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Unconstrained Methods for Generalized Complementarity Problems[*1)]

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Abstract

In this paper, the generalized complementarity problem is formulated as an unconstrained optimization problem. Our results generalize the results of [9]. The dimensionality of the unconstrained problem is the same as that of the original problem. If the mapping of generalized complementarity problem is differentiable, the objective function of the unconstrained problem is also differentiable. All the solutions of the original problem are global minimizers of the optimization problem. A generalized strict complementarity condition is given. Under certain assumptions, local properties of the correspondent unconstrained optimization problem are studied. Limited numerical tests are also reported.

1. Introduction

The complementarity problem, a special case of variational inequality problem, has many applications in different fields such as mathematical programming, game theory, economics. Generally, the standard complementarity problem has the following form:

$$y = F(x), \ x \geq 0, \ y \geq 0, \ \langle y, x \rangle = 0, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner products. When $F(x)$ is an affine function of x, it reduces to the *linear complementarity problem* which is denoted by LCP. Otherwise we call it *the nonlinear complementarity problem* or simply NCP. The complementarity problem has attracted many researchers since its appearance and many results have been given, a nice survey is given by [3]. The LCP problem, can be converted as a special linear programming or quadratic programming in the nonnegative orthant of R^n, thus many classical methods for linear programming are used to solve the LCP problem[10]. For the NCP problem, people often use so called NCP functions and formulate the NCP as a system of equations or unconstrained optimization problem, then classical methods for unconstrained optimization can be applied[4,7,8,11].

The generalized complementarity problem, denoted by $GCP(X, F)$, is to find a vector $x^* \in X$ such that:

$$F(x^*) \in X^*, \quad \text{and } \langle F(x^*), x^* \rangle = 0, \tag{1.2}$$

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where X is a convex cone in R^n, $X^* = -X^0$, and X^0 is the polar cone of X[12]

$$X^0 = \{y \in R^n : \ \langle y, x \rangle \leq 0, \ \forall \ x \in X\}. \tag{1.3}$$

It is obvious that when X is the nonnegative orthant of R^n, (1.2) reduces to the NCP. Although (1.2) was proposed and studied twenty-years ago [5], [6], little attention has paid to it. Traditionally it was considered as a variational inequality problem, which has the following standard form:

$$x^* \in X, \ \text{ and } \langle F(x^*), y - x^* \rangle \geq 0, \quad \forall \ y \in X, \tag{1.6}$$

which usually denoted by $VI(X, F)$. Then we use the same methods for VI problem to solve it. In doing so, through variational principle, a merit or gap function is applied, then we approach the solution of the GCP by minimizing the merit function.

Recently, some new interesting results for these problems are reported. In [2], through projection operators, the problem (1.4) is reconsidered as differentiable optimization problem. The objective function has some desirable global properties. In [9], for the NCP (1.1), the authors proposed unconstrained methods which mainly derived from an augmented Lagrangian formulation. Under certain conditions, the unconstrained problem has excellent local properties.

The main purpose of this paper is to generalize the results of [9] to the case where X is a convex cone. In the following section, we first describe some notations and concepts which will be used in the paper, similar to [9], we first consider a generalized augmented Lagrangian formulation. Then we show that this formulation equals to the difference of two functions defined in [2]. In section 3, some global properties of the optimization problem are discussed. A generalized strict complementarity condition is also considered. Under certain assumptions, we discuss the local properties of the optimization problem. Some numerical results are reported in the last section.

2. Preliminaries

First, we give some basic definitions [5], [6]:

Definition 2.1. *Let Ω be a nonempty subset of R^n; then*
(i): Ω is a cone if $x \in \Omega \Rightarrow \ \lambda x \in \Omega$ for all reals $\lambda \geq 0$;
(ii): Ω is a convex cone if $x \in \Omega, y \in \Omega \Rightarrow \lambda x + \mu y \in \Omega$ for all reals $\lambda \geq 0, \mu \geq 0$;
(iii): Ω is solid if it has nonempty interior relative to R^n.

In the rest of this paper, except for special description, we assume that the constraint set X is a closed solid convex cone, which means that problems (1.4) and (1.2) are equivalent. Similar to [5], [6], we define a partial ordering on R^n as follows: $x \overset{X}{\geq} y$ if and only if $x - y \in X$ and $x \overset{X}{>} y$ if and only if $x - y \in \text{int}(X)$. Now we can reformulate (1.2) as the following constrained minimization problem:

$$\min_x \Big\{ \langle F(x), x \rangle \Big| x \overset{X}{\geq} 0, F(x) \overset{X^*}{\geq} 0 \Big\}. \tag{2.1}$$

Obviously, the solution of GCP (1.2) is a global minimizer of (2.1). On the other hand, if x^* is a solution of (2.1) and $\langle x^*, F(x^*)\rangle = 0$ and, then x^* also solves (1.2). For any nonsingular matrix C, we define

$$X_C = \{y | y = Cx,\ x \in X\}. \tag{2.2}$$

The above definition gives

$$(X_C)^* = \{y | y = C^{-T}x,\ x \in X^*\}. \tag{2.3}$$

Furthermore, if X is closed solid convex cone, so are X_C and $(X_C)^*$ [12]. It is easy to see that (2.1) is equivalent to the following problem

$$\min_x \{\langle F(x), x\rangle | Cx \overset{X_C}{\geq} 0, C^{-T}F(x) \overset{(X_C)^*}{\geq} 0\}. \tag{2.4}$$

For any symmetric positive definite matrix G, let $\|x\|_G = \langle x, Gx\rangle^{\frac{1}{2}}$. For any closed convex set Ω, let $P_\Omega(x)$ be the projection of x onto Ω, and define $P_{\Omega,G}(x)$ as the unique solution of the problem

$$\min_{y\in\Omega} \|y - x\|_G. \tag{2.5}$$

Then it follows from the definition of X_C that

$$P_{X_C}(Cx) = CP_{X,G}(x), \tag{2.6}$$

where $G = C^T C$. Motivated by the results in [9], we consider a generalized augmented Lagrangian formulation for (2.4)

$$\begin{aligned} L(x,u,v,\alpha) = \langle F(x), x\rangle + \frac{1}{2\alpha}(\|P_{X_C}(u - \alpha C^{-T}F(x))\|^2 - \|u\|^2 \\ + \|P_{(X_C)^*}(v - \alpha Cx)\|^2 - \|v\|^2). \end{aligned} \tag{2.7}$$

Replace u by Cx and v by $C^{-T}F(x)$, we get the following *implicit Lagrangian* function:

$$\begin{aligned} M(x,\alpha) &= \langle F(x), x\rangle + \frac{1}{2\alpha}(\|P_{X_C}(Cx - \alpha C^{-T}F(x))\|^2 - \|x\|_G^2 \\ &\quad + \|P_{(X_C)^*}(C^{-T}F(x) - \alpha Cx)\|^2 - \|F(x)\|_{G^{-1}}^2) \\ &= \langle F(x), x\rangle + \frac{1}{2\alpha}(\|P_{X,G}(x - \alpha G^{-1}F(x))\|_G^2 - \|x\|_G^2 \\ &\quad + \|P_{X^*,G^{-1}}(F(x) - \alpha Gx)\|_{G^{-1}}^2 - \|F(x)\|_{G^{-1}}^2). \end{aligned} \tag{2.8}$$

Mangasarian and Solodov [9] have studied this interesting function when X is the nonnegative orthant and $C = I_n$. In this case, the dual polar of X equals to itself. However, when X is a general closed solid convex cone, this function demands the calculation of projections on two sets (X_C and $(X_C)^*$). This suggests that a direct application of (2.8) is difficult and urges us to consider it only in the constrained set X_C.

In what follows we give a relation between the projections onto a closed convex cone X and its polar cone X^0 and X^*:

Lemma 2.1. *Let X be a closed convex cone, X^0, X^* as defined in section 1, then we have*

$$P_{X^*}(x) = -P_{X^0}(-x) = x + P_X(-x), \quad \forall x \in R^n. \tag{2.9}$$

Proof. For any $x \in R^n$, the definition of $P_X(x)$ implies that

$$\langle x - P_X(x),\ y - P_X(x)\rangle \le 0,\ \forall y \in X. \tag{2.10}$$

Since X is a cone, the above inequality yields

$$\langle x - P_X(x),\ P_X(x)\rangle = 0. \tag{2.11}$$

It follows from (2.10) and (2.11) that

$$\langle x - P_X(x), y\rangle \le 0; \quad \forall y \in X, \tag{2.12}$$

which means $x - P_X(x) \in X^0$, consequently

$$\begin{aligned}\langle x - (x - P_X(x)), y - (x - P_X(x))\rangle &= \langle P_X(x), y\rangle - \langle P_X(x), x - P_X(x)\rangle \\ &\le 0,\ \forall y \in X^0. \end{aligned} \tag{2.13}$$

The above inequality shows that $P_{X^0}(x) = x - P_X(x)$. Because $X^* = -X^0$, we have

$$P_{X^0}(-x) = -P_{X^*}(x),\ \forall x \in R^n, \tag{2.14}$$

which gives (2.9). Thus the lemma is true.

Substitute the above relation into (2.8), by (2.6), we have

$$\begin{aligned} M(x,\alpha) &= \langle F(x), x\rangle + \frac{1}{2\alpha}(\|P_{X_C}(Cx - \alpha C^{-T}F(x))\|^2 - \|x\|_G^2 \\ &\quad + \| -\alpha Cx + C^{-T}F(x) + P_{X_C}(\alpha Cx - C^{-T}F(x))\|^2 - \|F(x)\|_{G^{-1}}^2) \\ &= \langle F(x), x\rangle + \frac{1}{2\alpha}(\|P_{X_C}(Cx - \alpha C^{-T}F(x))\|^2 - \|x\|\_G^2 \\ &\quad + \|\alpha Cx - C^{-T}F(x)\|^2 - \|P_{X_C}(\alpha Cx - C^{-T}F(x))\|^2 - \|F(x)\|_{G^{-1}}^2) \\ &= \frac{1}{2\alpha}(\|P_{X_C}(Cx - \alpha C^{-T}F(x))\|^2 - \|x\|_G^2 \\ &\quad - (\|P_{X_C}(\alpha Cx - C^{-T}F(x))\|^2 - \|\alpha x\|_G^2)) \\ &= \frac{1}{2\alpha}(\|P_{X,G}(x - \alpha G^{-1}F(x))\|_G^2 - \|x\|_G^2 \\ &\quad - (\|P_{X,G}(\alpha x - G^{-1}F(x))\|_G^2 - \|\alpha x\|_G^2)). \end{aligned} \tag{2.15}$$

It is pointed out in [9] that when the last two terms of (2.8) are dropped and $C = I_n, \alpha = 1$, (2.8) equals to (5.2) of Fukushima[[2]. For (1.4), Fukushima[2] developed a differentiable optimization method. The method mainly depends on the G-projection onto the constrained set X. If we define $H = P_{X,G}(x - G^{-1}F(x))$, then the merit function of Fukushima can be expressed as follows:

$$f(x) = -\langle F(x), H - x\rangle - \frac{1}{2}\langle H - x, G(H - x)\rangle. \tag{2.16}$$

For any constants $\alpha, \beta > 0$, problem (1.2) is unchanged if we replace x by αx and $F(x)$ by $\beta F(x)$. Let $H(\alpha, \beta) = P_{X,G}(\alpha x - \beta G^{-1}F(x))$, it follows that

$$\langle \alpha Gx - \beta F(x), H(\alpha, \beta)\rangle = \|H(\alpha, \beta)\|_G^2. \tag{2.17}$$

Define

$$f(x, \alpha, \beta) = -\langle \beta F(x), H(\alpha, \beta) - \alpha x\rangle - \frac{1}{2}\langle H(\alpha, \beta) - \alpha x, G(H(\alpha, \beta) - \alpha x)\rangle, \tag{2.18}$$

which can be viewed as a generalization of (2.16). Because X is a cone, it is not difficult to show that

$$f(x, \alpha, \beta) = \alpha^2 f(x, 1, \beta/\alpha). \tag{2.19}$$

The above homogeneous property shows that we only need to consider $f(x, \alpha, 1)$ or $f(x, 1, \alpha)$. The following result, connecting the implicit Lagrangian functions $M(x, \alpha)$ and the generalized merit functions $f(x, \alpha, \beta)$, is an interesting discovery:

Lemma 2.2. *For any $\alpha > 0$, we have*

$$M(x, \alpha) = \frac{f(x, 1, \alpha) - f(x, \alpha, 1)}{\alpha} = \frac{f(x, 1, \alpha)}{\alpha} - \alpha f\Big(x, 1, \frac{1}{\alpha}\Big). \tag{2.20}$$

Proof. From (2.15), (2.17) and (2.18), we have

$$\begin{aligned} f(x, 1, \alpha) - f(x, \alpha, 1) &= -\langle \alpha F(x), H(1, \alpha) - x\rangle - \frac{1}{2}\langle H(1, \alpha) - x, G(H(1, \alpha) - x)\rangle \\ &\quad + \langle F(x), H(\alpha, 1) - \alpha x\rangle + \frac{1}{2}\langle H(\alpha, 1) - \alpha x, G(H(\alpha, 1) - \alpha x)\rangle \\ &= \langle H(1, \alpha), Gx - \alpha F(x)\rangle - \frac{1}{2}\|H(1, \alpha)\|_G^2 - \frac{1}{2}\|x\|_G^2 \\ &\quad - \langle H(\alpha, 1), \alpha Gx - F(x)\rangle + \frac{1}{2}\|H(\alpha, 1)\|_G^2 + \frac{1}{2}\|\alpha x\|_G^2 \\ &= \frac{1}{2}(\|P_{X,G}(x - \alpha G^{-1}F(x))\|_G^2 - \|x\|_G^2 \\ &\quad - (\|P_{X,G}(\alpha x - G^{-1}F(x))\|_G^2 - \|\alpha x\|_G^2)) = \alpha M(x, \alpha), \end{aligned} \tag{2.21}$$

The later part of (2.20) follows from (2.19). Therefore, the lemma is true.

3. Unconstrained methods for GCP

In last section, we have stated some definition and discussed the relations between a generalized augmented Lagrangian method and the merit function in [2]. In this section, we mainly consider the properties of these functions and use them to construct unconstrained methods for (1.2). First we state some properties of the functions defined by (2.18). Because X is a closed convex solid cone, similar to [1], [2], we have the following result.

Lemma 3.1. *For each $x \in R^n$, let $H(\alpha,\beta)$ is the mapping defined in section 2. Then x solves the generalized complementarity problem (1.2) if and only if αx is a fixed point of $H(\alpha,\beta)$.*

Let $f(x,\alpha,\beta)$ be defined by (2.18), it can be easily verified that

$$\begin{aligned} f(x,\alpha,\beta) &= -\min_{y\in X}\langle \beta F(x) + \frac{1}{2}G(y-\alpha x), y-\alpha x\rangle \\ &= -\min_{y\in X_C}\langle \beta C^{-T}F(x) + \frac{1}{2}(y-\alpha Cx), y-\alpha Cx\rangle, \quad \alpha,\beta>0. \end{aligned} \tag{3.1}$$

Similar to Theorems 3.1 and 3.2 of Fukushima[2], we have the following results.

Lemma 3.2. *Let the function $f(x,\alpha,\beta) : R^n \to R$ be defined by (2.18). Then $f \ge 0$ for all $x \in X$, and $f = 0$ if and only if x solves the generalized complementarity problem (1.2). Furthermore, if $F(x)$ is continuously differentiable, then f is also continuously differentiable and its gradient is given by*

$$\nabla f = \alpha\beta F(x) - [\beta\nabla F^T(x) - \alpha G](H(\alpha,\beta) - \alpha x). \tag{3.2}$$

Let $H_1 = H(1,\alpha)$, $H_2 = H\Big(1,\frac{1}{\alpha}\Big) = \frac{1}{\alpha}H(\alpha,1)$, for any $\alpha > 1$, we have:

$$\begin{aligned} M(x,\alpha) &= \frac{f(x,1,\alpha)}{\alpha} - \alpha f\Big(x,1,\frac{1}{\alpha}\Big) \\ &= -\Big\langle F(x) + \frac{1}{2\alpha}G(H_1-x), H_1-x\Big\rangle - \alpha f\Big(x,1,\frac{1}{\alpha}\Big) \\ &\ge -\alpha\Big\langle \frac{1}{\alpha}F(x) + \frac{1}{2\alpha^2}G(H_2-x), H_2-x\Big\rangle - \alpha f\Big(x,1,\frac{1}{\alpha}\Big) \\ &= \Big(\frac{\alpha^2-1}{2\alpha}\Big)\|H_2-x\|_G^2 \ge 0 \end{aligned} \tag{3.3}$$

Combining Lemmas 3.1 and 3.2, we have the following result:

Theorem 3.1. *Let $M(x,\alpha)$ be defined by (2.15) with $\alpha > 1$, X as stated in section 2, then x^* solves the generalized complementarity (1.2) if and only if $M(x^*,\alpha) = 0$.*

It is obvious that $M(x,1) \equiv 0$. When $\alpha < 1$, it follows from (2.20) and (2.19) that $M(x,\alpha) = -M(x,1/\alpha)$, so $M(x,\alpha) \le 0$ and x^* is a global maximum solution of $M(x,\alpha)$.

From (3.2), we have that

$$\nabla M(x,\alpha) = -\Big(\nabla F^T - \frac{1}{\alpha}G\Big)(H(1,\alpha)-x) + (\nabla F^T - \alpha G)\Big(H\Big(1,\frac{1}{\alpha}\Big) - x\Big), \tag{3.4}$$

which is a generalization of (2.9) of [9]. One can show that, if x^* solves GCP (1.2) and the mapping $F(x)$ is continuously differentiable, it must hold $\nabla M(x^*,\alpha) = 0$.

In [9], under certain assumptions, the local properties of the method is discussed. In the following part of this section, we mainly concern about the differentiallity of $M(x,\alpha)$ in a neighbor ball of its global solution x^*.

Let x^* be the solution of (1.2), we define

$$N(x^*) = \text{Span}\ \{d : d \in R^n,\ x^* + \alpha d \in X, \text{for all sufficient small } \alpha \in R\}. \tag{3.5}$$

The definition implies that $N(x^*)$ is a subspace in R^n. For simplity, we denote it by N. It follows from (1.4) that

$$\langle F(x^*), d\rangle = 0, \quad \forall d \in N. \tag{3.6}$$

Let the orthogonal complement space of N be $N^\perp$, thus $F(x^*) \in N^\perp$. Define

$$\begin{aligned} X_1 &= P_N(X) = \{y|\ y = P_N(x),\ x \in X\}; \\ X_2 &= P_{N^\perp}(X) = \{y|\ y = P_{N^\perp}(x),\ x \in X\}. \end{aligned} \tag{3.7}$$

Then one can easily show that x^* is a relative interior of X_1 or equivalently $x^* \overset{X_1}{>} 0$. Furthermore, for any $x \in X$, it must hold:

$$x = x_1 + x_2,\ x_1 = P_N(x) \in X_1, x_2 = P_{N^\perp}(x) \in X_2. \tag{3.8}$$

Therefore, for any $x \in X$, (1.4) implies that:

$$\begin{aligned} 0 &\le \langle F(x^*), x - x^*\rangle = \langle F(x^*), P_N(x) - x^*\rangle + \langle F(x^*), P_{N^\perp}(x)\rangle \\ &= \langle F(x^*), P_{N^\perp}(x)\rangle. \end{aligned} \tag{3.9}$$

Let X_2^* be the dual cone of X_2 in subspace $N^\perp$ such that $X_2^* = \{x|x \in N^\perp,\ \langle x, y\rangle \ge 0,\ \forall y \in X_2\}$. To obtain a neat form of the Hessian of $M(x, \alpha)$, we make the following assumptions:

Assumption 1. 1) X_2^* is solid in $N^\perp$. 2) $F(x^*)$ is an interior of X_2^*.

It is obvious that this assumption equals to $F(x^*) \overset{X_2^*}{>} 0$. If Assumption 1 are satisfied, we call x^* a nondegenerate solution of the GCP (1.2), correspondently Assumption 1 is called a generalized strict complementarity condition. It is not hard to show that Assumption 1 reduce to the general strict complementarity conditions if X is the nonnegative orthant of R^n.

In what follows , we give an equivalent condition for Assumption 1. Define

$$N_1 = \text{Span}\ \{d : d \in R^n,\ F(x^*) + \alpha d \in X^*,\ \text{for all sufficient small } \alpha \in R\}. \tag{3.10}$$

We have the following result

Lemma 3.3. *If x^* is a solution of the GCP (1.2), then $N \perp N_1$, consequently*

$$dim\ (N) + dim\ (N_1) \le n. \tag{3.11}$$

Proof. From the definitions of N and N_1, we have

$$\langle x^* + td, F(x^*) + sy\rangle \ge 0, \quad \forall d \in N, y \in N_1, \tag{3.12}$$

if $t \in R$ and $s \in R$ are sufficiently small. The above inequality, (3.6) and $\langle x^*, F(x^*)\rangle = 0$ give that

$$st\langle d, y\rangle + t\langle x^*, y\rangle \ge 0, \tag{3.13}$$

for all sufficiently small s and t, which shows that

$$\langle x^*, y \rangle = 0; \tag{3.14}$$

$$\langle d, y \rangle = 0. \tag{3.15}$$

The later equality proves the lemma.

Our next result derives another equivalent condition for the generalized strict complementarity condition:

Lemma 3.4. *If x^* is a solution of the GCP* (1.2)*, then Assumption 1 hold if and only if*

$$N_1 = N^{\perp}. \tag{3.16}$$

Proof. 1) of Assumption 1 implies

$$\dim\,(X_2^*) = \dim\,(N^{\perp}) = n - dim(N). \tag{3.17}$$

The definitions of X_2 and X_2^* show that

$$X_2^* \subset X^*, \tag{3.18}$$

which, combining 2) of Assumption 1 and (3.10), yield that

$$\text{Span } X_2^* \subset N_1. \tag{3.19}$$

It follows that

$$\dim\,(N_1) \geq \dim\,(X_2^*) = \dim\,(N^{\perp}). \tag{3.20}$$

The above inequality and Lemma 3.3 give (3.16).

Assume (3.16) holds, it follows from the definition of X_2^* that

$$X_2^* = X^* \cap N^{\perp} = X^* \cap N_1. \tag{3.21}$$

This relation shows

$$\dim\,(X_2^*) = \dim\,(N_1) = \dim\,(N^{\perp}) \tag{3.22}$$

which says that X_2^* is solid in $N^{\perp}$. (3.10) and (3.21) imply that $F(x^*)$ is an interior of X_2^* in the subspace N_1.

Define

$$\begin{aligned} N_C &= \text{Span } \{d | Cx^* + \alpha d \in X_C, \text{ for all sufficiently small } \alpha \in R, \} \\ &= \text{Span } \{d | d = Cd_1, \forall d_1 \in N\}. \end{aligned} \tag{3.23}$$

By lemma 3.4, if assumption 1 is true, one can show that

$$\begin{aligned} N_{C1} &= \text{Span } \{d : d \in R^n,\ C^{-T}F(x^*) + \alpha d \in (X_C)^*, \text{ for all sufficient small } \ \alpha \in R\} \\ &= N_C^{\perp} = \text{Span } \{d | d = C^{-T}d_1, \forall d_1 \in N^{\perp}\}. \end{aligned} \tag{3.24}$$

If we define similar sets X_{C1} and $X_{C2}, (X_{C2})^*$, it follows that

$$Cx^* \overset{X_{C1}}{>} 0, \quad C^{-T}F(x^*) \overset{(X_{C2})^*}{>} 0. \tag{3.25}$$

By the definition of N_C, there exists a projection matrix A and $A = A^T = A^2$ such that

$$P_{N_C}(x) = Ax, \ \forall x \in R^n. \tag{3.26}$$

Because x^* is a solution of (2.4), we have

$$P_{N_C}(Cx^*) = ACx^* = Cx^*, \quad P_{N_C^\perp}(C^{-T}F(x^*)) = AC^{-T}F(x^*) = 0. \tag{3.27}$$

Assume x^* is a nondegenerate solution of the GCP (1.2). Let $B(x, \epsilon)$ be the neighbor ball of x defined as follows:

$$B(x, \epsilon) = \{y : \|y - x\| < \epsilon, \ y \in R^n\}. \tag{3.28}$$

For continuous mapping $F(x)$, Assumption 1 and the fact that Cx^* is a relative interior of X_{C1} imply

$$P_{N_C^\perp}(C^{-T}F(x) - Cx) \in (X_{C2})^*, \ P_{N_C}(Cx - C^{-T}F(x)) \in X_{C1}, \ \forall x \in B(x^*, \epsilon), \tag{3.29}$$

if $\epsilon > 0$ is sufficiently small. For such $x \in B(x^*, \epsilon)$ and any $y \in X_C$, we have:

$$\begin{aligned}
&\langle Cx - C^{-T}F(x) - P_{N_C}(Cx - C^{-T}F(x)), y - P_{N_C}(Cx - C^{-T}F(x))\rangle \\
&\quad = \langle P_{N_C^\perp}(Cx - C^{-T}F(x)), P_{N_C}(y) - P_{N_C}(Cx - C^{-T}F(x))\rangle \\
&\qquad + \langle P_{N_C^\perp}(Cx - C^{-T}F(x)), P_{N_C^\perp}(y)\rangle = -\langle P_{N_C^\perp}(F(x) - x), P_{N_C^\perp}(y)\rangle \le 0,
\end{aligned} \tag{3.30}$$

which implies that

$$P_{X_C}(Cx - C^{-T}F(x)) = P_{N_C}(Cx - C^{-T}F(x)) = A(Cx - C^{-T}F(x)), \ \forall x \in B(x^*, \epsilon), \tag{3.31}$$

The above relation, with (2.6) and (3.4) shows that for any $x \in B(x^*, \epsilon)$

$$\begin{aligned}
\nabla M(x, \alpha) = &-\Big(\nabla F^T - \frac{1}{\alpha}G\Big)[C^{-1}ACx - \alpha C^{-1}AC^{-T}F - x] \\
&+ (\nabla F^T - \alpha G)\Big[C^{-1}ACx - \frac{1}{\alpha}C^{-1}AC^{-T}F - x\Big],
\end{aligned} \tag{3.32}$$

which indicates that $M(x, \alpha)$ is twice differentiable near x^* if $F(x)$ is twice differentiable. So it follows from (3.27) that

$$\begin{aligned}
\nabla^2 M(x^*, \alpha) = &-\Big(\nabla F^T - \frac{1}{\alpha}G\Big)[C^{-1}AC - \alpha C^{-1}AC^{-T}\nabla F - I] \\
&+ (\nabla F^T - \alpha G)\Big[C^{-1}AC - \frac{1}{\alpha}C^{-1}AC^{-T}\nabla F - I\Big] \\
= &\ \frac{\alpha^2 - 1}{\alpha}[\nabla F^T C^{-1}AC^{-T}\nabla F + G - C^T AC].
\end{aligned} \tag{3.33}$$

Because A is a projection matrix, it holds

$$A = A^T = A^2, \quad I - A = (I - A)^T = (I - A)^2. \tag{3.34}$$

It follows that

$$\nabla^2 M(x^*, \alpha) = \frac{\alpha^2 - 1}{\alpha}[\nabla F^T C^{-1} A, C^T(I - A)] \begin{bmatrix} AC^{-T}\nabla F \\ (I - A)C \end{bmatrix}. \tag{3.35}$$

Thus, we have the following result:

Theorem 3.2. *Let $x^* \in R^n$ be a nondegenerate solution of the GCP* (1.2), *if $F(x)$ is twice differentiable at x^* and if*

$$[\nabla F^T C^{-1} A, C^T(I - A)] \tag{3.36}$$

is of rank n, then x^* is a strictly local minimizer of $M(x, \alpha)$ for any $\alpha > 1$.

When X is the nonnegative orthant and $C = I$, condition (3.36) reduces to the linear independence of the gradients of inactive constraints. When $X = R^n$, our method equals to minimizing the G-norm of $F(x)$.

4. Numerical results

A FORTRAN subroutine is designed to test our method. We use the BFGS method with inexact line searches to the unconstrained optimization problem

$$\min_{x \in R^n} M(x, \alpha), \;\; \alpha > 1. \tag{4.1}$$

The stopping criterion is that $\|\nabla M(x, \alpha)\| \leq 10^{-8}$. Due to $M(x, \alpha) = -M\Big(x, \dfrac{1}{\alpha}\Big)$, (5.1) is equivalent to

$$\min_{x \in R^n} -M(x, \alpha), \;\; \alpha < 1. \tag{4.2}$$

In our numerical tests, we let the matrix $G = I_n$ simplily. The two problems in [9] are tested by our method and the results are reported in Tables 1—2.

Problem 1.

$$\begin{aligned} F_1(x) &= 3x_1^2 + 2x_1x_2 + 2x_2^2 + x_3 + 3x_4 - 6, \\ F_2(x) &= 2x_1^2 + x_1 + x_2^2 + 10x_3 + 2x_4 - 2, \\ F_3(x) &= 3x_1^2 + x_1x_2 + 2x_2^2 + 2x_3 + 9x_4 - 9, \\ F_4(x) &= x_1^2 + 3x_2^2 + 2x_3 + 3x_4 - 3.0. \end{aligned}$$

This problem has two solution, one is a degenerate solution $\bar{x}^1 = (\sqrt{6}/2, 0, 0, 1/2)$, another one is a nondegenerate solution $\bar{x}^2 = (1, 0, 3, 0)$. For different α and initial points, we have the following result:

Tabel 1

Value of α	Initial point	Solution	NI/NF/NG	Val. of. $M(x,\alpha)$
1.5	$(0,0,0,0)$	$\bar{x}^1$	38/57/44	0.929372E-19
0.5		$\bar{x}^1$	37/63/47	0.21904E-18
0.2		$\bar{x}^1$	54/90/67	0.109251E-15
1.5	(1,-1,-1,1)	$\bar{x}^2$	22/37/26	0.627194E-20
0.5		$\bar{x}^2$	21/31/24	0.183154E-19
0.2		$\bar{x}^2$	21/35/26	0.208077E-19
0.5	(-1,1,1,-1)	$\bar{x}^1$	43/66/49	0.90732E-15
0.2		$\bar{x}^1$	55/80/62	0.203612E-16
100	(-2,-2,-2,-2)	$\bar{x}^1$	57/107/72	0.758048E-16
0.2		$\bar{x}^1$	38/67/50	0.294449E-16
1.5	(100,100,100,100)	$\bar{x}^2$	40/65/48	0.351688E-18
0.2		$\bar{x}^1$	35/59/41	0.689323E-18

In all the tables in this paper, NI is the number of iterations, NF and NG are the numbers of the evaluations of function $M(x,\alpha)$ and its gradient respectively. It is easy to see that, for most cases, our method converges to the solution of the original problem with properly chosen α.

Problem 2.

$$
\begin{aligned}
F_1(x) &= -x_2 + x_3 + x_4,\\
F_2(x) &= x_1 - 0.75(x_3 + \beta x_4)/x_2,\\
F_3(x) &= 1 - x_1 - 0.25(x_3 + \beta x_4)/x_3,\\
F_4(x) &= \beta - x_1.
\end{aligned}
$$

For different β and appropriately chosen α, we have the following result:

Tabel 2

Val. of β	Val. of α	Initial point	Solution	NI/NF/NG	Val. of. $M(x,\alpha)$
2.0	2.0	(1,1,1,1)	$(0.75, 1.663205, 1.663205, 0)$	10/13/11	0.459804E-17
	5.0		$(0.75, 1.679367, 1.679367, 0)$	10/16/11	0.191726E-18
	50	(10,10,10,10)	$(0.75, 10.49145, 10.49145, 0)$	15/28/18	0.366259E-18
0.5	2.0	(1,1,1,1)	$(0.5, 1.36502, 0.455, 0.91)$	12/16/14	0.962439E-17
	20		$(0.5, 0.871568, 0.290553, 0.581106)$	15/33/21	0.561812E-20
	5.0	(100,100,100,100)	$(0.5, 131.417973, 43.806, 87.61198)$	19/37/31	0.396077E-17
	20		$(0.5, 146.2952, 48.765, 97.53)$	14/39/29	0.842419E-19

We also use our method for a generalized linear complementarity problem which has the following form:

Problem 3.

$$F_1(x) = 10x_1 - 5x_2 - 1,$$

$$F_2(x) = x_1 + 5x_2,$$
$$F_3(x) = -3x_1 - 3x_2 + 8x_3 + 2x_4 - x_5,$$
$$F_4(x) = -4x_1 - 4x_2 + 2x_3 + 9x_4 + 2x_5,$$
$$F_5(x) = -5x_1 - 5x_2 - x_3 + 4x_4 + 15x_5.$$

with the constrained set $X = \{x : x_5 \geq 0,\ \sum_{i=1}^{4} x_i^2 \leq x_5^2,\ x \in R^5\}$. For different α and initial points, our method converges to the same point: $x^* = (0.0480333,\ -0.00309967,\ 0.00960245,\ 0.0031883,\ 0.0491851)$, the following is our result:

Tabel 3

Value of α	Initial point	NI/NF/NG	Val. of. $M(x,\alpha)$
1.5	(0,0,0,0,0)	24/50/33	0
5		28/62/37	0
1.5	(1,1,1,1,1)	23/41/29	0.60422E-17
5		36/64/40	0.1036E-16
1.5	(10,10,10,10,10)	26/47/33	0.4228E-17
5		33/51/35	0
1.5	(1000,1000,1000,1000,1000)	34/55/41	0
5		40/70/49	0
1.5	(-100,-100,-100,-100,-100)	27/45/33	0.3943E-17
5		24/49/30	0

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