(0)(\zeta-\xi)+f''(\rho_1)u(\zeta-\xi)+\frac{f''(\eta_1)}{2}(\zeta-\xi)^2,\zeta_x)|.$$

Since $(f'(0)(\zeta-\xi),\zeta_x)=0$ and according to Lemma 2,we obtain

$$\begin{aligned}|(f''(\rho_1)u(\zeta-\xi),\zeta_x)|&\leq C\|\zeta_x\|_{L^\infty}\|u\|(\|\zeta\|+\|\xi\|)\\&\leq C\|\zeta_{xx}\|\|u\|(\|\zeta\|+\|\xi\|)\leq\eta\|\zeta_{xx}\|^2+C\|u\|^2(\|\zeta\|^2+\|\xi\|^2).\end{aligned}\tag{3.2}$$

Analogously to (3.2), we have

$$\begin{aligned}&|(\frac{f''(\eta_1)}{2}(\zeta-\xi)^2,\zeta_x)|\leq C\|\zeta_x\|_{L^\infty}(\|\zeta\|^2+\|\xi\|^2)\\&\leq C\|\zeta_{xx}\|(\|u_N-F_Nu\|\|\zeta\|+\|\xi\|^2)\\&\leq\eta\|\zeta_{xx}\|^2+C((\|u_N\|^2+\|F_Nu\|^2)\|\zeta\|^2+\|\xi\|^4)\\&\leq\eta\|\zeta_{xx}\|^2+C((\|u_N\|^2+\|u\|^2)\|\zeta\|^2+\|\xi\|^4).\end{aligned}\tag{3.3}$$

Combining (3.2) and (3.3), we get

$$|(f(u_N)_x-f(u)_x,\zeta)|\leq 2\eta\|\zeta_{xx}\|^2+C_1\|\zeta\|^2(\|u\|^2+\|u_N\|^2)+C_2\|\xi\|^2(\|\xi\|^2+\|u\|^2).$$

Now we consider the $(\phi(u_N)_{xx}-\phi(u)_{xx},\zeta)=(\phi(u_N)-\phi(u),\zeta_{xx})$. Using Taylor formula and according to the assumption of Lemma 3 $\phi'(0)=0$, we have

$$\phi(u_N)-\phi(u)=\phi'(u)(u_N-u)+\frac{\phi''(\eta_2)}{2}(u_N-u)^2=\phi'(0)(u_N-u)$$
$$+\phi''(\rho_2)u(u_N-u)+\frac{\phi''(\eta_2)}{2}(u_N-u)^2=\phi''(\rho_2)u(\zeta-\xi)+\frac{1}{2}\phi''(\eta_2)(\zeta-\xi)^2.$$

So that

$$\begin{aligned}
&|(\phi(u_N)_{xx} - \phi(u)_{xx}, \zeta)| = |(\phi(u_N) - \phi(u), \zeta_{xx})| \\
&\le |(\phi''(\rho_2)u(\zeta - \xi), \zeta_{xx})| + |\frac{1}{2}(\phi''(\eta_2)(\zeta - \xi)^2, \zeta_{xx})| \\
&\le C\|\zeta_{xx}\|\|u\|_{L^\infty}(\|\zeta\| + \|\xi\|) + C(\|\zeta\|_{L^\infty} + \|\xi\|_{L^\infty})\|\zeta_{xx}\|(\|\zeta\| + \|\xi\|) \\
&\le \eta\|\zeta_{xx}\|^2 + C\|u\|^2_{L^\infty}(\|\zeta\|^2 + \|\xi\|^2) + C(\|\zeta\|^2_{L^\infty} + \|\xi\|^2_{L^\infty})(\|\zeta\|^2 + \|\xi\|^2).
\end{aligned}$$

Using the imbeding theorem $H^1 \hookrightarrow L^\infty$ and $\|F_N u\|_1 \le \|u\|_1$, we get

$$\begin{aligned}
|(\phi(u_N)_{xx} - \phi(u)_{xx}, \zeta)| \le{}& \eta\|\zeta_{xx}\|^2 \\
&+ C_3\|\zeta\|^2(\|u\|_1^2 + \|u_N\|_1^2) + C_4\|\xi\|^2(\|u\|_1^2 + \|u_N\|_1^2).
\end{aligned}$$

Finally, we have to estimate $(g(F_N u) - g(u_N), \zeta)$ and $(g(F_N u) - g(u), \zeta)$. By the assumption of Lemma 1

$$(g(u_N) - g(F_N u), \zeta) \le g_0\|\zeta\|^2,$$

in other hand

$$|(g(F_N u) - g(u), \zeta)| \le \|g'\|_{L^\infty}\|\xi\|\|\zeta\| \le \eta'\|\zeta\|^2 + C_5\|g'\|^2_{L^\infty}\|\xi\|^2.$$

Substituting above estimates into (3.1), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\zeta\|^2 + (\gamma - \frac{\alpha}{2} - 3\eta)\|\zeta_{xx}\|^2 \\
&\le (g_0 + \frac{\alpha}{2} + \eta')\|\zeta\|^2 + (C_1 + C_3)\|\zeta\|^2(\|u\|_1^2 + \|u_N\|_1^2) \\
&+ \|\xi\|^2(2C_2\|u\|^2 + C_4\|u\|_1^2 + C_4\|u_N\|_1^2 + C_5\|g'\|^2_{L^\infty}).
\end{aligned} \tag{3.4}$$

Since $\gamma - \frac{\alpha}{2} > 0$, we can take η very small, such that $\gamma - \frac{\alpha}{2} - 3\eta = \frac{1}{2}\gamma^* > 0$. Similar, since $g_0 + \frac{\alpha}{2} < 0$, we can take η' very small also, such that $g_0 + \frac{\alpha}{2} + \eta' = -\frac{s}{2} < 0$, $s > 0$. Therefore we have

$$\frac{d}{dt}\|\zeta\|^2 + \gamma^*\|\zeta_{xx}\|^2 \le R_1(t)\|\zeta\|^2 + R_2(t)\|\xi\|^2, \tag{3.5}$$

where

$$\begin{aligned}
R_1(t) &= 2(C_1 + C_3)(\|u(t)\|_1^2 + \|u_N\|_1^2), \\
R_2(t) &= 2(2C_2\|u(t)\|^2 + C_4\|u(t)\|_1^2 + C_4\|u_N(t)\|_1^2 + C_5\|g'\|^2_{L^\infty}).
\end{aligned}$$

Multiplying both sides of (3.5) by $\exp(-\int_0^t R_1(\tau)d\tau)$ and integrating from 0 to t, noting $\zeta(0) = 0$, we have

$$\exp(-\int_0^t R_1(\tau)d\tau)\|\zeta(t)\|^2 \le \int_0^t R_2(s)\|\xi(s)\|^2 \exp(-\int_0^s R_1(\tau)d\tau)ds,$$

i.e.

$$\|\zeta(t)\|^2 \le \int_0^t R_2(s)\|\xi(s)\|^2 \exp(\int_s^t R_1(\tau)d\tau)ds \tag{3.6}$$

According to the Theorem 1 and Lemma 3, $\int_0^{+\infty}\|u(t)\|_1^2dt$, $\int_0^{+\infty}\|u_N(t)\|_1^2dt \le C$, we can obtain

$$\exp(\int_s^t R_1(\tau)d\tau) \le \hat{E}_1, \qquad \sup_{0\le t<+\infty} R_2(t) \le \hat{E}_2, \qquad 0 \le s \le t, \forall t \ge 0.$$

Lemma 5.[12] *If* $0 \le \mu \le \sigma$, $u \in H_p^\sigma(\Omega)$, *then*

$$\|u - F_N u\|_\mu \le CN^{-(\sigma-\mu)}\|u\|_\sigma.$$

If $f \in C^2$, $\phi \in C^2$, $g \in C^1$, the solution of (1.2) $u(x,t) \in L^\infty(R^+; H_p^m(\Omega)) \cap L^2(R^+; H_p^m(\Omega))$, then according to the Lemma 5, (3.6) and triangle inequality we obtain

$$\begin{aligned}
&\sup_{0\le t<+\infty}\|u(t)-u_N(t)\|^2 \le 2\sup_{0\le t<+\infty}(\|\xi(t)\|^2+\|\zeta(t)\|^2)\\
&\le 2\sup_{0\le t<+\infty}\|\xi(t)\|^2 + C\int_0^{+\infty}N^{-2m}\|u(t)\|_m^2dt\\
&\le CN^{-2m}(\sup_{0\le t<+\infty}\|u(t)\|_m^2+\int_0^{+\infty}\|u(t)\|_m^2dt).
\end{aligned} \tag{3.7}$$

In order to estimate $\int_0^{+\infty}\|D_x^j\zeta(t)\|^2dt$, $0 \le j \le 2$, we integrating from 0 to t in both sides of (3.4), we have

$$\begin{aligned}
&\|\zeta(t)\|^2 + \gamma^*\int_0^t\|\zeta_{xx}(\tau)\|^2d\tau + s\int_0^t\|\zeta(\tau)\|^2d\tau\\
&\le \|\zeta(0)\|^2 + \int_0^t R_1(\tau)\|\zeta(\tau)\|^2d\tau + \int_0^t R_2(\tau)\|\xi(\tau)\|^2d\tau.
\end{aligned}$$

Since $\zeta(0) = 0$, therefore

$$\begin{aligned}
&\int_0^t(\|\zeta_{xx}(\tau)\|^2+\|\zeta(\tau)\|^2)d\tau\\
&\le C\sup_{0\le\tau<+\infty}\|\zeta(\tau)\|^2\int_0^{+\infty}R_1(\tau)d\tau + CN^{-2m}\int_0^{+\infty}R_2(\tau)\|u(\tau)\|_m^2d\tau.
\end{aligned}$$

let $t \to +\infty$, we have

$$\begin{aligned}
&\int_0^{+\infty}(\|\zeta_{xx}(t)\|^2+\|\zeta(t)\|^2)dt\\
&\le CN^{-2m}(\int_0^{+\infty}R_1(\tau)d\tau + \int_0^{+\infty}R_2(\tau)\|u(\tau)\|_m^2d\tau).
\end{aligned}$$

According to the Lemma 1, Lemma 3, and by using the interpolate inequality, we have

$$\int_0^{+\infty}\|D_x^j\zeta(t)\|^2dt \le CN^{-2m}, \qquad 0 \le j \le 2.$$

If $u(x,t) \in L^\infty(R^+; H_p^m(\Omega)) \cap L^2(R^+; H_p^{m+2}(\Omega))$,by using the triangle inequality we can obtain

$$\int_0^{+\infty} \|u_{xx} - u_{Nxx}\|^2 dt \le 2[\int_0^{+\infty} \|\xi_{xx}\|^2 dt + \int_0^{+\infty} \|\zeta_{xx}\|^2 dt]$$
$$\le CN^{-2m} \int_0^{+\infty} \|u(t)\|_{m+2}^2 \, dt + CN^{-2m}.$$

Summing up, we have

Theorem 3. *Suppose that the conditions of Lemma* 1 *and Lemma* 3 *are satisfied and* f, $\phi \in C^2$, $g \in C^1$, *the solution of* (1.2) $u(x,t) \in L^2(R^+; H_p^m(\Omega)) \cap L^\infty(R^+; H_p^m(\Omega))$, *then for the solution of* (2.1)-(2.2) $u_N(x,t)$ *we have*

$$\sup_{0 \le t < +\infty} \|u(.,t) - u_N(.,t)\| \le CN^{-m}.$$

If $u(x,t) \in L^\infty(R^+; H_p^m(\Omega)) \cap L^2(R^+; H_p^{m+j}(\Omega))$, *then we have*

$$\int_0^{+\infty} \|D_x^j(u(.,t) - u_N(.,t))\|^2 dt \le CN^{-2m}, \qquad 0 \le j \le 2,$$

where the constants C *are independent of* N.

References

[1] Y. Kuramoto, *Progr.Theoret.Phys.Suppl.*, 64(1978), 346-367.

[2] G. Sivashinsky, *Acta Astronant*, 4(1977), 1117-1206.

[3] M.A. Aimar and P. Dner, Universite de TOULON et da VAR, Preprint, 1982.

[4] T. Shang and G. Sivashinsky, *J. de Physique*, 43(1982), 459-466.

[5] B. Nicolaenko, B. Scherer and R. Temma, *Comm. Part. Diff. Equa.*, 14(1989), 245-297.

[6] B. Nicolaenko, B. Scherer and R. Temma, *C. R. Acad. Sci. Paris*, 298(1984), 23-25.

[7] B. Nicolaenko, B. Scherer and R. Temma, *Phys. D.*, 16(1985), 155-183.

[8] C. Foias, B. Nicoleanko, G.R. Sell and R. Temma, *C. R. Acad. Sci. Paris*, 301(1985), 285-288.

[9] B. Nicoleanko, *Nuclear Phys. B.*, (1987), 453-484.

[10] B. Guo, The global smooth solutions and universal attractors for some dissipative nonlinear evolution equations, in Proceedingers of the 6th International Workshop on Nonlinear Evolution Equations and Dynamical Systems, Springer Verlag, 1991.

[11] B. Guo, On the Generalized Kuramoto-Sivashinsky Type Equations with the Dispersive Effects, IAPCM, 90-105.

[12] C. Canuto and A. Quarteroni, Approximation results for orthogonal polynomials in Sobolev spaces, *Math. Comp.*, 38(1982), 67-86.