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CONVERGENCE OF A CONSERVATIVE DIFFERENCE SCHEME FOR THE ZAKHAROV EQUATIONS IN TWO DIMENSIONS*

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Abstract

A conservative difference scheme is presented for the initial-boundary-value problem of a generalized Zakharov equations. On the basis of a prior estimates in L_2 norm, the convergence of the difference solution is proved in order $O(h^2+r^2)$. In the proof, a new skill is used to deal with the term of difference quotient $(e^n_{j,k})t$. This is necessary, since there is no estimate of $E(x,y,t)$ in L_∞ norm.

1. Introduction

The Zakharov equations describe physical phenomena in Plasma[12]. The global existence of a weak solution for the Zakharov equations was considered by Sulem and Sulem in [11]. The existence and uniqueness of a smooth solution in one dimension are proved provided that smooth initial data are described. For small initial data, the existence of a weak solution for the Zakharov equations in two and three dimensions is obtained.

Numerical methods for the Zakharov equations in one dimension were considered in [1], [2], [4], [5] and [10]. A spectral method is used to compute solitary waves in [10]. In [4] and [5], Glassey considered an implicit difference scheme for the equations and proved its convergence in order $O(h+\tau)$. A new conservative difference scheme with a parameter $\theta, 0\le\theta\le\dfrac{1}{2}$ was presented in [2]. If $\theta=\dfrac{1}{2}$, the new scheme is identical to Glassey's scheme. For $\theta=0$ the new scheme is semi-explicit. In [1], we considered this semi-explicit scheme for generalized Zakharov equations and improved method of proof to get convergence in order$O(h^2+\tau^2)$. Numerical experiments demonstrate that the new scheme with $\theta=0$ is more accurate and efficient.

In this paper we consider the following periodic initial-value problem in two dimensions:

$$iE_t+E_{xx}+E_{yy}-NE=0, \quad \text{in } \Omega=(0,1)\times(0,1), \tag{1.1}$$

$$N_{tt}-N_{xx}-N_{yy}=(|E|^2)_{xx}, \quad \text{in } \Omega, \tag{1.2}$$

$$E|_{t=0}=E_0(x,y), \quad N|_{t=0}=N_0(x,y), \quad N_t|_{t=0}=N_1(x,y), \tag{1.3}$$

$$E(x+1,y,t)=E(x,y,t), \quad E(x,y+1,t)=E(x,y,t,)$$

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$$N(x+1,y,t) = N(x,y,t), \quad N(x,y+1,t) = E(x,y,t,), \tag{1.4}$$

where a complex unknown function E is the slowly varying envelope of highly oscillatory electric field and a real unknown function N denotes the fluctuation in the ion-density about its equilibrium value, $E_0(x,y), N_0(x,y)$ and $N_1(x,y)$ are periodic functions, $N_1(x,y)$ satisfies the compatibility condition:

$$\iint_\Omega N_1(x,y)dxdy = 0. \tag{1.5}$$

The periodic initial-value problem (1.1)–(1.5) possesses two conservative quantities:

$$\|E\|_{L^2}^2 = \text{const.} \tag{1.6}$$

and

$$\|E_x\|_{L^2}^2 + \|E_y\|_{L^2}^2 + \frac{1}{2}\|N\|_{L^2}^2 + \frac{1}{2}(\|u_x\|_{L^2}^2 + \|u_y\|_{L^2}^2) + \iint_\Omega N|E|^2 dxdy = \text{ Const.}, \tag{1.7}$$

where the potential function u is given by

$$u_{xx} + u_{yy} = N_t. \tag{1.8}$$

Assume that $E_0 \in H^1(\Omega), N_0 \in L_2(\Omega),\ N_1 \in H^{-1}(\Omega)$ and $\|E_0\|_{L_2} < \dfrac{1}{\sqrt{8}}$, then there exists a weak solution $E \in L^\infty(R^+,\ H^1(\Omega))$, $N \in L^\infty(R^+,\ L^2(\Omega))$ for the problem (1.1)–(1.5) (see [12]).

We propose an implicit conservative difference scheme for the problem (1.1)–(1.5) in this paper. We will prove the convergence of the difference solution in order $O(h^2+\tau^2)$. In the proof, a new skill is used to deal with the term $(e^n_{j,k})_t$. this is necessary, since there is no estimate of $E(x,y,t)$ in L_∞ norm.

In section 2, we describe the difference scheme and its basic properties. Some prior estimates and proof of the convergence of the difference solution are given in Section 3.

2. Finite difference Scheme

In this section, the finite difference method for the problem (1.1)–(1.5) is considered. As usual, the following notations are used

$$
\begin{aligned}
& h_x = \frac{1}{J}, \quad h_y = \frac{1}{K}, \\
& x_j = jh_x, \quad y_k = kh_y, \quad t^n = n\tau, \\
& E(j,k,n) \equiv E(x_j,y_k,t^n), \quad N(j,k,n) \equiv N(x_j,y_k,t^n), \\
& E_{j,k} \sim E(j,k,n), \quad N^n_{j,k} \sim N(j,k,n), \\
& (W^n_{j,k})_x = \frac{1}{h_x}(W^n_{j+1,k} - w^n_{j,k}), \quad (W^n_{j,k})_{\bar{x}} = \frac{1}{h_x}(W^n_{j,k} - W^n_{j-1,k}), \\
& W^{n+\frac{1}{2}}_{j,k} = \frac{1}{2}(W^{n+1}_{j,k} + W^n_{j,k}), \quad \|W^n\|_2^2 = h_x h_y \sum_{j=1}^{J}\sum_{k=1}^{K} |W^n_{j,k}|^2,
\end{aligned}
$$

$$\|W^n\|_\infty = \sup_{\substack{1\le j\le J\\1\le k\le K}} |W^n_{j,k}|,$$

and in this paper C denotes a general constant, which may have different values in different occurrences. Thus, the difference scheme for the Zakharov equations is given as

$$i(E^n_{j,k})_t + ((E^{n+\frac{1}{2}}_{j,k})_{y\bar{y}}) - N^{n+\frac{1}{2}}_{j,k} \cdot E^{n+\frac{1}{2}}_{j,k} = 0,$$

$$1 \le j \le J, 1 \le k \le K, n = 0, 1, \cdots, \left[\frac{T}{\tau}\right], \tag{2.1}$$

$$(N^n_{j,k})_{t\bar{t}} - \frac{1}{2}((N^{n+1}_{j,k})_{x\bar{x}} + (N^{n-1}_{j,k})_{x\bar{x}} + (N^{n+1}_{j,k})_{y\bar{y}} + (N^{n-1}_{j,k})_{y\bar{y}}) = (|E^n_{j,k}|^2)_{x\bar{x}} + (|E^n_{j,k}|^2)_{y\bar{y}},$$

$$1 \le j \le J, 1 \le k \le K, n = 0, 1, \cdots, \left[\frac{T}{\tau}\right]. \tag{2.2}$$

The initial conditions are approximated as

$$E^0_{j,k} = E^0(x_j, y_k), \quad N^0_{j,k} = N^0(x_j, y_k), \tag{2.3}$$

$$N^1_{j,k} = N^0_{j,k} + \tau N_1(x_j, y_k). \tag{2.4}$$

The periodic conditions are given as

$$\begin{aligned} E^n_{j+J,k} &= E^n_{j,k}, \qquad E^n_{j,k+K} = E^n_{j,k}, \\ N^n_{j+J,k} &= N^n_{j,k}, \qquad N^n_{j,k+K} = N^n_{j,k}. \end{aligned} \tag{2.5}$$

We define the potential function $u^{n+\frac{1}{2}}_{j,k}$ by

$$(u^{n+\frac{1}{2}}_{j,k})_{x\bar{x}} + (u^{n+\frac{1}{2}}_{j,k})_{y\bar{y}} = (N^n_{j,k})_t, u^{n+\frac{1}{2}}_{0,k} = u^{n+\frac{1}{2}}_{J,k} = u^{n+\frac{1}{2}}_{j,0} = u^{n+\frac{1}{2}}_{j,K} = 0. \tag{2.6}$$

In computation, $N^0_{j,k}, N^1_{j,k}, E^0_{j,k}$ are obtained from the initial conditions (2.3) and (2.4). Putting $n = 0$ in (2.1), $E^1_{j,k}$ is solved. Putting $n = 1$ in (2.1) and solve for $E^2_{j,k}$, etc. We note that the scheme (2.1)–(2.5) is implicit, but the equations (2.2) are linear for $N^{n+1}_{j,k}$. The scheme (2.1) is also linear for $E^{n+1}_{j,k}$ if $N^{n+1}_{j,k}$ is known. In practical computation, we need only to solve two five-diagonal systems of equations in each step of time.

Theorem 1. *Assume $E_0(x,y) \in H^1(\Omega)$, $N_0(x,y) \in L^2(\Omega)$, $N_1(x,y) \in L^2(\Omega)$. The difference problem (2.1)–(2.5) possesses the following invariants $\|E^n\|_2^2 =$ Const. and*

$$\begin{aligned} H^n_h = &\|E^n_x\|_2^2 + \|E^n_y\|_2^2 + \frac{1}{2}\|u_x^{n-\frac{1}{2}}\|_2^2 + \frac{1}{2}\|u_y^{n-\frac{1}{2}}\|_2^2 \\ &+ \frac{1}{4}(\|N^n\|_2^2 + \|N^{n-1}\|_2^2) + h_x h_y \sum_{j=1}^{J}\sum_{k=1}^{K} N^{n-\frac{1}{2}}_{j,k} |E^n_{j,k}|^2 = \text{ Const.} \end{aligned}$$

Proof. Computing the inner product of (2.1) with $(\overline{E^{n+1}_{j,k} + E^n_{j,k}})$ yields

$$i((E^n_{j,k})_t, \overline{E^{n+1}_{j,k} + E^n_{j,k}}) + ((E^{n+\frac{1}{2}}_{j,k})_{x\bar{x}} + (E^{n+\frac{1}{2}}_{j,k})_{y\bar{y}}, \overline{E^{n+1}_{j,k} + E^n_{j,k}})$$

$$- (N_{j,k}^{n+\frac{1}{2}} \cdot E_{j,k}^{n+\frac{1}{2}}, \overline{E_{j,k}^{n+1} + E_{j,k}^{n}}) = 0, \tag{2.7}$$

where

$$\mathrm{Re}((E_{j,k}^{n})_t, \overline{E_{j,k}^{n+1} + E_{j,k}^{n}}) = \mathrm{Re}(\frac{1}{\tau}(E_{j,k}^{n+1} - E_{j,k}^{n}), \overline{E_{j,k}^{n+1} + E_{j,k}^{n}}) = \frac{1}{\tau}(\|E^{n+1}\|_2^2 - \|E^{n}\|_2^2),$$

$$\begin{aligned}
((E_{j,k}^{n+\frac{1}{2}})_{x\bar{x}} + (E_{j,k}^{n+\frac{1}{2}})_{y\bar{y}}, \overline{E_{j,k}^{n+1} + E_{j,k}^{n}}) &= ((E_{j,k}^{n+\frac{1}{2}})_{x\bar{x}}, 2\overline{E_{j,k}^{n+\frac{1}{2}}}) + ((E_{j,k}^{n+\frac{1}{2}})_{y\bar{y}}, 2\overline{E_{j,k}^{n+\frac{1}{2}}}) \\
&= -2((E_{j,k}^{n+\frac{1}{2}})_x, (\overline{E_{j,k}^{n+\frac{1}{2}}})_x) - 2((E_{j,k}^{n+\frac{1}{2}})_y, (\overline{E_{j,k}^{n+\frac{1}{2}}})_y) \\
&= -2(\|E_x^{n+\frac{1}{2}}\|_2^2 + \|E_y^{n+\frac{1}{2}}\|_2^2),
\end{aligned}$$

$$(N_{j,k}^{n+\frac{1}{2}} \cdot E_{j,k}^{n+\frac{1}{2}}, \overline{E_{j,k}^{n+1} + E_{j,k}^{n}}) = 2h_x h_y \sum_{j=1}^{J}\sum_{k=1}^{K} N_{j,k}^{n+\frac{1}{2}} |E_{j,k}^{n+\frac{1}{2}}|^2.$$

Thus, we take imaginary part for (2.7) and use the formulae derived above to get

$$\|E^{n+1}\|_2^2 = \|E^{n}\|_2^2 = \|E^{0}\|_2^2 = \text{ Const.}$$

Computing the inner product of (2.1) with $\tau(\overline{E_{j,k}^{n}})_t$ and taking real part, we have

$$\begin{aligned}
\mathrm{Im}((E_{j,k}^{n})_t, \tau(\overline{E_{j,k}^{n}})_t) + \ \mathrm{Re}((E_{j,k}^{n+\frac{1}{2}})_{x\bar{x}} + (E_{j,k}^{n+\frac{1}{2}})_{y\bar{y}}, \overline{E_{j,k}^{n+1}} - \overline{E_{j,k}^{n}}) \\
-\mathrm{Re}(N_{j,k}^{n+\frac{1}{2}} \cdot E_{j,k}^{n+\frac{1}{2}}, \overline{E_{j,k}^{n+1}} - \overline{E_{j,k}^{n}}) = 0.
\end{aligned} \tag{2.8}$$

Direct computation yields $\mathrm{Im}((E_{j,k}^{n})_t, \tau(\overline{E_{j,k}^{n}})_t) = \tau \cdot \ \mathrm{Im}(\|E_t^{n}\|_2^2) = 0$ and

$$\begin{aligned}
\mathrm{Re}(N_{j,k}^{n+\frac{1}{2}} \cdot E_{j,k}^{n+\frac{1}{2}}, \overline{E_{j,k}^{n+1}} - \overline{E_{j,k}^{n}}) &= \frac{1}{2}\mathrm{Re}(N_{j,k}^{n+\frac{1}{2}} \cdot (E_{j,k}^{n+1} + E_{j,k}^{n}), \overline{E_{j,k}^{n+1}} - \overline{E_{j,k}^{n}}) \\
=&\frac{1}{2}h_x h_y \sum_{j=1}^{J}\sum_{k=1}^{K} N_{j,k}^{n+\frac{1}{2}}(|E_{j,k}^{n+1}|^2 - |E_{j,k}^{n}|^2) \\
=&\frac{1}{2}h_x h_y \sum_{j=1}^{J}\sum_{k=1}^{K} N_{j,k}^{n+\frac{1}{2}}|E_{j,k}^{n+1}|^2 - |E_{j,k}^{n}|^2) \\
=&\frac{1}{2}h_x h_y \sum_{j=1}^{J}\sum_{k=1}^{K} N_{j,k}^{n+\frac{1}{2}}|E_{j,k}^{n+1}|^2 - \frac{1}{2}h_x h_y \sum_{j=1}^{J}\sum_{k=1}^{K} N_{j,k}^{n-\frac{1}{2}}|E_{j,k}^{n}|^2 \\
&- \frac{1}{2}h_x h_y \sum_{j=1}^{J}\sum_{k=1}^{K}(N_{j,k}^{n-\frac{1}{2}} - N_{j,k}^{n-\frac{1}{2}})|E_{j,k}^{n}|^2
\end{aligned}$$

Summing by parts, we obtain

$$\begin{aligned}
\mathrm{Re}((E_{j,k}^{n+\frac{1}{2}})_{x\bar{x}} + (E_{j,k}^{n+\frac{1}{2}})_{y\bar{y}}, \overline{E_{j,k}^{n+1}} - \overline{E_{j,k}^{n}}) &= -\frac{1}{2}\mathrm{Re}((E_{j,k}^{n+1} + E_{j,k}^{n})_x, (\overline{E_{j,k}^{n+1}} - \overline{E_{j,k}^{n}})_x) \\
&- \frac{1}{2}\mathrm{Re}((E_{j,k}^{n+1} + E_{j,k}^{n})_y, (\overline{E_{j,k}^{n+1}} - \overline{E_{j,k}^{n}})_y) \\
=& - \frac{1}{2}(\|E_x^{n+1}\|_2^2 - \|E_x^{n}\|_2^2 + \|E_y^{n+1}\|_2^2 - \|E_y^{n}\|_2^2).
\end{aligned}$$

Thus, it follows from (2.8) that

$$\begin{aligned}&\|E_x^{n+1}\|_2^2-\|E_x^n\|_2^2+\|E_y^{n+1}\|_2^2-\|E_y^n\|_2^2+h_xh_y\sum_{j=1}^{J}\sum_{k=1}^{K}N_{j,k}^{n+\frac12}|E_{j,k}^{n+1}|^2\\&\quad-h_xh_y\sum_{j=1}^{J}\sum_{k=1}^{K}N_{j,k}^{n-\frac12}|E_{j,k}^{n}|^2=h_xh_y\sum_{j=1}^{J}\sum_{k=1}^{K}(N_{j,k}^{n+\frac12}-N_{j,k}^{n-\frac12})|E_{j,k}^{n}|^2\\&=\frac12h_xh_y\sum_{j=1}^{J}\sum_{k=1}^{K}(N_{j,k}^{n+1}-N_{j,k}^{n-1})|E_{j,k}^{n}|^2.\end{aligned}\tag{2.9}$$

Computing the inner product of (2.2) with $(u_{j,k}^{n+\frac12}+u_{j,k}^{n-\frac12})$ and summing by parts, we have

$$\begin{aligned}&((N_{j,k}^n)_{t\bar t},u_{j,k}^{n+\frac12}+u_{j,k}^{n-\frac12})-\frac12(N_{j,k}^{n+1}+N_{j,k}^{n-1},(u_{j,k}^{n+\frac12}+u_{j,k}^{n-\frac12})_{x\bar x}+(u_{j,k}^{n+\frac12}+u_{j,k}^{n-\frac12})_{y\bar y})\\&=(|E_{j,k}^n|^2,(u_{j,k}^{n+\frac12}+u_{j,k}^{n-\frac12})_{x\bar x}+(u_{j,k}^{n+\frac12}+u_{j,k}^{n-\frac12})_{y\bar y}),\end{aligned}$$

which is equivalent to

$$\begin{aligned}&-((u_{j,k}^{n+\frac12})_{\bar t x},(u_{j,k}^{n+\frac12}+u_{j,k}^{n-\frac12})_x)-((u_{j,k}^{n+\frac12})_{\bar t y},(u_{j,k}^{n+\frac12}+u_{j,k}^{n-\frac12})_y)\\&-\frac12(N_{j,k}^{n+1}+N_{j,k}^{n-1},(N_{j,k}^n)_t+(N_{j,k}^{n-1})_t)=(|E_{j,k}^n|^2,(N_{j,k}^n)_t+(N_{j,k}^{n-1})_t),\end{aligned}$$

where the definition of $u_{j,k}^{n+\frac12}$ is used. Direct computation shows that this equation equals

$$\begin{aligned}&\|u_x^{n+\frac12}\|_2^2-\|u_x^{n-\frac12}\|_2^2+\|u_y^{n+\frac12}\|_2^2-\|u_y^{n-\frac12}\|_2^2+\frac12\|N^{n+1}\|_2^2-\frac12\|N^{n-1}\|_2^2\|_2^2\\&=-h_xh_y\sum_{j=1}^{J}\sum_{k=1}^{K}(N_{j,k}^{n+1}-N_{j,k}^{n-1})|E_{j,k}^n|^2.\end{aligned}\tag{2.10}$$

Combining (2.9) with (2.10), we have

$$\begin{aligned}&\|E_x^{n+1}\|_2^2+\|E_y^{n+1}\|_2^2+\frac12(\|u_x^{n+\frac12}\|_2^2+\|u_y^{n+\frac12}\|_2^2)+\frac14(\|N^{n+1}\|_2^2+\|N^n\|_2^2)\\&\quad+h_xh_y\sum_{j=1}^{J}\sum_{k=1}^{K}N_{j,k}^{n+\frac12}\cdot|E_{j,k}^{n+1}|^2=\|E_x^n\|_2^2+\|E_y^n\|_2^2+\frac12(\|u_x^{n-\frac12}\|_2^2+\|u_y^{n-\frac12}\|_2^2)\\&+\frac14(\|N^n\|_2^2+\|N^{n-1}\|_2^2)+h_xh_y\sum_{j=1}^{J}\sum_{k=1}^{K}N_{j,k}^{n-\frac12}\cdot|E_{j,k}^n|^2,\end{aligned}$$

i.e., $$H_h^{n+1}=H_h^n=H_h^0=\text{ Const.}$$
Comparing (1.6), (1.7) with invariants given in the Theorem 1, we know that the difference scheme (2.1), (2.2) keeps two conservative laws that the differential equations possess.

Theorem 2. *Assume the solutions of the differential problem* (1.1)-(1.5), $E(x,y,t) \in C^5(\Omega\times(0,T)), N(x,\,y,t)\in C^5(\Omega\times(0,T))$. *Then the truncation errors of the difference scheme* (2.1)-(2.2) *are given as*

$$iE_t+E_{xx}+E_{yy}-NE=-\frac{i\tau^2}{24}E_{ttt}-\frac{\tau^2}{8}E_{xxtt}-\frac{\tau^2}{8}E_{yytt}-\frac{h_x^2}{12}E_{xxxx}-\frac{h_y^2}{12}E_{yyyy}$$
$$+\frac{\tau^2}{8}EN_{tt}+\frac{\tau^2}{8}NE_{tt}+O(h_x^3+h_y^3+\tau^3), \tag{2.11}$$

and

$$N_{tt}-N_{xx}-N_{yy}-(|E|^2)_{xx}+(|E|^2)_{yy}=\frac{i\tau^2}{12}N_{tttt}+\frac{\tau^2}{2}N_{xxtt}+\frac{\tau^2}{2}N_{yytt}$$
$$+\frac{h_x^2}{12}N_{xxxx}+\frac{h_y^2}{12}N_{yyyy}+\frac{h_x^2}{12}(|E|^2)_{xxxx}+\frac{h_y^2}{12}(|E|^2)_{yyyy}+O(h_x^3+h_y^3+\tau^3), \tag{2.12}$$

Proof. substituting the solutions of the differential problem (1.1)-(1.5) into the difference scheme and using Taylor's expansion, we obtain the formulae (2.11) and (2.12).

3. Convergence of Difference Scheme

In this section, the convergence of the difference problem (2.1)-(2.5) is considered. We begin by defining the standard errors

$$e^n_{j,k}=E(j,k,n)-E^n_{j,k} \text{ and } \eta^n_{j,k}=N(j,k,n)-N^n_{j,k}. \tag{3.1}$$

Let
$$(U^{n+\frac12}_{j,k})_{x\bar x}+(U^{n+\frac12}_{j,k})_{y\bar y}=(\eta^{n+1}_{j,k})_{\bar t},$$
$$U^{n+\frac12}_{o,k}=U^{n+\frac12}_{j,k}=U^{n+\frac12}_{j,o}=U^{n+\frac12}_{j,K}=0. \tag{3.2}$$

Then error equations are deduced as follows:

$$i(e^n_{j,k})_t+((e^{n+\frac12}_{j,k})_{x\bar x}+(e^{n+\frac12}_{j,k})_{y\bar y})-\frac14(N(j,k,n+1)+N(j,k,n))$$
$$\cdot(E(j,k,n+1)+E(j,k,n))+\frac14(N^{n+1}_{j,k}+N^n_{j,k})(E_{j,k}n+1+E^n_{j,k})=R^e, \tag{3.3}$$

$$(\eta^n_{j,k})_{t\bar t}-\frac12((\eta^{n+1}_{j,k})_{x\bar x}+(\eta^{n-1}_{j,k})_{x\bar x}+(\eta^{n+1}_{j,k})_{y\bar y}+(\eta^{n-1}_{j,k})_{y\bar y}) \tag{3.4}$$
$$=(|E(j,k,n)|^2)_{x\bar x}+(|E(j,k,n)|^2)_{y\bar y}-(|E^n_{j,k}|^2)_{x\bar x}-(|E^n_{n,k}|^2)_{y\bar y}+R^n,$$

$$e^0_{j,k}=0,\qquad \eta^0_{j,k}=0, \tag{3.5}$$
$$\eta^1_{j,k}=O(\tau^2), \tag{3.6}$$

where

$$R^e=-\frac{i\tau^2}{14}E_{ttt}-\frac{\tau^2}{8}E_{xxtt}-\frac{\tau^2}{8}E_{yytt}-\frac{h_x^2}{12}E_{xxxx}-\frac{h_y^2}{12}E_{yyyy}$$

$$+\frac{\tau^2}{8}EN_{tt}+\frac{\tau^2}{8}NE_{tt}+O(h_x^3+h_y^3+\tau^3), \tag{3.7e}$$

$$R^\eta=\frac{\tau^2}{12}N_{tttt}+\frac{\tau^2}{2}N_{xxtt}+\frac{\tau^2}{2}N_{yytt}+\frac{h_x^2}{12}N_{xxxx}+\frac{h_y^2}{12}N_{yyyy}$$
$$+\frac{h_x^2}{12}(|E|^2)_{xxxx}+\frac{h_y^2}{12}(|E|^2)_{yyyy}+O(h_x^3+h_y^3+\tau^3). \tag{3.7 η}$$

Lemma 1. (Sobolev estimate[3]) *Suppose* $W\in L_q(R^n), D^mW\in L_q(R^n), D^mW\in L_r(R^n), 1\le q,r<\infty$. *Then for* $0\le j\le m, \frac{j}{m}\alpha\le 1$, *we have* $\|D^jW\|_{L_p}\le C\|D^MW\|_{L_r}^{\alpha}\cdot\|W\|_{L_q}^{1-\alpha}$, *where* $\frac{1}{p}=\frac{j}{n}+\alpha(\frac{1}{r}-\frac{m}{n})+(1-\alpha)\frac{1}{q}$.

Lemma 2. *Assume that complex function* $u(x,y)=u_1(x,y)+iu_2(x,y)$ *and* $v(x,y)=v_1(x,y)+iv_2(x,y)$, *where the* $u_1(x,y),u_2(x,y),v_1(x,y)$, *and* $v_2(x,y)$ *are smooth real functions of compact support in* R^2. *Then the inequality*

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|u|^2|v|^2dxdy\le(2+\varepsilon)\|u\|_{L_2}^2(\|v_x\|_{L_2}^2+\|v_y\|_{L_2}^2)$$
$$+\Big(2+\frac{1}{\varepsilon}\Big)\|v\|_{L_2}^2(\|u_x\|_{L_2}^2+\|u_y\|_{L_2}^2)$$
$$+\frac{2}{\varepsilon}\|u\|_{L_2}^2\|v\|_{L_2}^2+\varepsilon(\|u_x\|_{L_2}^2\|v_y\|_{L_2}^2+\|u_y\|_{L_2}^2\|v_x\|_{L_2}^2)$$

holds, where ε *is a positive constant.*

Proof. First, we consider two real functions $f(x,y)$ and $g(x,y)$. Because of that equality

$$f(x,y)\cdot g(x,y)=\int_{-\infty}^{x}(fg)_xdx=\int_{-\infty}^{y}(fg)_ydy,$$

we have
$$\max_x|f(x,y)\cdot g(x,y)|\le\int_{-\infty}^{+\infty}|f_xg+fg_x|dx,$$
and
$$\max_y|f(x,y)\cdot g(x,y)|\le\int_{g\infty}^{+\infty}|f_yg+fg_y|dy.$$
Then using Schwartz inequality, we obtain

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f^2g^2dxdy\le\int_{-\infty}^{+\infty}\max_y|f\cdot g|dx\int_{-\infty}^{+\infty}\max_x|f\cdot g|dy$$
$$\le\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|f_yg+fg_y|dxdy\cdot\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|f_xg+fg_x|dxdy$$
$$\le(\|f_y\|_{L_2}\|g\|_{L_2}+\|f\|_{L_2}\|g_y\|_{L_2})(\|f_x\|_{L_2}\|g\|_{L_2}+\|f\|_{L_2}\|g_x\|_{L_2})$$
$$=\|g\|_{L_2}^2\|f_x\|_{L_2}\|f_y\|_{L_2}+\|f_y\|_{L_2}\|g\|_{L_2}\|f\|_{L_2}\|g_x\|_{L_2}$$
$$+\|f\|_{L_2}\|f_x\|_{L_2}\|g\|_{L_2}\|g_y\|_{L_2}+\|f\|_{L_2}^2\|g_x\|_{L_2}\|g_y\|_{L_2}$$
$$\le\frac{1}{2}\|g\|_{L_2}^2(\|f_x\|_{L_2}^2+\|f_y\|_{L_2}^2)+\frac{1}{2}\|f\|_{L_2}^2(\|g_x\|_{L_2}^2+\|g_y\|_{L_2}^2)$$
$$+\frac{1}{4}(\frac{1}{\varepsilon}\|f\|_{L_2}^2+\varepsilon\|f_x\|_{L_2}^2)(\|g\|_{L_2}^2+\|g_y\|_{L_2}^2)$$
$$+\frac{1}{4}(\frac{1}{\varepsilon}\|f\|_{L_2}^2+\varepsilon\|f_y\|_{L_2}^2)(\|g\|_{L_2}^2+\|g_x\|_{L_2}^2)$$

$$= \Big(\frac{1}{2}+\frac{\varepsilon}{4}\Big)\|g\|_{L_2}^2(\|f_x\|_{L_2}^2+\|f_y\|_{L_2}^2)+\Big(\frac{1}{2}+\frac{1}{4\varepsilon}\Big)\|f\|_{L_2}^2(\|g_x\|_{L_2}^2+\|g_y\|_{L_2}^2)$$
$$+\frac{1}{2\varepsilon}\|f\|_{L_2}^2\|g\|_{L_2}^2+\frac{\varepsilon}{4}(\|f_x\|_{L_2}^2\|g_y\|_{L_2}^2+\|f_y\|_{L_2}^2\|g_y\|_{L_2}^2).$$

While for the complex functions, it follows from the inequality derived above that

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|u|^2|v|^2dxdy=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(|u_1|^2+|u_2|^2)(|v_1|^2+|v_2|^2)dxdy$$
$$\le (2+\varepsilon)\|u\|_{L_2}^2(\|v_x\|_{L_2}^2+\|v_y\|_{L_2}^2)+\Big(2+\frac{1}{\varepsilon}\Big)\|v\|_{L_2}^2(\|u_x\|_{L_2}^2+\|u_y\|_{L_2}^2)$$
$$+\frac{2}{\varepsilon}\|u\|_{L_2}^2\|v\|_{L_2}^2+\varepsilon(\|u_x\|_{L_2}^2\|v_y\|_{L_2}^2+\|u_y\|_{L_2}^2\|v_x\|_{L_2}^2).$$

Lemma 3. *Assume* $E_0(x,y)\in H^1(\Omega)$, $N_0(x,y)\in L_2(\Omega)$, $N_1(x,y)\in L_2(\Omega)$ *and* $\|E^0\|_2<\frac{1}{2\sqrt{2}}$. *Then we have estimates:*

$$\|E_x^n\|_2\le C,\quad \|E_y^n\|_2\le C,\quad \|u_x^{n-\frac{1}{2}}\|_2\le C,\quad \|N^n\|_2\le C.$$

Proof. First, we estimate that

$$\Big|h_xh_y\sum_{j=1}^{J}\sum_{k=1}^{K}N_{j,k}^{n-\frac{1}{2}}|E_{j,k}^n|^2\Big|\le\frac{1}{4}h_xh_y\sum_{j=1}^{J}\sum_{k=1}^{K}|N_{j,k}^{n-\frac{1}{2}}|^2+h_xh_y\sum_{j=1}^{J}\sum_{k=1}^{K}|E_{j,k}^n|^4.$$

Using Lemma 2 and interpolation formula[9], we have

$$\Big|h_xh_y\sum_{j=1}^{J}\sum_{k=1}^{K}N_{j,k}^{n-\frac{1}{2}}|E_{j,k}^n|^2\Big|\le\frac{1}{4}\|N^{n-\frac{1}{2}}\|_2^2+8\|E^n\|_2^2\cdot(\|E_x^n\|_2^2+\|E_y^n\|_2^2)$$
$$=\frac{1}{4}\|N^{n-\frac{1}{2}}\|_2^2+8\|E^0\|_2^2\cdot(\|E_x^n\|_2^2+\|E_y^n\|_2^2).$$

Thus, it follows from Theorem 1 and $\|E_0\|_2<\frac{1}{2\sqrt{2}}$ that

$$\|E_x^n\|_2\le C,\quad \|E_y^n\|_2\le C,\quad \|u_x^{n-\frac{1}{2}}\le C,\quad \|u_y^{n-\frac{1}{2}}\|_2\le C,\quad \|N^n\|_2\le C.$$

Theorem 3. *Assume the solution of the differential problem* (1.1)–(1.5), $E(x,y,t)\in C^5(\Omega\times(0,T))$, $N(x,\,y,t)\in C^5(\Omega\times(0,T))$, *and the initial data* $E_0(x,y)\in H^1(\Omega)$, $N_0(x,y)\in L_2(\Omega)$, $N_1(x,y)\in L_2(\Omega)$; $\|E_0\|_2\le\frac{1}{2\sqrt{2}}$, $\|E^0\|_2\le\frac{1}{2\sqrt{2}}$. *Then the solution of the difference equations* (2.1)–(2.6) *convergence to the solution of the problem* (1.1)–(1.5) *with order* $O(h_x^2+h_y^2+\tau^2)$ *in* L_2 *norm.*

Proof. Computing the inner product of (3.3) with $(e_{j,k}^{n+1}+e_{j,k}^n)$ and taking the imaginary part, we have

$$\frac{1}{\tau}(\|e^{n+1}\|_2^2-\|e^n\|_2^2)=\frac{1}{4}\mathrm{Im}((N(j,k,n+1)+N(j,k,n))(E(j,k,n+1)+E(j,k,n))$$

$$-(N_{j,k}^{n+1}+N_{j,k}^{n})(E_{j,k}^{n+1}+E_{j,k}^{n}),e_{j,k}^{n+1}+e_{j,k}^{n})+\mathrm{Im}(R^e,e_{j,k}^{n+1}+e_{j,k}^{n}), \tag{3.8}$$

where

$$|(R^e,e_{j,k}^{n+1}+e_{j,k}^{n})|\le C(\tau^2+h_x^2+h_y^2)^2+\|e^{n+1}\|_2^2+\|e^n\|_2^2,$$

$$\begin{aligned}\mathrm{Im}(&(N(j,kn+1)+N(j,k,n))(E(k,k,n+1)+E(j,k,n))\\&-(N_{j,k}^{n+1}+N_{j,k}^{n})(E_{j,k}^{n+1}+E_{j,k}^{n}),e_{j,k}^{n+1}+e_{j,k}^{n})\\&=\mathrm{Im}((\eta_{j,k}^{n+1}+\eta_{j,k}^{n})(E(j,k,n+1)+E(j,k,n))+(N_{j,k}^{n+1}\\&\quad+N_{j,k}^{n})(e_{j,k}^{n+1}+e_{j,k}^{n}),\overline{e_{j,k}^{n+1}}+e_{j,k}^{n})\\&=\mathrm{Im}((\eta_{j,k}^{n+1}+\eta_{j,k}^{n})(E(j,k,n+1)+E(j,k,n)),\overline{e_{j,k}^{n+1}}+\overline{e_{j,k}^{n}}),\end{aligned}$$

since

$$\mathrm{Im}((N_{j,k}^{n+1}+N_{j,k}^{n})(e_{j,k}^{n+1}+e_{j,k}^{n}),e_{j,k}^{n+1}+e_{j,k}^{n})=0.$$

Thus, we use the formulae derived above in (3.8) to get

$$\begin{aligned}\|e^{n+1}\|_2^2-\|e^n\|_2^2&\le C\tau(\tau^2+h_x^2+h_y^2)^2+(\|e^{n+1}\|_2^2+\|e^n\|_2^2)+\tau\|E(j,k,n+1)\\&\quad+E(j,k,n)\|_{L-\infty}\cdot(\|\eta^{n+1}\|_2^2+\|\eta^n\|_2^2+\|e^{n+1}\|_2^2+\|e^n\|_2^2)\\&\le C\tau(\tau^2+h_x^2+h_y^2)^2+C\tau(\|e^{n+1}\|_2^2+\|e^n\|_2^2+\|\eta^{n+1}\|_2^2+\|\eta^n\|_2^2),\end{aligned} \tag{3.9}$$

where $(x,y,t)\in C^5(\Omega\times(0,T))$ is used.

Computing the inner product of (3.4) with $U_{j,k}^{n+\frac12}+U_{j,k}^{n-\frac12}$ and summing by parts, we have

$$\begin{aligned}&\|U_x^{n+\frac12}\|_2^2-\|U_x^{n-\frac12}\|_2^2+\|U_y^{n+\frac12}\|_2^2-\|U_y^{n-\frac12}\|_2^2+\frac12(\|\eta^{n+1}\|_2^2-\|\eta^{n-1}\|_2^2)\\&\qquad=-(|E(j,k,n)|^2-|E_{j,k}^n|^2,\eta_{j,k}^{n-1}-\eta_{j,k}^{n-1})-\tau(R^\eta,U_{j,k}^{n+\frac12}+U_{j,k}^{n-\frac12}),\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\|U_x^{n+\frac12}\|_2^2-\|U_x^{n-\frac12}\|_2^2+\|U_y^{n+\frac12}\|_2^2-\|U_y^{n-\frac12}\|_2^2+\frac12(\|\eta^{n+1}\|_2^2-\|\eta^{n-1}\|_2^2)\\&\quad+(|E(j,k,n+1)|^2-|E_{j,k}^{n+1}|^2,\eta_{j,k}^{n+1}+\eta_{j,k}^{n})-(|E(j,k,n)|^2-|E_{j,k}^{n}|^2,\eta_{j,k}^{n}+\eta_{j,k}^{n-1})\\&=P_2^{n+\frac12}-\tau(R^\eta,U_{j,k}^{n+\frac12}+U_{j,k}^{n-\frac12}),\end{aligned} \tag{3.10}$$

where

$$P_2^{n+\frac12}=(|E(j,k,n+1)|^2-|E_{j,k}^{n+1}|^2-|E(j,k,n)|^2+|E_{j,k}^{n}|^2,\eta_{j,k}^{n+1}+\eta_{j,k}^{n-1}).$$

Computing the inner product of (3.3) with $\tau(e_{j,k}^n)_t$ and taking real part, we obtain

$$\frac12(\|e_x^{n+1}\|_2^2-\|e_x^n\|_2^2+\|e_y^{n+1}\|_2^2-\|e_y^n\|_2^2)=\frac14P_1^{n+\frac12}-\tau Re(R^e,(e_{j,k}^n)_t), \tag{3.11}$$

where

$$P_1^{n+\frac{1}{2}} = - Re((N(j,k,n+1) + N(j,k,n))(E(j,k,n+1) + E(j,k,n)) \\ - (N_{j,k}^{n+1} + N_{j,k}^{n})(E_{j,k}^{n+1} + E_{j,k}^{n}), e_{j,k}^{n+1} - e_{j,k}^{n}).$$

It follows from (3.10) and (3.11) that

$$2(\|e_x^{n+1}\|_2^2 - \|e_x^n\|_2^2 + \|e_y^{n+1}\|_2^2 - \|e_y^n\|_2^2) + \|U_x^{n+\frac{1}{2}}\|_2^2 + \|U_y^{n+\frac{1}{2}}\|_2^2 - \|U_x^{n-\frac{1}{2}}\|_2^2 - \|U_y^{n-\frac{1}{2}}\|_2^2 \\ + \frac{1}{2}(\eta^{n+1}\|_2^2 - \|\eta^{n-1}\|_2^2) + (|E(j,k,n+1)|^2 - |E_{j,k}^{n+1}|^2, \eta_{j,k}^{n+1} + \eta_{j,k}^{n}) \\ - (|E(j,k,n)|^2 - |E_{j,k}^{n}|^2, \eta_{j,k}^{n} + \eta_{j,k}^{n-1}) \\ = - \tau((R^\eta, U_{j,k}^{n+\frac{1}{2}} + U_{j,k}^{n-\frac{1}{2}}) - 4\tau(R^e, (\overline{e_{j,k}^n})_t) + P_1^{n+\frac{1}{2}} + P_2^{n+\frac{1}{2}}. \quad (3.12)$$

Using direct computation, we have

$$P_1^{n+\frac{1}{2}} + P_2^{n+\frac{1}{2}} = -\mathrm{Re}\Big\{h_x h_y \sum_{j=1}^{J}\sum_{k=1}^{K}[(N(j,k,n+1) + N(j,k,n)) \\ \cdot (|E(j,k,n+1)|^2 - |E(j,k,n)|^2) - (N(j,k,n+1) + N(j,k,n)) \\ \cdot (E(j,k,n+1) + E(j,k,n))(\overline{E_{j,k}^{n+1} - E_{j,k,n}^{n}}) \\ - (N_{j,k}^{n+1} + N_{j,k}^{n})(E_{j,k}^{n+1} + E_{j,k}^{n}) \cdot \overline{(E(j,k,n+1) - E(j,k,n))} \\ + (N_{j,k}^{n+1} + N_{j,k}^{n})(|E_{j,k}^{n+1}|^2 - |E_{j,k}^{n}|^2)\Big\} \\ + h_x h_y \sum_{j=1}^{J}\sum_{k=1}^{K}[|E(j,k,n+1)|^2 - |E(j,k,n)|^2 - |E_{j,k}^{n+1}|^2 + |E_{j,k}^{n}|^2] \\ \cdot [N(j,k,n+1) + N(j,k,n) - N_{j,k}^{n+1} - N_{j,k}^{n}] \\ = \mathrm{Re}\Big\{h_x h_y \sum_{j=1}^{J}\sum_{k=1}^{K}[(N(j,k,n+1) + N(j,k,n))(\overline{E_{j,k}^{n+1} - E_{j,k}^{n}}) \\ - (N_{j,k}^{n+1} + N_{j,k}^{n})\overline{(E(j,k,n+1) - E(j,k,n))}] \\ \cdot (E(j,k,n+1) + E(j,k,n) - E_{j,k}^{n+1} - E_{j,k}^{n})\Big\} \\ = \mathrm{Re}\Big\{h_x h_y \sum_{j=1}^{J}\sum_{k=1}^{K}\overline{[(E(j,k,n+1) - E(j,k,n))}(\eta_{j,k}^{n+1} + \eta_{j,k}^{n}) \\ - \overline{(e_{j,k}^{n+1} - e_{j,k}^{n})}(N(j,k,n+1) + N(j,k,n))](e_{j,k}^{n+1} + e_{j,k}^{n})\Big\}. \quad (3.13)$$

It follows from $E(x,y,t) \in C^5$ that

$$\left|\mathrm{Re}\ \Big[h_x h_y \sum_{j=1}^{J}\sum_{k=1}^{K}\overline{(E(j,k,n+1) - E(j,k,n))}(\eta_{j,k}^{n+1} + \eta_{j,k}^{n})(e_{j,k}^{n+1} + e_{j,k}^{n})\Big]\right|$$

$$\begin{aligned}
&\le C\tau\Big|\mathrm{Re}\ \Big[h_xh_y\sum_{j=1}^{J}\sum_{k=1}^{K}(\eta_{j,k}^{n+1}+\eta_{j,k}^{n})(e_{j,k}^{n+1}+e_{j,k}^{n})\Big]\Big|\\
&\le C\tau(\|\eta^{n+1}\|_2^2+\|\eta^{n}\|_2^2+\|e^{n+1}\|_2^2+\|e^{n}\|_2^2). \qquad (3.14)
\end{aligned}$$

Using the error equation (3.3) and $E(x,y,t)\in C^5, N(x,y,t)\in C^5$, we obtain

$$\begin{aligned}
&\Big|\mathrm{Re}\Big\{h_xh_y\sum_{j=1}^{J}\sum_{k=1}^{K}(N(j,k,n+1)+N(j,k,n))(e_{j,k}^{n+1}+e_{j,k}^{n})\overline{(e_{j,k}^{n+1}-e_{j,k}^{n})}\Big\}\Big|\\
=&\Big|\tau\mathrm{Re}\Big\{h_xh_y\sum_{j=1}^{J}\sum_{k=1}^{K}(N(j,k,n+1)+N(j,k,n))\overline{e_{j,k}^{n+1}+e_{j,k}^{n})}\\
&\cdot i[\frac{1}{2}((e_{j,k}^{n+\frac12})_{x\bar x}+(e_{j,k}^{n+\frac12})_{y\bar y})\\
&+\frac14(N(j,k,n+1)+N(j,k,n))(E(j,k,n+1)+E(j,k,n))\\
&-\frac14(N_{j,k}^{n+1}+N_{j,k}^{n})(E_{j,k}^{n+1}+E_{j,k}^{n})+R^e]\Big\}\Big|\\
=&\Big|\tau\mathrm{Im}\ \Big\{h_xh_y\sum_{j=1}^{J}\sum_{k=1}^{K}(N(j,k,n+1)+N(j,k,n))\overline{(e_{j,k}^{n+1}+e_{j,k}^{n})}\\
&\cdot[\frac14((e_{j,k}^{n+1})_{x\bar x}+(e_{j,k}^{n})_{x\bar x}+(e_{j,k}^{n+1})_{y\bar y}+(e_{j,k}^{n})_{y\bar y})\\
&+\frac14(\eta_{j,k}^{n+1}+\eta_{j,k}^{n})(E(j,k,n+1)+E(j,k,n))\\
&+\frac14(N_{j,k}^{n+1}+N_{j,k}^{n})(e_{j,k}^{n+1}+e_{j,k}^{n})+R^e]\Big\}\Big|\\
=&\frac{\tau}{4}\Big|\Big\{-h_xh_y\sum_{j=1}^{J}\sum_{k=1}^{K}[(N(j,k,n+1)+N(j,k,n))\overline{e_{j,k}^{n+1}+e_{j,k}^{n})}]_x(e_{j,k}^{n+1}+e_{j,k}^{n})_x\\
&-h_xh_y\sum_{j=1}^{J}\sum_{k=1}^{K}[(N(j,k,n+1)+N(j,k,n))\overline{(e_{j,k}^{n+1}+e_{j,k}^{n})}]_y(e_{j,k}^{n+1}+e_{j,k}^{n})_y\\
&+h_xh_y\sum_{j+1}^{J}\sum_{k=1}^{K}(N(j,k,n+1)\\
&+N(j,k,n))\overline{(e_{j,k}^{n+1}+e_{j,k}^{n})}(\eta_{j,k}^{n+1}+\eta_{j,k}^{n})(E(j,k,n+1)+E(j,k,n))\\
&+h_xh_y\sum_{j=1}^{J}\sum_{k=1}^{K}(N(j,k,n+1)+N(j,k,n))\overline{(e_{j,k}^{n+1}+e_{j,k}^{n})}\cdot 4R^e\Big\}\Big|\\
\le& C\tau(\|e_x^{n+1}\|_2^2+\|e_x^{n}\|_2^2+\|e^{n+1}\|_2^2+\|e^{n}\|_2^2\\
&+\|\eta^{n+1}\|_2^2+\|\eta^{n}\|_2^2+(h_x^2+h_y^2+\tau^2)^2). \qquad (3.15)
\end{aligned}$$

Using the formulae (3.7e), (3.7η) and the error equation (3.3),we have

$$|(R^\eta,U_{j,k}^{n+\frac12}+U_{j,k}^{n-\frac12})|\le C(h_x^2+h_y^2+\tau^2)^2+C(\|U^{n+\frac12}\|_2^2+\|U_x^{n-\frac12}\|_2^2)$$

$$\begin{aligned}\le & C(h_x^2+h_y^2+\tau^2)^2\\ & +C(\|U_x^{n+\frac{1}{2}}\|_2^2+\|U_y^{n+\frac{1}{2}}\|_2^2+\|U_x^{n-\frac{1}{2}}\|_2^2+\|U_y^{n-\frac{1}{2}}\|_2^2),\end{aligned} \tag{3.16}$$

where the formula (3.2) are used,

$$\begin{aligned}|(R^e,&(e_{j,k}^n)_t| = \Big| -\frac{i\tau^2}{24}E_{ttt}-\frac{\tau^2}{8}E_{xxtt}-\frac{\tau^2}{8}E_{yytt}-\frac{h_x^2}{12}E_{xxxx}-\frac{h_y^2}{12}E_{yyyy}\\ &+\frac{\tau^2}{8}EN_{tt}+\frac{\tau^2}{8}NE_{tt}+O(h_x^3+h_y^3+\tau^3),(e_{j,k}^n)_t)\Big|\\ &=\Big|\Big(-\frac{i\tau^2}{24}E_{ttt}-\frac{\tau^2}{8}E_{xxtt}-\frac{\tau^2}{8}E_{yytt}-\frac{h_x^2}{12}E_{xxxx}-\frac{h_y^2}{12}E_{yyyy}\\ &\quad+\frac{\tau^2}{8}EN_{tt}+\frac{\tau^2}{8}NE_{tt},-\frac{1}{4}((e_{j,k}^{n+1})_{x\bar{x}}+(e_{j,k}^n)_{x\bar{x}}+(e_{j,k}^{n+1})_{y\bar{y}}+(e_{j,k})_{y\bar{y}}\\ &\quad+\frac{1}{4}(\eta_{j,k}^{n+1}+\eta_{j,k}^n))(E(j,k,n+1)+E(j,k,n))+\frac{1}{4}(N_{j,k}^{n+1}+N_{j,k}^n)(e_{j,k}^{n+1}+e_{j,k}^n)\\ &\quad+O(h_x^2+h_y^2+\tau^2)\Big)+(O(h_x^3+h_y^3+\tau^3),\tau(e_{j,k}^{n+1}-e_{j,k}^n))\Big|\\ &\le C(h_x^2+h_y^2+\tau^2)^2+C(\|e_x^{n+1}\|_2^2+\|e_x^n\|_2^2+\|e_y^{n+1}\|_2^2+\|e_y^n\|_2^2\\ &\quad+\|\eta^{n+1}\|_2^2+\|\eta^n\|_2^2).\end{aligned} \tag{3.17}$$

Substituting (3.13), (3.14), (3.15), (3.16) and (3.17) and (3.12) yields

$$L^{n+\frac{1}{2}}\le L^{n-\frac{1}{2}}+C\tau(h_x^2+h_y^2\tau^2)^2+C\tau G^{n+\frac{1}{2}}, \tag{3.18}$$

where

$$\begin{aligned}L^{n+\frac{1}{2}}=&2(\|e_x^{n+1}\|_2^2+\|e_y^{n+1}\|_2^2)+\|U_x^{n+\frac{1}{2}}\|_2^2+\|U_y^{n+\frac{1}{2}}\|_2^2\\ &+\frac{1}{2}\|\eta^{n+1}\|_2^2+\frac{1}{2}\|\eta^n\|_2^2+(|E(j,k,n+1)|^2-|E_{j,k}^{n+1}|^2,\eta_{j,k}^{n+1}+\eta_{j,k}^n)\end{aligned}$$

and

$$\begin{aligned}G^{n+\frac{1}{2}}=&\|e^{n+1}\|_2^2+\|e^n\|_2^2+\|e_x^{n+1}\|_2^2+\|e_x^n\|_2^2+\|e_y^{n+1}\|_2^2+\|e_y^n\|_2^2\\ &+\|U_x^{n+\frac{1}{2}}\|_2^2+\|U_x^{n-\frac{1}{2}}\|_2^2+\|U_y^{n+\frac{1}{2}}\|_2^2+\|U_y^{n-\frac{1}{2}}\|_2^2+\|\eta^{n+1}\|_2^2+\|\eta^n\|_2^2.\end{aligned}$$

Multiplying (3.9) by C_ε and summing it with (3.18), we have

$$L^{n+\frac{1}{2}}+C_\varepsilon\|e^{n+1}\|_2^2\le L^{n-\frac{1}{2}}+C_\varepsilon\|e^n\|_2^2+C\tau(h_x^2+h_y^2+\tau^2)^2+C\tau G^{n+\frac{1}{2}}, \tag{3.19}$$

where the constant C_ε will be chosen later.

Thus, it is easy to get that

$$L^{n+\frac{1}{2}}+C_\varepsilon\|e^{n+1}\|_2^2\le L^{-\frac{1}{2}}+C_\varepsilon\sum_{l=0}^{n}G^{l+\frac{1}{2}}$$

$$\leq C(h_x^2+h_y^2+\tau^2)^2+C\tau\sum_{l=0}^{n}G^{l+\frac{1}{2}}, \tag{3.20}$$

On the other hand, it follows from Schwarz' inequality that

$$\begin{aligned}
|(|E(j,k,n+1)|^2-|E_{j,k}^{n+1}|^2,\eta_{j,k}^{n+1}+\eta_{j,k}^{n})| &\leq \frac{1}{4}(\|\eta^{n+1}\|_2^2+\|\eta^n\|_2^2)\\
&+h_xh_y\sum_{j=1}^{J}\sum_{k=1}^{K}(E(j,k,n+1)|^2-|E_{j,k}^{n+1}|^2)^2=\frac{1}{4}(\|\eta^{n+1}\|_2^2\\
&+\|\eta^n\|_2^2)+h_xh_y\sum_{j=1}^{J}\sum_{k+1}^{K}[\mathrm{Re}((E(j,k,n+1)+E_{j,k}^{n+1})+E_{j,k}^{n+1})\overline{e_{j,k}^{n+1}})]^2\\
&\leq(\|\eta^{n+1}\|_2^2+\|\eta^n\|_2^2)+h_xh_y\sum_{j=1}^{J}\sum_{k=1}^{K}|E(j,k,n+1)+E_{j,k}^{n+1}|^2|e_{j,k}^{n+1}|^2.
\end{aligned}$$

Using Lemma 2 and interpolation formula [9], we have

$$\begin{aligned}
|(E(j,k,n+1)|^2-|E_{j,k}^{n+1}|^2,\eta_{j,k}^{n+1}+\eta_{j,k}^{n})| &= |(E(j,k,n+1)+E_{j,k}^{n+1})e_{j,k}^{n+1},\eta_{j,k}^{n+1}+\eta_{j,k}^{n})|\\
&\leq \frac{1}{4}(\|\eta^{n+1}\|_2^2+\|\eta^n\|_2^2)\\
&+\Big(2+\frac{1}{\varepsilon}\Big)\|e^{n+1}\|_2^2(\|E_x(\cdot,\cdot,n+1)\|_2^2\\
&+\|E_y(\cdot,\cdot,n+1)\|_2^2+\|E_x^{n+1}\|_2^2+\|E_y^{n+1}\|_2^2+\|E_y^{n+1}\|_2^2)\\
&+(2+\varepsilon)(\|E(\cdot,\cdot,n+1)\|_2^2+\|E^{n+1}\|_2^2)(\|e_x^{n+1}\|_2^2+\|e_y^{n+1}\|_2^2)\\
&+\frac{2}{\varepsilon}\|e^{n+1}\|_2^2(\|E(\cdot,\cdot,n+1)\|_2^2+\|E^{n+1}\|_2^2)\\
&+\varepsilon\|e_x^{n+1}\|_2^2(\|E_y(\cdot,\cdot,n+1)\|_2^2+\|E_y^{n+1}\|_2^2)\\
&+\varepsilon\|e_y^{n+1}\|_2^2(\|E_x(\cdot,\cdot,n+1)\|_2^2+\|E_x^{n+1}\|_2^2).
\end{aligned}$$

Choosing the $\varepsilon=\dfrac{2}{8C_0+1}$ implies that

$$\begin{aligned}
&|(|E(j,k,n+1)|^2-|E_{j,k}^{n+1}|^2,\eta_{j,k}^{n+1}+\eta_{j,k}^{n})|\\
&\quad\leq\frac{1}{4}(\|\eta^{n+1}\|_2^2+\|\eta^n\|_2^2)+\Big(8C_0+\frac{8C_0}{\varepsilon}\Big)\|e^{n+1}\|_2^2\\
&\quad\quad+[(2+\varepsilon)(\|E(\cdot,\cdot,0)\|_2^2+\|E^0\|_2^2)+2C_0\varepsilon](\|e_x^{n+1}\|_2^2+\|e_y^{n+1}\|_2^2)\\
&\quad\leq\frac{1}{4}(\|\eta^{n+1}\|_2^2+\|\eta^n\|_2^2)+(12C_0+32C_0^2)\|e^{n+1}\|_2^2+\|e_x^{n+1}\|_2^2+\|e_y^{n+1}\|_2^2,
\end{aligned} \tag{3.21}$$

where $\|E_0\|\leq\dfrac{1}{2\sqrt{2}}$ and $\|E^0\|\leq\dfrac{1}{2\sqrt{2}}$ are used, and $C_0\geq 1$ is a constant such that $\|E_x(\cdot,\cdot,n)\|_{L_2}^2\leq c_0$, $\|E_y(\cdot,\cdot,n)\|_{L_2}^2\leq C_0$ and $\|E_y^n\|_{L_2}^2\leq C_0$. Choosing $C_\varepsilon=12C_0+32C_0^2+1$, we obtain

$$L^{n+\dfrac{1}{2}}+C_\varepsilon\|e^{n+1}\|_2^2\geq\|e_x^{n+1}\|_2^2+\|e_y^{n+1}\|_2^2+\|U_x^{n+\frac{1}{2}}\|_2^2+\|U_y^{n+\frac{1}{2}}\|_2^2$$

$$+\frac{1}{4}\|\eta^{n+1}\|_2^2+\frac{1}{4}\|\eta^n\|_2^2+\|e^{n+1}\|_2^2,$$

which is equivalent to

$$L^{n+\frac{1}{2}}+C_\varepsilon\|e^{n+1}\|_2^2+L^{n-\frac{1}{2}}+C_\varepsilon\|e^n\|_2^2\ge\frac{1}{C}G^{n+\frac{1}{2}}. \tag{3.22}$$

It follows from (3.20) and (3.22) that

$$\begin{aligned}G^{n+\frac{1}{2}}&\le C(L^{n+\frac{1}{2}}+C_\varepsilon\|e^{n+1}\|_2^2+L^{n-\frac{1}{2}}+C_\varepsilon\|e^n\|_2^2)\\&\le 2C^2\Big((h_x^2+h_y^2+\tau^2)^2+\tau\sum_{l=0}^{n}G^{l+\frac{1}{2}}\Big),\end{aligned}$$

i.e.,

$$G^{n+\frac{1}{2}}\le C^2\Big((h_x^2+h_y^2+\tau^2)^2+C\tau\sum_{l=0}^{n}G^{l+\frac{1}{2}}\Big).$$

Using discrete Gronwall's inequality [7], we obtain

$$G^{n+\frac{1}{2}}\le C^2(h_x^2+h_y^2+\tau^2)^2,\quad 0\le n\le\frac{T}{\tau}.$$

This completes the proof.

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