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H-SPLITTINGS AND ASYNCHRONOUS PARALLEL ITERATIVE METHODS*

C.L. Wang and Z.Y. You
(*Department of Mathematics Xi'an Jiaotong University*)

Abstract

The paper discusses H-splitting and H-compatible splitting furthermore, some properties are given. Asynchronous parallel multisplitting algorithm and its generalization for linear systems $Ax = b$ are established. Convergence of these algorithms is proved under given conditions. The convergent range of relaxation factor ω is given, numerical example is shown.

1. Properties of H-splitting

A. Frommer and D.B. Szyld[3] proposed H-splitting and H-compatible splitting for two-stage methods. But they didn't discuss two splitting furthermore. We will show some properties of H-splitting and H-compatible splitting before we apply them to establish asynchronous parallel multisplitting algorithm.

Definition 1. ([3]) *Given $A \in L(R^n)$, $A = M - N(M,\ N \in L(R^n))$, which is called H-splitting, if $\langle M\rangle - |N|$ is an M-matrix; which is called H-compatible splitting, if $\langle A\rangle = \langle M\rangle - |N|$. Where $\langle A\rangle$ is Ostrowski matrix, $|N|$ is absolution matrix.*

Obviously, for an H-matrix, an H-compatible splitting is an H-splitting, but an H-splitting is not necessarily an H-compatible splitting. For example:

$$A = \begin{bmatrix} 1 & 0.25 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -0.25 \\ -1 & 0 \end{bmatrix} = M - N$$
$$\langle M\rangle - |N| = \begin{bmatrix} 1 & -0.25 \\ -3 & 1 \end{bmatrix} \neq \langle A\rangle.$$

But $\langle M\rangle - |N|$ is an M-matrix.

Property 1. *Given $A \in L(R^n)$, let $A = M - N$ be an H-splitting. Then A is an H-matrix.*

Proof. By definition, $\langle M\rangle - |N|$ is an M-matrix.

$$\langle A\rangle = \langle M - N\rangle \geq \langle M\rangle - |N|$$

By comparison property of M-matrix, $\langle A\rangle$ is an M-matrix.

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Hence, A is an H-matrix.

Property 2. *Given $A \in L(R^n)$, let $A = M - N$ be an H-splitting. Then $\rho(\langle M\rangle^{-1}|N|) < 1$ and $\rho(|M^{-1}N|) < 1$.*

By definition, $\langle M\rangle - |N|$ is an M-matrix, hence $\langle M\rangle$ is an M-matrix. This implies that $\langle M\rangle - |N|$ is a convergent regular splitting, then $\rho(\langle M\rangle^{-1}|N|) < 1$.

By $\langle M\rangle^{-1} \geq |M^{-1}|$, we have

$$|M^{-1}N| \leq |M^{-1}||N| \leq \langle M\rangle^{-1}|N|.$$

Hence, $\rho(|M^{-1}N|) < 1$.

Property 3. *Let A be a nonsingular H-matrix, let $A = M_i - N_i (i = 1, 2)$ be H-compatible splittings. If $|N_1 \geq |N_2|$, then $\rho(\langle M_1\rangle^{-1}|N_1|) \geq \rho(\langle M_2\rangle^{-1}|N_2|)$.*

Proof. By definition of H-compatible splitting and comparison property of regular splittings [1], conclusion is proved directly.

But it is not true for H-splitting, for example:

$$\begin{aligned}
&A = \begin{bmatrix} 1 & -0.25 \\ -1 & 1 \end{bmatrix} = M_1 - N_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0.25 \\ 1 & 0 \end{bmatrix}, \\
&A = \begin{bmatrix} 1 & -0.25 \\ -1 & 1 \end{bmatrix} = M_2 - N_2 = \begin{bmatrix} 1 & 0 \\ -1.5 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0.25 \\ -0.5 & 0 \end{bmatrix}, \\
&|N_1| \geq |N_2|, \quad \rho(\langle M_2\rangle^{-1}|N_2|) = \frac{3 + \sqrt{41}}{16} > 0.56, \quad \rho(\langle M_1\rangle^{-1}|N_1|) = 0.5.
\end{aligned}$$

2. Asynchronous Parallel Algorithm

The parallel multisplitting iterative method for solving large systems of linear algebraic equations

$$Ax = b, \quad A \in L(R^n) \quad x, b \in R^n \tag{2.1}$$

was first presented by O'Leary and White[4] in 1985. Since then, many papers have dealt with parallel multisplitting iterative methods for linear and nonlinear problems[4]–[10] etc. In order to make use of parallel computer efficiently, a great deal of research is currently being focused on asynchronous implementation, in which computation and communication are performed independently in each processor so that processor idle time is reduced, time of convergence is shorted and so on.

In this section, we will establish asynchronous parallel iterative methods based on H-splitting.

Let $A = M_i - N_i (i = 1, 2, \cdots, l)$ be H-splittings, $E_i (i = 1, 2, \cdots, l)$ are nonnegative diagonal matrices and $\sum_{i=1}^{l} E_i = I$, where I is the identity matrix. (2.1) changes into following equivalent form:

$$x = \sum_{i=1}^{l} E_i M_i^{-1} N_i x + \sum_{i=1}^{l} E_i M_i^{-1} b, \tag{2.2}$$

which is the fixed-point equation that must be satisfied by solution of (2.1)[4].

Synchronous parallel multisplitting iterative method:

$$x_i^{k+1} = E_i M_i^{-1} N_i x^k + E_i M_i^{-1} b, \quad i = 1, 2, \cdots, l,$$
$$x^{k+1} = \sum_{i=1}^{l} x_i^{k+1}$$

Definition 2. *For $k = 1, 2, \cdots$ and $i = 1, 2, \cdots, l$, let there be given nonempty sets $I(k) \in\subset \{1, \cdots, l\}$ and n-tuples $s^i(k) \equiv (s_1^i(k), \cdots, s_n^i(k)) \in (N \cup \{0\})^n$ (N represents positive integer). Suppose that the following assumption hold.*

(i) For $i \in \{1, \cdots, l\}$ and $j \in \{1, \cdots, n\}$, there exists a positive integer T_1 such that

$$k - T_1 \le s_j^i(k) \le k \tag{2.3}$$

(ii) For $i \in \{1, \cdots, l\}$, there exists a positive integer T_2 such that

$$i \in (I(k), \cdots, I(k - T_2)) \tag{2.4}$$

We change synchronous parallel multisplitting method into asynchronous parallel algorithm (I).

$$x_i^{k+1} = E_i M_i^{-1} N_i x^{s^i(k)} + E_i M_i^{-1} b \quad i = 1, 2, \cdots, l$$
$$x^{k+1} = \sum_{i \in I(k)} x_i^{k+1} + \sum_{i \notin I(k)} E_i x^{s^i(k)}$$

We introduce the relaxed factor $\omega > 0$ and generalize algorithm (I) into algorithm (II)

$$x_i^{k+1} = E_i M_i^{-1} N_i x^{s^i(k)} + E_i M_i^{-1} b \quad i = 1, 2, \cdots, l$$
$$x^{k+1} = (1 - \omega) x^{s^i(k)} + \omega \Big(\sum_{i \in I(k)} x_i^{k+1} + \sum_{i \notin I(k)} E_i x^{s^i(k)} \Big)$$

The above two asynchronous parallel models generalize that of [8] or [9].

3. Convergence

For convenience, $A = D_A - B_A$ is Jacobi splitting, $J_A = D_A^{-1} B_A$ unless specification.

Lemma 1. *Given $A \in L(R^n)$, let $A = M - N$ be H-splitting. If there exists a strictly diagonally dominant matrix G such that $\langle M \rangle - |N| \ge \langle G \rangle$. Then $\langle M \rangle - |N|$ is a strictly diagonally dominant matrix. Furthermore, if diag$(M) = D_G$, then $\| \langle M \rangle^{-1} |N| \, \|_\infty \le \|J_G\|_\infty < 1$.*

Proof. By assumption, we have $\langle G \rangle, \langle M \rangle - |N|$ are M-matrices. Then $\langle M \rangle - |N| \ge \langle G \rangle$ implies that $\langle M \rangle - |N|$ is a strictly diagonally dominant matrix.

By assumption, $\langle M \rangle^{-1} |N| \le I - \langle M \rangle^{-1} \langle G \rangle$. Let $e = (1, 1, \cdots, 1)^T$.

$$\langle M \rangle^{-1} |N| e \le e - \langle M \rangle^{-1} \langle G \rangle e \le e - |D_G^{-1}| \langle G \rangle e \le J_G e.$$

Hence, $\|\langle M\rangle^{-1}|N|\,\|_\infty \leq \|J_G\|_\infty$. On thd other hand, by assumption of G, we have, $\|J_G\|_\infty < 1$.

Theorem 1. *Given $A \in L(R^n)$, let $A = M_i - N_i$ $(i = 1, 2, \cdots, l)$ be H-splittings, there exists a strictly diagonally dominant matrix G such that $\langle M_i\rangle - |N_i| \geq \langle G\rangle$, and $diag(\langle M_i\rangle) = D_G$, $(i = 1, 2, \cdots, l)$. Then $\{x^k\}$ generated by algorithm (I) converges to the solution of* (2.1).

Proof. Let $e^k = x^k - x^*$, where x^* is the solution of (2.1).

By algorithm (I), we have

$$\begin{aligned}
e^{k+1} &= \sum_{i\in I(k)} E_i M_i^{-1} N_i e^{s^i(k)} + \sum_{i\notin I(k)} E_i e^{s^i(k)} \\
&= \sum_{i\in I(k)} E_i M_i^{-1} N_i e^{s^i(k)} + \sum_{i\in I(k-t_i)} E_i M_i^{-1} N_i e^{s^i(k-t_i)} \\
|e^{k+1}| &\leq \Big|\sum_{i\in I(k)} E_i M_i^{-1} N_i\Big|\, |e^{s^i(k)}| + \Big|\sum_{i\in I(k-t_i)} E_i M_i^{-1} N_i\Big|\, |e^{s^i(k-t_i)}| \\
&\leq \sum_{i\in I(k)} E_i \langle M_i\rangle^{-1} |N_i|\, |e^{s^i(k)}| + \sum_{i\in I(k-t_i)} E_i \langle M_i\rangle^{-1} |N_i|\, |e^{s^i(k-t_i)}| \\
&\leq \sum_{i=1}^{l} E_i \langle M_i\rangle^{-1} |N_i| \max_{1\leq i\leq l}\{|e^{s^i(k)}|, |e^{s^i(k-t_i)}|\},
\end{aligned}$$

thus, $\|e^{k+1}\|_\infty \leq \Big\|\sum_{i=1}^{l} E_i \langle M_i\rangle^{-1}|N_i|\Big\|_\infty \cdot \max_{1\leq i\leq l}\{\|e^{s^i(k)}\|_\infty,\ \|e^{s^i(k-t_i)}\|_\infty\}$.

By assumption, we have $k - T_1 \leq s^i(k) \leq k$, $0 \leq t_i \leq T_2$, $i = 1, \cdots, l$. Hence, let $T = T_1 + T_2$.

$$\|e^{k+1}\|_\infty \leq \|J_G\|_\infty \max_{0\leq j\leq T} \|e^{k-j}\|_\infty \tag{3.1}$$

By $\|J_G\|_\infty < 1$, we have $\lim_{k\to\infty} \|e^{k+1}\|_\infty = 0$, that is, $\lim_{k\to\infty} x^k = x^*$.

Corollary 1. *Let A be a strictly diagonally dominant matrix, let $A = M_i - N_i$ $(i = 1, 2, \cdots, l)$ be H-compatible splittings. If $diag(\langle M_i\rangle) = diag(\langle A\rangle)$, then $\{x^k\}$ generated by algorithm (I) converges to the solution of* (2.1).

Theorem 2. *Given $A \in L(R^n)$, let $A = M_i - N_i (i = 1, 2, \cdots, l)$ be H-splittings, and there exists an H-matrix G such that $\langle M_i\rangle - |N_i| \geq \langle G\rangle (i = 1, 2, \cdots, l)$. If $diag(\langle M_i\rangle) = D_G$, then $\{x^k\}$ generated by algorithm (I) converges to the solution of* (2.1).

Proof. By definition of H-matrix, there exists a positive diagonal matrix P such that GP is strictly diagonally dominant matrix. Obviously, $\langle M_i\rangle P$ and $\langle M_i\rangle P - |N_i|P$ are M-matrices.

By lemma 1, $\langle M_i\rangle P - |N_i|P$ are also strictly diagonal dominant matrices, and

$$\|(\langle M_i\rangle P)^{-1}|N_i|P\|_\infty \leq \|P^{-1}J_G P\|_\infty < 1 \quad (i = 1, 2, \cdots, l).$$

Let $Pe^k = x^k - x^*$, where x^* is the solution of (2.1). By algorithm (I), we have

$$Pe^{k+1} = \sum_{i\in I(k)} E_i M_i^{-1} N_i P e^{s^i(k)} + \sum_{i\notin I(k)} E_i P e^{s^i(k)}$$

$$
\begin{aligned}
&= \sum_{i\in I(k)} E_i M_i^{-1} N_i P e^{s^i(k)} + \sum_{i\in I(k-t_i)} E_i M_i^{-1} N_i P e^{s^i(k-t_i)} \\
|Pe^{k+1}| &\le \Big| \sum_{i\in I(k)} E_i M_i^{-1} N_i |P| e^{s^i(k)} \Big| + \sum_{i\in I(k-t_i)} E_i |M_i^{-1} N_i |P| e^{s^i(k-t_i)}| \\
&\le \sum_{i\in I(k)} E_i \langle M_i\rangle^{-1} |N_i| P |e^{s^i(k)}| + \sum_{i\in I(k-t_i)} E_i \langle M_i\rangle^{-1} |N_i| P |e^{s^i(k-t_i)}| \\
&\le \sum_{i=1}^{l} E_i \langle M_i\rangle^{-1} |N_i| P \max_{1\le i\le l} \{|e^{s^i(k)}|, |e^{s^i(k-t_i)}|\},
\end{aligned}
$$

Similar proving processing as theorem 1, we have

$$||e^{k+1}||_\infty \le ||P^{-1} J_G P||_\infty \max_{0\le j\le T} ||e^{k-j}||_\infty. \tag{3.2}$$

Hence, $\lim_{k\to\infty} ||e^{k+1}||_\infty = 0$, that is, $\lim_{k\to\infty} x^k = x^*$.

Corollary 2. *Let A be a nonsingular H-matrix, let $A = M_i - N_i (i = 1, 2, \cdots, l)$ be H-compatible splittings, if $diag(\langle M_i\rangle) = diag(\langle A\rangle)$, then $\{x^k\}$ generated by algorithm (I) converges to the solution of* (2.1).

Lemma 2. *Suppose that A is an H-matrix, $A = D_A - B_A$ is Jacobi splitting. Then there exists a positive diagonal matrix P such that $||P^{-1} J_A P||_\infty = \rho(|J_A|)$.*

Proof. By property of H-matrix [11], there exists an optimally scaled matrix P such that AP satisfies the following equations:

$$\frac{\sum_{j\ne i} |a_{ij} P_j|}{|a_{ii} P_i|} = \rho(|J_A|), \quad i = 1, 2, \cdots, n,$$

which implies $||P^{-1} J_A P||_\infty = \rho(|J_A|)$.

Theorem 3. *Given $A \in L(R^n)$, let $A = M_i - N_i (i = 1, 2, \cdots, l)$ be H-splittings. If there exists an H-matrix G such that $\langle M_i\rangle - |N_i| \ge \langle G\rangle$, $diag(\langle M_i\rangle) = D_G$, $(i = 1, 2, \cdots, l)$, if $0 < \omega < \dfrac{2}{1+\rho(|J_G|)}$, then $\{x^k\}$ generated by algorithm (II) converges to the solution of* (2.1).

Proof. Let $Pe^k = x^k - x^*$, where P is the optimally scaled matrix, x^* is the solution of (2.1).

The same proving proceeding as that of theorem 2, we have

$$
\begin{aligned}
||e^{k+1}||_\infty &\le |1-\omega| + \omega \Big\| \sum_{i=1}^{l} E_i \langle M_i\rangle^{-1} |N_i| \Big\|_\infty \max_{0\le j\le T} ||e^{k-j}||_\infty \\
&\le |1-\omega| + \omega ||P^{-1} J_G P||_\infty \max_{0\le j\le T} ||e^{k-j}||_\infty,
\end{aligned}
$$

By lemma 2, we have

$$||e^{k+1}||_\infty \le |1-\omega| + \omega\rho(|J_G|) \max_{0\le j\le T} ||e^{k-j}||_\infty.$$

When $0 < \omega < \dfrac{2}{1+\rho(|J_G|)}$,

$$|1-\omega| + \omega\rho(|J_G|) < 1.$$

Hence $\lim\limits_{k\to\infty} ||e^{k+1}||_\infty = 0 \Rightarrow \lim\limits_{k\to\infty} x^k = x^*$.

Corollary 3. *Let A be a nonsingular H-matrix, let $A = M_i - N_i$ $(i = 1, 2, \cdots, l)$ be H-compatible splittings, diag$(\langle M_i\rangle)$ = diag$(\langle A\rangle)$ $(i = 1, 2, \cdots, l)$. When $0 < \omega < \dfrac{2}{1+\rho(|J_A|)}$, then $\{x^k\}$ generated by algorithm (II) converges to the solution of* (2.1).

Following discusses the convergence rate of algorithm (I).

Definition 3. *Let $\{x^k\}_{k=0}^{\infty}$ be a sequence in R^n such that $\lim\limits_{k\to\infty} x^k = x^*$. Then $\sigma(\{x^k\}_{k=0}^{\infty}) = \lim\limits_{k\to\infty}\sup ||x^k - x^*||^{\frac{1}{k}}$ is the R_1-factor of the sequence $\{x^k\}_{k=0}^{\infty}$.*

This factor is independent of choice of the norm $||\cdot||$.

Theorem 4. *Let A be a nonsingular H-matrix, and $A = M_i - N_i (i = 1, \cdots, l)$ be H-splittings. $T = T_1 + T_2$. If there exists a nonsingular H-matrix G such that $\langle G\rangle \le \langle M_i\rangle - |N_i|$, and diag$(\langle M_i\rangle)$ = diag$(\langle G\rangle)$, then $\sigma(\{x^k\}_{k=0}^{\infty}) \le \sqrt[T]{\rho(|J_G|)}$.*

Proof. Let $e^k = P^{-1}(x^k - x^*)$, where P is the optimally scaled matrix, x^* is the solution of (2.1). Similar proving proceeding as (3.2), we have

$$||e^{k+1}||_\infty \le ||P^{-1}J_G P||_\infty \max_{0\le j\le T} ||e^{k-j}||_\infty. \tag{3.3}$$

By lemma 2, we obtain

$$||e^{k+1}||_\infty \le \rho(|J_G|) \max_{0\le j\le T} ||e^{k-j}||_\infty.$$

Hence,

$$\begin{aligned}
\sigma(\{x^k\}_{k=0}^{\infty}) &= \lim_{k\to\infty}\sup(||x^k - x^*||)^{\frac{1}{k}} = \lim_{k\to\infty}\sup(||Pe^k||)_\infty))^{\frac{1}{k}} \\
&\le \lim_{k\to\infty}\sup(||P||_\infty)^{\frac{1}{k}} \cdot (||e^k||_\infty)^{\frac{1}{k}} \\
&\le \lim_{k\to\infty}\sup\ (\rho(|J_G|)^{\frac{k}{T}} \max_{0\le j\le T} ||e^j||_\infty ||P||_\infty)^{\frac{1}{k}} \\
&\le \lim_{k\to\infty}\sup\ \sqrt[T]{\rho(|J_G|)}\Big(\max_{0\le j\le T} ||e^j||_\infty ||P||_\infty\Big)^{\frac{1}{k}} = \sqrt[T]{\rho(|J_G|)}.
\end{aligned}$$

Corollary 4. *Let A be a nonsingular H-matrix, and $A = M_i - N_i (i = 1, \cdots, l)$ be H-compatible splittings, $T = T_1 + T_2$. If diag(M_i) = diag(A) $(i = 1, \cdots, l)$, then $\sigma(\{x^k\}_{k=0}^{\infty}) \le \sqrt[T]{\rho(|J_A|)}$.*

4. Numerical Example

We use our algorithm to solve usually block tridiagonal linear equations as follows.

$$Ax = b$$

Where

$$A=\begin{bmatrix} A_{11} & A_{12} & & & & \\ A_{21} & A_{22} & A_{23} & & & \\ & A_{32} & A_{33} & A_{34} & & \\ & & A_{43} & A_{44} & A_{45} & \\ & & & A_{54} & A_{55} & A_{56} \\ & & & & A_{65} & A_{66} \end{bmatrix}$$

$$A_{11}=\begin{bmatrix} 2 & -0.4 & & & \\ -0.2 & 2 & -0.4 & & \\ & -0.2 & 2 & -0.4 & \\ & & -0.2 & 2 & -0.4 \\ & & & -0.2 & 2 \end{bmatrix}$$

$$A_{21}=\begin{bmatrix} 0 & 0 & -0.1 & -0.2 & -0.2 \\ -0.2 & 0 & 0 & -0.1 & -0.2 \\ -0.2 & -0.2 & 0 & 0 & -0.1 \\ -0.1 & -0.2 & -0.2 & 0 & 0 \\ 0 & -0.1 & -0.2 & -0.2 & 0 \end{bmatrix}$$

$$A_{ii}=\begin{bmatrix} 1 & -0.2 & 0 & 0 & 0 \\ -0.1 & 1 & -0.2 & 0 & 0 \\ 0 & -0.1 & 1 & -0.2 & 0 \\ 0 & 0 & -0.1 & 1 & -0.2 \\ 0 & 0 & 0 & -0.1 & 1 \end{bmatrix}, \quad 2\le i\le 6$$

$$A_{ii-1}=\begin{bmatrix} 0 & 0 & 0.05 & 0.1 & 0.1 \\ 0.1 & 0 & 0 & 0.05 & 0.1 \\ 0.1 & 0.1 & 0 & 0 & 0.05 \\ 0.05 & 0.1 & 0.1 & 0 & 0 \\ 0 & 0.05 & 0.1 & 0.1 & 0 \end{bmatrix}, \quad 3\le i\le 6$$

$$A_{ii+1}=\begin{bmatrix} -0.2 & -0.1 & -0.1 & -0.05 & 0 \\ 0 & -0.2 & -0.1 & -0.1 & -0.05 \\ -0.05 & 0 & -0.2 & -0.1 & -0.1 \\ -0.1 & -0.05 & 0 & -0.2 & -0.1 \\ -0.1 & -0.1 & -0.05 & 0 & -0.2 \end{bmatrix}, \quad 1\le i\le 5$$

$b=(1.15,\ 0.95,\ 0.95,\ 0.95,\ 1.35,\ -0.15,\ -0.25,\ -0.25,\ -0.25,\ -0.05,\ 0.6,\ 0.5,\ 0.5,\ 0.5,\ 0.7,\ 0.6;\ 0.5,\ 0.5,\ 0.5,\ 0.7,\ 0.6,\ 0.5,\ 0.5,\ 0.5,\ 0.7,\ 1.05,\ 0.95,\ 0.95,\ 0.95,\ 1.15)^T$.

We use three splitting as follows.

(1) Guass-Seidel splitting;

(2) Jacobi splitting;

(3) SOR splitting.

The power matrices $E_1=\mathrm{diag}\,(\underbrace{1,\cdots,1}_{10},\underbrace{0,\cdots,0}_{20})$, $E_2=\mathrm{diag}\,(\underbrace{0,\cdots,0}_{10},\underbrace{1,\cdots,1}_{10},\underbrace{0,\cdots,0}_{10})$, $E_3=\mathrm{diag}\,(\underbrace{0,\cdots,0}_{20},\underbrace{1,\cdots,1}_{10})$.

The initial value $x^{(0)} = (50,\ 100,\ 150,\ -200,\ -100,\ 0,\ -90,\ 200,\ 200,\ -250,\ 110,\ 120,\ 130,\ 140,\ 150,\ 120,\ 170,\ 100,\ -190,\ -500,\ -10,\ -20,\ -30,\ 30,\ 50,\ -50,\ -80,\ -10,\ 0,\ 110)^T$, the accuracy $\epsilon = 0.0001$.

method	$G-S$	SOR ($\omega = 1.0$)	method in [4] ($\omega = 1.0$)	algorithm (I) ($\omega = 0.9$)	Jacobi
iterative number	15	15	24	15	29
waiting amount			3960	740	
computing amount	3970	3970	6360	3970	7470

Notes: Waiting time is replaced by waiting multiplying amount, computing time is replaced by multiplying amount. In the table, the parameter ω is the optimal parameter.

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