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NUMERICAL ANALYSIS FOR A MEAN-FIELD EQUATION FOR THE ISING MODEL WITH GLAUBER DYNAMICS*[1)]

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Abstract

In this paper, a mean-field equation of motion which is derived by Penrose (1991) for the dynamic Ising model with Glauber dynamics is considered. Various finite difference schemes such as explicit Euler scheme, predictor-corrector scheme and some implicit schemes are given and their convergence, stabilities and dynamical properties are discussed. Moreover, a Lyapunov functional for the discrete semigroup $\{S\}_{n>0}$ is constructed. Finally, numerical examples are computed and analyzed. it shows that the model is a better approximation to Cahn-Allen equation which is mentioned in Penrose (1991).

1. Introduction

We consider the following mean-field equation of motion for the dynamic Ising model on a periodic lattice Λ:

$$\begin{cases} \mathbf{u}_t + \mathbf{u} = \tanh(\beta \mathbf{A}\mathbf{u}) \quad t > 0 & (1.1a) \\ \mathbf{u}(0) = \mathbf{u}_0 \in V_\Lambda & (1.1b) \\ \mathbf{u}_{a+N\mathbf{e}^i} = \mathbf{u}_a \qquad a \in \Lambda,\ 1 \le i \le d & (1.1c) \end{cases}$$

where Λ denotes the lattice of $\mathbf{Z}^d$ with N^d sites defined by $\Lambda := \Big\{a : a = \sum_{i=1}^{d} a_i \mathbf{e}^i,$ $a_i \in \mathbf{Z},\ 1 \le a_i \le N\Big\}$ with $\{\mathbf{e}^i\}$ being the standard unit vectors of $\mathbf{Z}^d$. We say that Λ is a d-dimensional lattice. We denote by V_Λ the N^d dimensional space of lattice vectors $\mathbf{v} = (v_a)_{a\in\Lambda^*}$ satisfying $v_{a+N\mathbf{e}^i} = v_a$. Here $\mathbf{u} = (u_a)_{a\in\Lambda}$ and u_a denotes the expectation of the spin at site a of the lattice and Λ^* is defined by $\Big\{a : a = \sum_{i=1}^{d} a_i \mathbf{e}^i,\ a_i \in Z\Big\}$.

The $N^d \times N^d$ symmetric matrix $\mathbf{A}$ is defined by, for $v \in V_\Lambda$

$$\{\mathbf{A}\mathbf{v}\}_a := \sum_{b\in\Lambda} E_{ab} v_b \tag{1.2}$$

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where $J_{ab} = JE_{ab}$ $(J > 0)$ is the Ising interaction between sites a and b satisfying, for all $a, b \in \Lambda$

$$(i).\quad E_{ab} \geq 0 \qquad (ii).\quad E_{ab} > 0 \Longleftrightarrow b \in N(a) \qquad (iii).\quad E_{ab} \leq 1. \tag{1.3}$$

Here $N(a)$ denotes the neighborhood of the site a defined by $N(a) = \Big\{b : \sum_{i=1}^{d} |a_i - b_i| = 1\Big\}$. The parameter $\beta = J/\theta$, where $\theta(> 0)$ is the absolute temperature. Furthermore throughout the paper we use the convention that for any lattice vector $\mathbf{u}$, the component at site a in $(\mathbf{u})_a = u_a$ and for any $f : \mathbf{R} \to \mathbf{R}$, $\{f(\mathbf{u})\}_a = f(u_a)$. The dynamical system (1.1) was derived by Penrose[1] from an Ising model on the lattice Λ. It approximately represents the behavior in the mean of the Ising model with Glauber (spin-flip) stochastic dynamics, Glauber[2]. Existence and bounded of absorbing sets, global attractor for (1.1) are studied by Lu Bainian[3], and the bifurcation solutions for the steady-state equation of the equation (1.1) also are discussed in [3].

In this paper we shall construct some explicit and implicit finite difference approximations and their convergence, stability, dynimical properties and long time behavior for the equation (1.1).

For simplicity, we shall use the same notations and abbreviations as used in [3]

$$\theta_a := J \sum_{b \in N(a)} E_{ab} \tag{1.4a}$$

$$\theta_c := \max_{a \in \Lambda} \theta_a = J \|\mathbf{A}\|_\infty, \tag{1.4b}$$

where $\|\mathbf{A}\|_\infty$ is the infinity norm of the matrix $\mathbf{A}$ and given by $\|\mathbf{A}\|_\infty := \max_{a \in \Lambda} \sum_{b \in N(a)} E_{ab}$.

The discrete weighted L^2 inner product and L^2 norm are defined as

$$(\mathbf{u}, \mathbf{v}) = h^d \sum_{a \in \Lambda} u_a v_a \qquad \forall \mathbf{u}, \mathbf{v} \in V_\Lambda, \tag{1.5}$$

$$\|\mathbf{v}\| = (\mathbf{v}, \mathbf{v})^{1/2}, \ \forall \mathbf{v} \in V_\Lambda. \tag{1.6}$$

and discrete maximum norm is defined as

$$\|\mathbf{v}\|_\infty = \max_{a \in \Lambda} |v_a|, \qquad \forall \mathbf{v} \in V_\Lambda \tag{1.7}$$

The inverse of $\tanh(\cdot)$ is denoted by $\phi(\cdot)$ so that $\phi(r) = \dfrac{1}{2} \ln \dfrac{1+r}{1-r}$.

We introduce the homogeneous 'free energy' functions for $r \in (-1, 1)$

$$\psi(r) := \frac{1}{2}((1+r)\ln(1+r) + (1-r)\ln(1-r)) \tag{1.8}$$

Then as noted by Penrose[1], an important feature of the system (1.1) is the existence of a Lyapunov functional given in our notation by

$$I(\mathbf{u}) := \frac{\beta}{2}(\mathbf{A}\,\mathbf{u}, \mathbf{u}) + (\mathbf{e}, \psi(\mathbf{u})) \tag{1.9}$$

where $\{\mathbf{e}\}_a = 1$.

Remark 1.1. In order to study the Lyapunov functional for discrete finite difference schemes, we introduce the (1.9). It is easy to check that the functional (1.9) is equivalent to (1.8) in Lu Bainian[3].

In the following, we shall review the theory of dissipative dynamical system sufficient for our needs. Let $\{S^n\}_{n\geq 0}$ be a family of operators from a complete metric space H into itself satisfying the semigroup properties $S^{n+m} = S^n S^m$, $\forall n, m \in \mathcal{Z}$, $S^0 = I$ and $S^n : \mathbf{R}_+ \times H \to H$ is continuous.

For each $\mathbf{v} \in H$ we denote the positive orbit $\bigcap\limits_{n>0} S^n\mathbf{v}$ by $\gamma^+(\mathbf{v})$ and the ω-limit set $\omega(\mathbf{v})$ defined by $\bigcup\limits_{k\geq 0} \overline{\bigcap\limits_{n\geq k} S^n\mathbf{v}}$.

We use the finite difference notation for time variable t: $\partial f^{n+1} = (f^{n+1} - f^n)/k$, where $k > 0$ is the time step. It is convenient to note the following

Lemma 1.1. *Suppose that $f^n \in R$, $n \geq 0$ and that*

$$\partial f^{n+1} < c - df^{n+1},\ (c, d > 0) \quad f^n > 0 \tag{1.10}$$

For any $\varepsilon > 0$, there exists an n_0 such that

$$f^n < (c + \varepsilon)/d \quad \forall n \geq n_0. \tag{1.11}$$

Proof. Clearly if $f^n < c/d$, then we have $f^{n+1} < \dfrac{ck + f^n}{1 + dk} < \dfrac{ck + c/d}{1 + dk} = c/d$. Furthermore if $f^0 - c/d = \delta > 0$, then it holds that

$$f^n - c/d < (1 + dk)^{-n}\delta. \tag{1.12}$$

This proves the Lemma.

2. The Explicit Euler Approximation

In this section we consider the explicit Euler scheme

$$\begin{cases} \partial\mathbf{u}^{n+1} + \mathbf{u}^n = \tanh(\beta A\mathbf{u}^n) \quad n = 1, 2, \cdots & \text{(2.1a)} \\ \mathbf{u}^0 = \mathbf{u}_0 & \text{(2.1b)} \end{cases}$$

where $\mathbf{u}^n \in \mathbf{V}_\Lambda$ $\forall n \geq 0$.

Lemma 2.1. *Let $\mathbf{u}^n$ solve* (2.1). *Then for any $\varepsilon_1, \varepsilon_2 > 0$ and $k_0 < 1$, provided*

$$k < \min\Big\{k_0, 1 - \varepsilon_2 - \Big(\tanh\Big(\frac{\theta_c}{\theta}\frac{1 + \varepsilon_1}{1 - k_0}\Big)\Big)^2\Big\}$$

there exists an $n_0(\|\mathbf{u}_0\|_\infty)$ such that $\|\mathbf{u}^n\|_\infty < 1$ for all $n > n_0$.

Proof. Similar to the proof of Lemma 4.2 in [3], multiplication the a's component of the both sides of the equation (2.1) by $\mathbf{u}_a^{n+1}$, and by the fact

$$\mathbf{u}_a^{n+1}\partial\mathbf{u}_a^{n+1} = \frac{1}{2}(\partial|\mathbf{u}_a^{n+1}|^2 + k|\partial\mathbf{u}_a^{n+1}|^2), \tag{2.2}$$

we have

$$\frac{1}{2}\partial|\mathbf{u}_a^{n+1}|^2+\frac{k}{2}|\partial\mathbf{u}_a^{n+1}|^2+|\mathbf{u}_a^{n+1}|^2=k\mathbf{u}_a^{n+1}\partial\mathbf{u}_a^{n+1}+\mathbf{u}_a^{n+1}\{\tanh(\beta A\mathbf{u}^n)\}_a \tag{2.3}$$

Note that $\alpha\cdot\beta\leq\frac{1}{2}(\alpha^2+\beta^2)$ and $\tanh(x)<1$, by (2.3), we have

$$\partial|\mathbf{u}_a^{n+1}|^2\leq 1-(1-k)|\mathbf{u}_a^{n+1}|^2 \qquad \forall a\in\Lambda \tag{2.4}$$

Assume that $|\mathbf{u}_a^{n+1}|=\|\mathbf{u}^{n+1}\|_\infty$, then clearly $|\mathbf{u}_a^n|\leq\|\mathbf{u}^n\|_\infty$. So by (2.4), implies

$$\partial\|\mathbf{u}^{n+1}\|_\infty^2<1-(1-k)\|\mathbf{u}^{n+1}\|_\infty^2 \tag{2.5}$$

By Lemma 1.1, we have, for any $\varepsilon_1>0$, then there is an $n_1>0$, when $n>n_1$, we have

$$\|\mathbf{u}^n\|_\infty<\frac{1+\varepsilon_1}{1-k} \tag{2.6}$$

First we fix $k<1$ and $\varepsilon_1>0$ and write k as k_0. Set $q=\dfrac{1+\varepsilon_1}{1-k_0}$. Then $\tanh(\theta_c/\theta q)<1$. For simplicity, say $C_0=(\tanh(\theta_c/\theta q))^2$.

Note that the monotonicity of tanh, (2.6) and (1.4b), implies

$$\{\tanh(\beta\mathbf{A}\,\mathbf{u}^n)\}_a\leq\tanh\|\beta\mathbf{A}\,\mathbf{u}^n\|_\infty\leq\tanh(\beta\|\mathbf{A}\,\|_\infty\|\mathbf{u}^n\|_\infty)<\tanh(\theta_c/\theta q) \tag{2.7}$$

By (2.3), similar to the proof of (2.5), we get $\partial\|\mathbf{u}^{n+1}\|_\infty^2<C_0-(1-k)\|\mathbf{u}^{n+1}\|_\infty^2$. By Lemma 1.1, implies, for any $\varepsilon_2>0$, then there exists an $n_1(\|\mathbf{u}_0\|_\infty,\,k_0)>0$, when $n>\max\{n_0,\,n_1\}$, $\|\mathbf{u}^n\|_\infty^2<\dfrac{C_0+\varepsilon_2}{1-k}$. When $k<1-\varepsilon_2-C_0$, we have $\|\mathbf{u}^n\|_\infty<1$. This completes the proof of the Lemma.

Theorem 2.1. *For any $\mathbf{u}_0\in K$ defined by $\{\mathbf{v}\mid\|\mathbf{v}\|_\infty\leq C\}$ (here C is an any positive constant). Then when the time mesh parameter k satisfies the condition of Lemma* 2.1, *then there exists a unique solution, $\mathbf{u}^n$, for* (2.1). *Moreover, there exists a $n_0(K,k_0,\,\|\mathbf{u}_0\|_\infty)>0$, when $n>n_0$, $\mathbf{u}^n\in K_1=\{\mathbf{v}\mid\|\mathbf{v}\|_\infty<1\}$. The mapping $\mathbf{u}_0\to\mathbf{u}^n$ is continuous. Therefore the family solution operators $\{S^n\}_{n>0}$ defined by $S^n\mathbf{u}_0\equiv\mathbf{u}^n$, forms a continuous semigroup on K.*

Proof. Clearly, by Lemma 2.1, there exists $n_0(K,k_0,\,\|\mathbf{u}^0\|_\infty)>0$, when $n>n_0$, $\mathbf{u}^n\in K_1$. So for any $\mathbf{u}_0\in K$, then $\mathbf{u}^n\in K$.

In the following we shall prove the continuity of $\mathbf{u}^n$ with respect to initial data.

Let $\mathbf{u}^n$ and $\mathbf{v}^n$ are solutions of (2.1) under initial data $\mathbf{u}_0$ and V_0, respectively and set $e^n=\mathbf{u}^n-\mathbf{v}^n$. Then by (2.1a) we have

$$\partial e^{n+1}+e^{n+1}=\tanh(\beta A\mathbf{u}^n)-\tanh(\beta A\mathbf{v}^n). \tag{2.8}$$

Note that $|\tanh(x)-\tanh(y)|\leq|\tanh'(x+\eta(x-y))||(x-y)|<|x-y|$

Multiplying the a'th component of the both sides of (2.8) by e_a^{n+1} and similar to the proof (2.4), we have

$$\frac{1}{2}\partial|e_a^{n+1}|^2+\frac{k}{2}|\partial e_a^{n+1}|^2+|e_a^{n+1}|^2<\frac{k}{2}(|e_a^{n+1}|^2+|\partial e_a^{n+1}|^2)+\frac{1}{2}(|e_a^{n+1}|^2+\frac{\theta_c}{\theta}|e_a^n|^2)$$

So
$$\partial\|e^{n+1}\|_\infty < (k-1)\|e^{n+1}\|_\infty^2 + \frac{\theta_c}{\theta}\|e^n\|_\infty^2 \tag{2.9}$$

Summation of (2.9) with respect to n from 1 to m and note that $1 < 1+k(1-k)$ for $k<1$, implies $\|e^m\|_\infty^2 < \|e^0\|_\infty^2 + k\Big(\frac{\theta_c}{\theta}+k-1\Big)\sum_{n=0}^{m-1}\|e^n\|_\infty$ by the discrete Grönwall's inequality, we have $\|e^m\|_\infty^2 < \exp\Big(T_1\frac{\theta_c}{\theta}\Big)\|e^0\|_\infty$, where $km < T_1$. Therefore we get the mapping $\mathbf{u}_0 \to \mathbf{u}^n$ is continuous. So the Theorem is completed.

Theorem 2.2. *For the mesh parameter k, under assumption of Theorem 2.1, the ball $B=\{\mathbf{v} \mid \|\mathbf{v}\|_\infty < 1\}$ is an absorbing set for the semigroup $\{S^n\}_{n>0}$.*

Proof. By the proof of Theorem 2.1, immediately implies that the Theorem holds.

Theorem 2.3. *There exists a global attractor $\mathcal{A}^n \subset V_\Lambda$ for the semigroup $\{S^n\}_{n>0}$. Furthermore, $\mathcal{A}^n$ is connected.*

Proof. Since V_Λ is finite dimensional, so $\{S^n\}_{n>0}$ is uniformly compact. By the result of Theorem 2.2, the existence of a global attractor $\mathcal{A}^n = \omega(\mathbf{B})$ is an immediate consequence of theorem 1.1 in Temam[4]. Therefore the Theorem is completed.

Lemma 2.2.[5] *Suppose that $\gamma^+(\mathbf{u}_0)$ is relatively compact for each $\mathbf{u}_0 \in \mathbf{V}_\Lambda$ and that there exists a Lyapunov functional I on $\mathbf{V}_\Lambda$ under S^n. Then $\omega(\mathbf{u}_0) \subset \mathcal{E}$ for each $\mathbf{u}_0 \in \mathbf{V}_\Lambda$. Furthermore, if $\mathcal{E}$ is bounded then S^n is point dissipative on $\mathbf{V}_\Lambda$. Where $\mathcal{E}$ is the set of equilibria for the steady-state of the equation* (1.1) (*See Lu Bainian* [3]).

Theorem 2.4. *For any $\varepsilon > 0$, if the time mesh k satisfies*
$$k < \frac{1-\varepsilon}{1+\frac{\theta_c}{\theta}/2} \tag{2.10}$$
then the functional defined by (1.9) *is a Lyapunov functional for the semigroup $\{S^n\}_{n>0}$. In addition, for any $\mathbf{u}_0 \in V_\Lambda$, the ω-limit set, $\omega(\mathbf{u}_0)$, is contained in $\mathcal{E}$.*

Proof. (i). First we shall prove that $I(\mathbf{u}^n)$ is a non-increasing functional with respect to n. By (1.9) for any $\mathbf{u}^{n+1}$ and $\mathbf{u}^n$ we have
$$\begin{aligned} I(\mathbf{u}^{n+1}) - I(\mathbf{u}^n) &= (\psi(\mathbf{u}^{n+1}) - \psi(\mathbf{u}^n), \mathbf{e}) - \frac{\beta}{2}[(\mathbf{A}\,\mathbf{u}^{n+1}, \mathbf{u}^{n+1}) - (\mathbf{A}\mathbf{u}^n, \mathbf{u}^n)] \\ &= k(\phi(\mathbf{u}^n + k \text{ diam } \xi_n \partial\mathbf{u}^{n+1}) - \beta\mathbf{A}\mathbf{u}^n, \partial\mathbf{u}^{n+1}) - \frac{\beta k^2}{2}(\mathbf{A}\partial\mathbf{u}^{n+1},\ \partial\mathbf{u}^{n+1}) \end{aligned} \tag{2.11}$$

Where $|(\xi_n)_{aa}| < 1$, so $\|\mathbf{u}^n + k \text{ diag } \xi_n\partial\mathbf{u}^{n+1}\|_\infty < \max(\|\mathbf{u}^n\|_\infty, \|\mathbf{u}^{n+1}\|_\infty)$. By (2.1a), we have $\beta\mathbf{A}\mathbf{u}^n = \phi(\mathbf{u}^n + \partial\mathbf{u}^{n+1})$, so
$$\phi(\mathbf{u}^n + k \text{ diag } \xi_n\partial\mathbf{u}^{n+1}) - \phi(\mathbf{u}^n + \partial\mathbf{u}^{n+1}) = \text{ diag } \phi'(\eta_n)(k \text{ diag } \xi_n - Id)\partial\mathbf{u}^{n+1} \tag{2.12}$$
where Id is a $J^d \times J^d$ unit matrix and η_n is a vector between $\mathbf{u}^n + k \text{ diag } \xi_n\partial\mathbf{u}^{n+1}$ and $\tanh(\beta A\mathbf{u}^n)$, so $\|\eta_n\|_\infty < \max(1, \|\mathbf{u}^n\|_\infty, \|\mathbf{u}^{n+1}\|_\infty)$. Therefore, by (2.11) and (2.12), we have $I(\mathbf{u}^{n+1}) - I(\mathbf{u}^n) = k\Big((k \,\mathrm{diag}\, \xi_n - Id)\,\mathrm{diag}\,\phi'(\eta_n) - \frac{\beta k}{2}\mathbf{A}\Big)\partial\mathbf{u}^{n+1}, \partial\mathbf{u}^{n+1})$. Let $B_1 = (k\,\mathrm{diag}\,\xi_n - Id)\,\mathrm{diag}\,\phi'(\eta_n) - \frac{\beta k}{2}\mathbf{A}$ and $\mathbf{v} = \partial\mathbf{u}^{n+1}$, then

If there exists a $a \in \Lambda$, such that $\mathbf{v}_a \neq 0$, then by Lemma 2.4, we have

$$k(B_1\mathbf{v}, \mathbf{v}) \leq kh^d\Big((k-1)\sum_{a\in\Lambda}(\mathbf{v}_a)^2 + \frac{k\beta}{2}\sum_{a\in\Lambda}\sum_{b\in N(a)}|E_{ab}|\,|\mathbf{v}_a|\,|\mathbf{v}_b|\Big)$$
$$\leq k\Big(k-1+k\frac{\theta_c}{\theta}\Big)\|\partial\mathbf{u}^{n+1}\|_\infty^2.$$

Under the assumption of (2.10), we have $(\mathbf{B_1 v}, \mathbf{v}) < 0$. So $I(\mathbf{u}^{n+1}) < I(\mathbf{u}^n)$, when $\mathbf{v} \neq \mathbf{0}$ i.e. $\mathbf{u}^{n+1} \neq \mathbf{u}^n$.

If for any $a \in \Lambda$, have $\mathbf{v}_a = 0$, then $\mathbf{u}^{n+1} = \mathbf{u}^n$, so $\mathbf{u}^n \equiv \mathbf{u}^* \in \mathcal{E}$.

(ii). Next we shall prove that if $I(\mathbf{u}^n) = I(\mathbf{u}_0)$ then $\mathbf{u}_0 \in \mathcal{E}$.

If $I(\mathbf{u}^n) = I(\mathbf{u}_0)$, then $I(\mathbf{u}^{n+1}) = I(\mathbf{u}^n)$, by (2.10) and (2.11) we have $0 \leq -\dfrac{\varepsilon(1-\varepsilon)}{1+\frac{\theta_c}{\theta}/2}\|\partial\mathbf{u}^{n+1}\|_\infty^2$, then $\|\partial\mathbf{u}^{n+1}\|_\infty^2 = 0$. So $\mathbf{u}^{n+1} = \mathbf{u}^n \equiv \mathbf{u}^* \in \mathcal{E}$.

(iii). Finally we shall prove that $I(\mathbf{u}^n)$ is bounded.

For any $n \in Z$, we have $I(\mathbf{u}^n) \geq -\ln 2\dfrac{1}{1-k} - \dfrac{\theta_c}{\theta} = -C_4$. This completes the $I(\mathbf{u}^n)$ is a Lyapunov functional. By Lemma 2.2, clearly the Theorem holds.

Remark 2.1. The restrictions on the time step k are independent of the lattice size.

3. The Predictor-Corrector Approximation

In this section we shall give other explicit finite difference scheme: predictor-corrector scheme given by

$$\begin{cases} \dfrac{\tilde{\mathbf{u}}^{n+1}-\mathbf{u}^n}{k} + \mathbf{u}^n = \tanh(\beta A\mathbf{u}^n) & n = 1, 2, \cdots \qquad (3.1a) \\ \partial\mathbf{u}^{n+1} + \mathbf{u}^{n+1/2} = \tanh(\beta A\mathbf{u}^{n+1/2}) & n = 1, 2, \cdots \qquad (3.1b) \\ \mathbf{u}^0 = \mathbf{u}_0 & \qquad (3.1c) \end{cases}$$

Where $\mathbf{u}^{n+1/2} = (\tilde{\mathbf{u}}^{n+1} + \mathbf{u}^n)/2$.

Lemma 3.1. *Let* $f(k) = 1 - k\Big(\dfrac{5}{2} - k + \dfrac{k^2}{4}\Big)$*, then there exists a unique positive root* $k^* \in \Big(0, \dfrac{1}{2}\Big)$ *and for any* $k \in (0, k^*)$ *we have* $f(k) > 0$*. Where the approximative value of* $k^* = 0.481608$.

Proof. Because $f'(k) = -\dfrac{5}{2} + 2k - \dfrac{3k^2}{4}$, implies $f'(k)$ does not exist zero point. So by $f'(0) = -\dfrac{5}{2} < 0$, we have $f'(k) < 0$, $\forall k \in R$. Therefore $f(k)$ is a strictly decreasing function. Because $f(0)f\Big(\dfrac{1}{2}\Big) < 0$, then the Lemma holds.

Theorem 3.1. *For any* $\mathbf{u}_0 \in K$ *defined by* $\{\mathbf{v} \mid \|\mathbf{v}\|_\infty \leq C\}$*. Then there exists a unique solution,* $\mathbf{u}^n$*, for the schemes* (3.1)*. Moreover, for any* $\delta \in (0, k^*)$ *and* $\varepsilon > 0$*, when the time mesh parameter satisfies*

$$k < \min\Big\{\delta, \frac{4(1-C_0(1+\varepsilon))}{2C_0+10-4\delta}\Big\}$$

there exists an $N > 0$*, when* $n > N$*,* $\mathbf{u}^n \in K_1 = \{\mathbf{v} \,|\, \|\mathbf{v}\|_\infty < 1\}$*. The mapping* $\mathbf{u}_0 \to \mathbf{u}^n$ *is continuous. Therefore the family solution operators* $\{S^n\}_{n>0}$ *defined by* $S^n\mathbf{u}_0 \equiv \mathbf{u}^n$ *for* (3.1)*, forms a continuous semigroup on* K*. Where* $q = [(1+\delta/2+\varepsilon)/(1-\delta(5/2-\delta+\delta^2/4)]^{1/2}$*,* $C_0^{1/2} = \tanh\Big(\dfrac{\theta_c}{\theta}q\Big)$*.*

Proof. By (3.1), we have

$$\tilde{\mathbf{u}}^{n+1} = (1-k)\mathbf{u}^n + k\tanh(\beta A\mathbf{u}^n) \tag{3.2}$$

Substitute (3.2) into (3.1b), follows

$$\partial \mathbf{u}^{n+1} + (1-k/2)\mathbf{u}^n + \frac{k}{2}\tanh(\beta A\mathbf{u}^n) = \tanh\Big(\beta A\frac{\mathbf{u}^n + \tilde{\mathbf{u}}^{n+1}}{2}\Big) \tag{3.3}$$

Multiplying $\mathbf{u}_a^{n+1}$ to the a'th component of the both sides of (3.3), similar to (2.2), implies

$$\begin{aligned}\frac{1}{2}\partial|\mathbf{u}_a^{n+1}|^2 + \frac{k}{2}|\partial\mathbf{u}_a^{n+1}|^2 + \Big(1-\frac{k}{2}\Big)|\mathbf{u}_a^{n+1}|^2 &= k\Big(1-\frac{k}{2}\Big)\mathbf{u}_a^{n+1}\partial\mathbf{u}_a^{n+1}\\ &- \frac{k}{2}\mathbf{u}_a^{n+1}(\tanh(\beta A\mathbf{u}^n))_a + \mathbf{u}_a^{n+1}(\tanh(\beta A\mathbf{u}^{n+1/2}))_a\end{aligned} \tag{3.4}$$

Therefore, by (3.4) we can get

$$\partial\|\mathbf{u}^{n+1}\|_\infty^2 < 1 + \frac{k}{2} - \Big[1 - k\Big(\frac{5}{2} - k + \frac{k^2}{4}\Big)\Big]\|\mathbf{u}^{n+1}\|_\infty^2 \tag{3.5}$$

By Lemmas 1.1 and 3.1, when $k < k^*$, there exists an $N > 0$, when $n > N$, we have

$$\|\mathbf{u}^n\|_\infty < \sqrt{\frac{1+\frac{k}{2}+\varepsilon}{1-k(\frac{5}{2}-k+\frac{k^2}{4})}} \tag{3.6}$$

under the assumption of the Theorem, first we take $k = \delta$, we have $\|\mathbf{u}^n\|_\infty \le q$.

Similar to the proof of Lemma 2.1, there exists an $N > 0$, when $n > N$ and $k \le \dfrac{4(1-C_0(1+\varepsilon))}{2C_0+10-4\delta}$, we have $\|\mathbf{u}^n\|_\infty < \sqrt{\dfrac{(1+\frac{k}{2}+\varepsilon)C_0}{1-k(\frac{5}{2}-k+\frac{k^2}{4})}} < 1$.

In the following we shall prove that S^n is continuous: let $e^n = \mathbf{u}^n - \mathbf{v}^n$, by (3.2) and (3.3), we have

$$\partial e^{n+1} + \Big(1-\frac{k}{2}\Big)e^n = \frac{k\beta}{2}[\beta f'(\eta_1^n)\mathbf{A}\, f'(\eta_2^n) - f'(\eta_3^n)]\mathbf{A}e^n + \Big(1-\frac{k}{2}\Big)\beta f'(\eta_1^n)\mathbf{A}e^n \tag{3.7}$$

Where $f(\mathbf{u}) = \tanh(\beta A\mathbf{u})$, $f'(\eta_j^n) = \mathrm{diag}\,(f'(\eta_{j\,a}^n))_{a\in\Lambda}$, $j = 1,2,3$.

Similar to the proof of the Theorem 2.1, follows

$$\partial\|e^{n+1}\|_\infty^2 + k\Big(1-k\Big(\frac{1}{2}-\frac{k}{4}\Big)\Big)\|\partial e^{n+1}\|_\infty^2 \le \|e^{n+1}\|_\infty^2 + \Big(\frac{k\{\frac{\theta_c}{\theta}\}^2}{2} + \frac{\theta_c}{\theta}\Big)\|e^n\|_\infty^2 \tag{3.8}$$

By the discrete Gronwall's inequality, we have $\|e^n\|_\infty^2 \le \dfrac{1}{1-\delta}\exp\Big(\Big(\dfrac{k\{\frac{\theta_c}{\theta}\}^2}{2} + \dfrac{\theta_c}{\theta}\Big)T_1\Big)$ $\|e(0)\|_\infty^2$, $\forall t \in (0, T_1)$. Then we can get the semigroup $\{S^n\}_{n\ge 0}$ is continuous.

Similar to the proof of the Theorems 2.2 and 2.3, we can get the following results:

Theorem 3.2. *Under the assumption of Theorem* 3.1*, the ball* $B = \{\mathbf{v} \mid \|\mathbf{v}\|_\infty < 1\}$ *is a absorbing set for the semigroup* $\{S^n\}_{n>0}$ *of* (3.1)*.*

Theorem 3.3. *There exists a global attractor* $\mathcal{A}^n \subset V_\Lambda$ *for the semigroup* $\{S^n\}_{n>0}$ *of* (3.1)*. Furthermore,* $\mathcal{A}^n$ *is connected.*

Lemma 3.2. *Assume that* $\tilde{\mathbf{u}}^{n+1}$, $\mathbf{u}^n$ *are the solutions of* (3.1)*, For any* $\mathbf{u}_0 \in \mathbf{V}_\Lambda$ *and any* $\delta \in (0, k^*)$*, then there exists an* $N(\mathbf{u}_0) > 0$*, when* $n > N$*, for any* $C_1 \in (C_0, 1)$*, when the time size satisfies:*

$$k \leq \min\Big\{\frac{1}{2}, \frac{4(C_1 - C_0)}{2C_0 + (10 - 4\delta)C_1}\Big\}$$

the following result hold $\max\{\|\tilde{\mathbf{u}}^{n+1}\|_\infty, \|\mathbf{u}^n\|_\infty, \|\mathbf{u}^{n+1/2}\|_\infty\} \leq C_1 < 1$*. Where* C_0 *is defined in the Theorem* 3.1*.*

Proof. First, we shall prove that $\|\mathbf{u}^n\|_\infty$ and $\|\tilde{\mathbf{u}}^{n+1}\|_\infty$ are bounded.

Similar to the proof of (2.3), we have

$$\frac{\|\tilde{\mathbf{u}}^{n+1}\|^2 - \|\mathbf{u}^n\|^2}{k} < 1 - (1 - k)\|\tilde{\mathbf{u}}^{n+1}\|^2 \tag{3.9}$$

So by the Lemmas 1.1, there exists an $N(\mathbf{u}_0)$, when $n > N$ and $k < 1$, we have $\|\tilde{\mathbf{u}}^{n+1}\|_\infty^2 < 1/(1 - k)$. By calculation, we have $1/(1 - k) < q$, where q is defined in Theorem 3.1. So by (3.6) and (3.9) and the assumption of the Theorem, when

$$k < \min\Big\{\frac{4(1 - C_0)}{2C_0 + 10 - 4\delta}, \frac{1}{2}\Big\}$$

we have $\|\mathbf{u}^n\|_\infty < q$.

Then we shall prove that $\|\mathbf{u}^n\|_\infty$ and $\|\mathbf{u}\|_\infty$ are less then one.

By the definition of C_0, similar to the estimate of (3.5), we have

$$\partial\|\mathbf{u}^{n+1}\|_\infty^2 < \Big(1 + \frac{k}{2}\Big)C_0 - \Big[1 - k\Big(\frac{5}{2} - k + \frac{k^2}{4}\Big)\Big]\|\mathbf{u}^{n+1}\|_\infty^2 \tag{3.10}$$

By Lemmas 1.1, we have, for any $\varepsilon > 0$, there exists an $N > 0$, when $n > N$

$$\|\mathbf{u}^n\|_\infty^2 < \frac{(1 + \frac{k}{2} + \varepsilon)C_0}{1 - k(\frac{5}{2} - k + \frac{k^2}{4})} \tag{3.11}$$

when $k \leq \dfrac{4(C_1 - C_0)}{2C_0 + (10 - 4\delta)C_1}$, implies, $\|\mathbf{u}^n\|_\infty^2 \leq C_1 < 1$. Because

$$\frac{4(C_1 - C_0)}{2C_0 + (10 - 4\delta)C_1} < \frac{4(1 - C_0)}{2C_0 + 10 - 4\delta},$$

similar to the proof above, implies when $k < 1 - \frac{C_0}{C_1}$, then, $\|\mathbf{u}\|_\infty^2 \leq \dfrac{C_0}{1 - k} < C_1$.

Because
$$\frac{4(C_1 - C_0)}{2C_0 + (10 - 4\delta)C_1} < 1 - \frac{C_0}{C_1}$$

So when the time size satisfies the assumption of the Theorem, we have the Theorem holds. Therefore this Lemma is completed.

Theorem 3.4. *Under the condition of Lemma* 3.2, *the functional defined by* (1.9) *is a Lyapunov functional for the semigroup* $\{S^n\}_{n>0}$ *of* (3.1) *under restriction for the time mesh parameter*

$$k < \min\Big\{\frac{2-\delta}{\frac{\theta_c}{\theta}+1}, \frac{2\delta^2(1-C_2^2)}{(2+\frac{\theta_c}{\theta})\delta^2(1-C_2^2)+4\frac{\theta_c}{\theta})}\Big\} \quad \forall\delta > 0.$$

Where $C_2 = \phi'(C_1)$ *and* C_1 *is defined in the Theorem* 3.1. *In addition, for any* $\mathbf{u}_0 \in V_\Lambda$, *the* ω*-limit set,* $\omega(\mathbf{u}_0)$*, is contained in* $\mathcal{E}$.

Proof. By (3.1a), we have

$$\beta\mathbf{A}\mathbf{u}^n = \phi\Big(\mathbf{u}^n + \frac{\mathbf{u}-\mathbf{u}^n}{k}\Big) \tag{3.12}$$

So, follows

$$\begin{aligned}\phi(\mathbf{u}^n + k\,\mathrm{diag}\,\xi_n\partial\mathbf{u}^{n+1}) - \beta\mathbf{A}\mathbf{u}^n &= \phi(\mathbf{u}^n + k\,\mathrm{diag}\;\xi_n\partial\mathbf{u}^{n+1}) - \phi\Big(\mathbf{u}^n + \frac{\mathbf{u}-\mathbf{u}^n}{k}\Big)\\ &= \mathrm{diag}\,\phi'(\eta_n)\Big(k\,\mathrm{diag}\,\xi_n\partial\mathbf{u}^{n+1} - \frac{\mathbf{u}-\mathbf{u}^n}{k}\Big)\end{aligned} \tag{3.13}$$

substitute (3.1a) from (3.1b), yields

$$\frac{\mathbf{u}^{n+1}-\mathbf{u}}{k} + \frac{\mathbf{u}-\mathbf{u}^n}{2} = \mathrm{diag}\,\phi'(\eta_n)\beta\mathbf{A}\frac{\mathbf{u}-\mathbf{u}^n}{2} \tag{3.14}$$

rewrite (3.14) as

$$\frac{\mathbf{u}^{n+1}-\mathbf{u}}{k} = (\mathrm{diag}\,\phi'(\eta_n)\ beta\mathbf{A} - \mathbf{Id})\frac{\mathbf{u}-\mathbf{u}^n}{2} \tag{3.15}$$

By (3.15), we have

$$\frac{\mathbf{u}-\mathbf{u}^n}{k} = \partial\mathbf{u}^{n+1} - \frac{\mathbf{u}^{n+1}-\mathbf{u}}{k} = \partial\mathbf{u}^{n+1} - \frac{k}{2}(\mathrm{diag}\ \tanh'(\eta_n)\beta\mathbf{A} - \mathbf{Id})\frac{\mathbf{u}-\mathbf{u}^n}{k} \tag{3.16}$$

By (3.16), we have

$$[\mathbf{Id} - \frac{k}{2}(\mathbf{Id} - \mathrm{diag}\ \tanh'(\eta_n)\beta\mathbf{A})]\frac{\mathbf{u}-\mathbf{u}^n}{k} = \partial\mathbf{u}^{n+1} \tag{3.17}$$

Let $\mathbf{D} = \dfrac{k}{2-k}\ \mathrm{diag}\ \tanh'(\eta_n)\beta\mathbf{A}$, and $\mathbf{B_2} = \Big(1-\dfrac{k}{2}\Big)(\mathbf{Id}+\mathbf{D})$. When the time size $k < \min\Big\{\dfrac{2-\delta}{\frac{\theta_c}{\theta}+1},\ 2-\delta\}$, $\forall\delta > 0$ we have $\mathbf{B_2}$ is a invertible matrix. And $\mathbf{B}_2^{-1} = \dfrac{2}{2-k}(\mathbf{Id}-\mathbf{D})^{-1} = \dfrac{2}{2-k}(\mathbf{Id}+\mathbf{D}(\mathbf{Id}-\mathbf{D})^{-\mathbf{1}})$. So, by (3.17), implies

$$\frac{\mathbf{u}-\mathbf{u}^n}{k} = \mathbf{B}_2^{-1}\partial\mathbf{u}^{n+1} \tag{3.18}$$

Substitution (3.18) into (3.14), we have

$$\phi(\mathbf{u}^n + k \text{ diag } \xi_n \partial \mathbf{u}^{n+1}) - \beta \mathbf{A}\mathbf{u}^n = \text{ diam } \phi'(\eta_n)(k \text{ diam } \xi_n - \mathbf{B}_2^{-1})\partial \mathbf{u}^{n+1} \quad (3.19)$$

Let $\mathbf{B}_3 = \text{ diam } \phi'(\eta_n)(k \text{ diam } \xi_n - \mathbf{B}_2^{-1}) - \frac{\beta k}{2}\mathbf{A}$.

Then by (1.9) and (2.11), similar to the proof of Theorem 2.4, we have

$$I(\mathbf{u}^{n+1}) - I(\mathbf{u}^n) = k(\mathbf{B}_3\mathbf{v}, \mathbf{v}) \quad (3.20)$$

Where $\mathbf{v} = \partial \mathbf{u}^{n+1}$.

Because

$$-(\text{ diam } \phi'(\eta_n)\mathbf{B}_2^{-1}\mathbf{v}, \mathbf{v}) = -\frac{2}{2-k}[(\text{ diam } \phi'(\eta_n)\mathbf{v}, \mathbf{v}) + (\text{ diam } \phi'(\eta_n)\mathbf{D}(\mathbf{Id}-\mathbf{D})^{-1}\mathbf{v}, \mathbf{v})] \quad (3.21)$$

Therefore by (3.20) and (3.21), we have

$$(\mathbf{B}_3\mathbf{v}, \mathbf{v}) = \Big(\text{ diam } \phi'(\eta_n)\Big(k \text{ diag } \xi_n - \frac{2}{2-k}\mathbf{Id}\Big)\mathbf{v}, \mathbf{v}\Big) - \Big(\Big(\frac{2}{2-k}\text{diag } \phi'(\eta_n)\mathbf{D}(\mathbf{Id}-\mathbf{D})^{-1} - \frac{k\beta}{2}\mathbf{A}\Big)\mathbf{v}, \mathbf{v}\Big) \quad (3.22)$$

Because $\mathbf{A}$ is a symmetric matrix, then by Lemma 1.1, we have $\|\mathbf{D}\| \le \frac{k}{2-k}\|\beta\mathbf{A}\|$ and $\rho(\beta\mathbf{A}) = \|\beta\mathbf{A}\| \le \frac{\theta_c}{\theta}$. Then $\|\mathbf{D}\| \le \frac{k}{2-k}\frac{\theta_c}{\theta}$, $\|(\mathbf{Id}-\mathbf{D})^{-1}\| \le \frac{1}{1-\|D\|} \le \frac{2-k}{2-k-k\frac{\theta_c}{\theta}}$.

Similar to the proof of Theorem 2.4, by (3.22), when $k<1$,we have

$$(\mathbf{B}_3\mathbf{v}, \mathbf{v}) < \Big(-\frac{2}{2-k} + k + \frac{2k\frac{\theta_c}{\theta}}{(2-k)(1-C_2^2)(2-k-k\frac{\theta_c}{\theta})} + \frac{k\frac{\theta_c}{\theta}}{2}\Big)\|\mathbf{v}\|^2 \quad (3.23)$$

when the time size k satisfies the restriction of the Theorem, we have

$$(\mathbf{B}_3\mathbf{v}, \mathbf{v}) < 0 \quad (3.24)$$

Theorem $I(\mathbf{u}^{n+1}) < I(\mathbf{u}^n)$.

4. Some Implicit Schemes

We first give some implicit schemes as following

$$\begin{cases} \partial \mathbf{u}^{n+1} + \mathbf{u}^{n+1} = \tanh(\beta A \mathbf{u}^n) \quad n = 1, 2, \cdots & (4.1a) \\ \mathbf{u}^0 = \mathbf{u}_0 & (41b) \end{cases}$$

And

$$\begin{cases} \partial \mathbf{u}^{n+1} + \mathbf{u}^{n+1} = \tanh(\beta A \mathbf{u}^{n+1}) \quad n = 1, 2, \cdots & (4.2a) \\ \mathbf{u}^0 = \mathbf{u}_0 & (4.2b) \end{cases}$$

In the following we shall discuss absorbing sets and Lyapunov functional for the schemes above:

Theorem 4.1. *For any* $\mathbf{u}_0 \in K$ *defined by* $\{\mathbf{v} \mid \|\mathbf{v}\|_\infty \le C\}$. *Then there exists a unique solution for the schemes* (4.1) *and* (4.2) *provided that* $k < \min\Big\{\dfrac{C}{1+C}, \dfrac{1}{1+\frac{\theta_c}{\theta}}\Big\}$ *and* $k < \min\{1/C, 1\}$ *respectively, say* $\mathbf{u}^n$. *Moreover, there exists an* $N > 0$, *when* $n > N$, $\mathbf{u}^n \in K_1 = \{\mathbf{v} \mid \|\mathbf{v}\|_\infty < 1\}$ $\forall n > 0$. *The mapping* $\mathbf{u}_0 \to \mathbf{u}^n$ *is continuous. Therefore the family solution operators* $\{S^n\}_{n>0}$ *defined by* $S^n\mathbf{u}_0 \equiv \mathbf{u}^n$ *for* (4.1) *and* (4.2), *forms a continuous semigroup on* K.

Proof. Fist we shall prove the existence for the schemes (4.1) and (4.2). Let $K_2 = \{\mathbf{v} \mid \|\mathbf{v} - \mathbf{u}^n\|_\infty \le C\}$ then by (4.1) and (4.2), we imply for any $\mathbf{u}^{n+1} \in K$, let $f(\mathbf{u}) = \mathbf{u}^n + k(\tanh(\beta AW) - \mathbf{u}^{n+1})$, where $W = \mathbf{u}^n$ or $\mathbf{u}^{n+1}$. then it is not difficult to prove $f(\mathbf{u})$ is a contractive mapping provided $k < \min\Big\{\dfrac{C}{1+C}, \dfrac{1}{1+\frac{\theta_c}{\theta}}\Big\}$ for (4.1) and $k < \min\{1/C, 1\}$ for (4.2). Therefore there exists a unique solution $\mathbf{u}^n$ for the schemes (4.1) and (4.2).

Then, we shall prove $\mathbf{u}^n \in K_1$. Multiplying the both sides of the a'th component of (4.1a) and (4.2a), we have

$$\frac{1}{2}\partial|\mathbf{u}_a^{n+1}|^2 + |\mathbf{u}_a^{n+1}|^2 = \mathbf{u}_a^{n+1}(\tanh(\beta AW))_a \tag{4.3}$$

Similar to the proof of Lemma 2.1, we have

$$\partial\|\mathbf{u}^{n+1}\|_\infty^2 < 1 - \|\mathbf{u}^{n+1}\|_\infty^2 \tag{4.4}$$

by the Lemmas 1.1 and 2.1, similar to the proof of §2, we get $\mathbf{u}^n \in K_1$.

Similar to the proof of §2, the Theorem is completed.

Theorem 4.2. *The ball* $B = \{\mathbf{v} \mid \|\mathbf{v}\|_\infty < 1\}$ *is an absorbing set for the semigroup* $\{S^n\}_{n>0}$ *of* (4.1) *and* (4.2).

Theorem 4.3. *There exists a global attractor* $\mathcal{A}^n \subset V_\Lambda$ *for the semigroup* $\{S^n\}_{n>0}$ *of* (4.1) *and* (4.2). *Furthermore,* $\mathcal{A}^n$ *is connected.*

Theorem 4.4. *The functional defined by* (1.9) *is a Lyapunov functional for the semigroup* $\{S^n\}_{n>0}$ *of* (4.1) *under restriction for the time mesh parameter* $k < 1/\frac{\theta_c}{\theta}$ *and of* (4.2) *under restriction for the time parameter* $k < 1/(1 + \frac{\theta_c}{\theta})$, *respectively. In addition, for any* $\mathbf{u}_0 \in \mathbf{V}_\Lambda$, *the* ω*-limit set,* $\omega(\mathbf{u}_0)$, *is contained in* $\mathcal{E}$.

Proof. For the scheme (4.1) similar to the proof of the Theorem 2.4, from (4.1a), we have $\beta\mathbf{A}\,\mathbf{u}^n = \phi(\partial\mathbf{u}^{n+1} + \mathbf{u}^{n+1})$. So, $\phi(\mathbf{u}^n + k \text{ diam } \xi_n\partial\mathbf{u}^{n+1}) - \phi(\mathbf{u}^{n+1} + \partial\mathbf{u}^{n+1}) = \phi'(\eta_n)(k \text{ diam } \xi_n - (1+k)Id)\partial\mathbf{u}^{n+1}$. Therefore we have

$$\begin{aligned} I(\mathbf{u}^{n+1}) - I(\mathbf{u}^n) &= k\Big(\Big((k \text{ diam } \xi_n - (1+k)Id)\text{diag } \phi'(\eta_n) - \frac{\beta k}{2}\mathbf{A}\Big)\partial\mathbf{u}^{n+1},\ \partial\mathbf{u}^{n+1}\Big) \\ &< k\Big(1 - k\frac{\theta_c}{\theta}\Big)\|\partial\mathbf{u}^{n+1}\|^2. \end{aligned}$$

Therefore, when $k \le 1/\dfrac{\theta_c}{\theta}$, we have $I(\mathbf{u}^{n+1}) < I(\mathbf{u}^n)$.

Similar to the prove above we have the result holds for the scheme (4.2).

5. Numerical Results

In this section some results of the simulations on the equation set (1.1) using the numerical approximation schemes are mentioned in this paper. Where $E_{ab} = 1$ for any $a \in N(b)$. The numerical results are almost the same. However we prefer the predictor-corrector approximation because the local error is smaller. Explicit methods are particularly useful because the time step restrictions are independent of the size of the lattice and they do not require the solution of algebraic equation. The mesh has been used on the square $[0, 100] \times [0, 100]$. We take that $h = 0.01$ and $k = 0.1$ and initial data as random numbers. The simulations show that there exists property of the moving by mean curvature[6] and the features of rapid phase separation[7]. It illustrates that the equation (1.1) is a better approximation to Cahn-Allen equation[1]. **Figure** 1 shows the picture of the initial data as random number. **Figure** 2 to **Figure** 13 show the pictures of the numerical solution at different time levels respectively t=20, t=40, t=60, t=80, t=100, t=200, t=300, t=400, t=500, t=5300, t=5400 and t=5500. Last figure **Figure** 14 shows that the Lyapunov function i.e. energy function is decreasing with time t and when $t > 5500$, it becomes a constant. It just shows that the Lyapunov function arrives to minimum value. So the constant is a attractor.

We have simulated a lot of examples in one dimension. We shall ignore those pictures. The figures show that irregular wells will first tend to circles and then tend to a constant with the time t increase.

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