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FINITE ELEMENT APPROXIMATION OF EIGENVALUE PROBLEM FOR A COUPLED VIBRATION BETWEEN ACOUSTIC FIELD AND PLATE

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Abstract

We formulate a coupled vibration between plate and acoustic field in mathematically rigorous fashion. It leads to a non-standard eigenvalue problem. A finite element approximation is considered in an abstract way, and the approximate eigenvalue problem is written in an operator form by means of some Ritz projections. The order of convergence is proved based on the result of Babuška and Osborn. Some numerical example is shown for the problem for which the exact analytical solutions are calculated. The results shows that the convergence order is consistent with the one by the numerical analysis.

1. Introduction

In this paper, we study a numerical method to calculate eigen-frequencies of a coupled vibration between acoustic field and plate. A typical application of this research is to reduce a noise inside a car caused by an engine or other sources of the sound. Our study was motivated by the work of Hagiwara et al.[5]. The background of the research and some applications can be seen in [5]. We restrict our research to the problems where exact solutions can be given in a special case.

The main feature of our research is the mathematically rigorous approach to the problem. We formulate the problem as an eigenvalue problem in some Hilbert space and approximate it using the finite element method. We prove the convergence of the approximate eigenvalues. We show some numerical example for a two-dimensional test problem and check the validity of our method and analysis.

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2. Formulation of a Problem

We study the vibrations of an acoustic field coupled with a plate which consists of a part of the boundary (Fig. 1). We assume that a shape of the plate is rectangular. This

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condition together with a special boundary condition enables us to reduce the problem to a two-dimensional one (Fig. 2).

Fig. 1 Fig. 2

The time evolution problem for this coupled system is described by the following system of partial differential equations(cf. [3], where the boundary conditions are slightly different):

$$
\begin{cases}
\dfrac{\partial^2}{\partial t^2} P_0 - c^2 \nabla^2_{x,y,z}\ P_0 = 0 & \text{in} \quad \Omega_0,\ \nabla_{x,y,z} = \Big(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z}\Big), \\
\qquad \dfrac{\partial P_0}{\partial n} = -\rho_0 \dfrac{\partial^2 U_0}{\partial t^2} & \text{on} \quad S_0, \\
\qquad P_0|_{\Gamma_0} = 0 & \text{on} \quad \Gamma_0, \\
\rho_1 \dfrac{\partial^2 U_0}{\partial t^2} + D \nabla^4_{y,z} U_0 = P_0|_{S_0} & \text{on} \quad S_0,\ \nabla_{y,z} = \Big(\dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z}\Big), \\
U_0|_{\partial S_0} = \dfrac{\partial^2 U_0}{\partial \sigma^2}\Big|_{\partial S_0} = 0 & \text{on} \quad \partial S_0,
\end{cases}
\tag{1}
$$

where

Ω :	two-dimensional bounded region,
$\Omega_0 = \Omega \times (0, \pi)$:	three-dimensional acoustic field,
$S_0 = S \times (0, \pi)$:	domain of plate,
$\Gamma_0 = \Gamma \times (0, \pi)$:	a part of the boundary of acoustic field, $\partial\Omega_0 = S_0 \cup \Gamma_0$,
∂S_0 :	boundary of plate,
P_0 :	acoustic pressure in Ω_0,
U_0 :	vertical plate displacement,
c :	sound velocity,
ρ_0 :	air mass density,
D :	flexural rigidity of plate,
ρ_1 :	plate mass density,
n :	outward normal vector on $\partial\Omega_0$,
σ :	outward normal vector on ∂S_0.

Let us consider the following eigen-oscillation solution to (1) with a time frequency ω:

$$\begin{cases} P_0 = e^{i\omega t}P(x,y,z) & \text{in} \quad \Omega_0, \\ U_0 = e^{i\omega t}U(y,z) & \text{on} \quad S_0. \end{cases}$$

This leads to the eigenvalue problem:

$$\begin{cases} -c^2\nabla^2_{x,y,z}P - \omega^2 P = 0 & \text{in} \quad \Omega_0, \\ \qquad \dfrac{\partial P}{\partial n}|_{S_0} = \rho_0\omega^2 U & \text{on} \quad S_0, \\ \qquad P|_{\Gamma_0} = 0 & \text{on} \quad \Gamma_0, \\ D\nabla^4_{y,z}U - \omega^2\rho_1 U = P|_{S_0} & \text{on} \quad S_0, \\ U|_{\partial S_0} = \dfrac{\partial^2 U}{\partial\sigma^2}|_{\partial S_0} = 0 & \text{on} \quad \partial S_0. \end{cases} \tag{2}$$

Because of a symmetry of the domain Ω_0 and the boundary conditions, we can make the Fourier mode decomposition in z direction:

$$\begin{cases} P = p\sin mz, \\ U = u\sin mz, \end{cases}$$

with the Fourier mode number m (= *integer*). Then the problem (2) is transformed into the following problem (3):

$$\begin{cases} -\nabla^2_{x,y}p + (-\omega^2/c^2 + m^2)p = 0 & \text{in} \quad \Omega, \\ \qquad p|_{\Gamma} = 0 & \text{on} \quad \Gamma, \\ \qquad \dfrac{\partial p}{\partial x}|_S = \rho_0\omega^2 u & \text{on} \quad S \ (= (0,\pi)), \\ D(u'''' - 2m^2u'' + m^4u) - \omega^2\rho_1 u = p|_S & \text{on} \quad S, \\ u(0) = u''(0) = 0, \\ u(\pi) = u''(\pi) = 0. \end{cases} \tag{3}$$

3. Operator Theoretical Approach to a Two-Dimensional Eigenvalue Problem

For a rather general 2-dimensional bounded domain Ω with its boundary $\partial\Omega = \Gamma \cup S$, the eigenvalue problem (3) for a coupled system has a weak formulation:

$$\begin{cases} \dfrac{1}{\rho_0}[\ll p, q \gg + m^2((p,q))] = \omega^2\Big[\dfrac{1}{\rho_0 c^2}((p,q)) + (u, \gamma_S q)\Big], \\ D[(u'', v'') + 2m^2(u', v') + m^4(u,v)] - (\gamma_S p, v) = \omega^2\rho_1(u,v), \end{cases} \tag{4}$$

where $\gamma_S p$ denotes a restriction of p onto the set S and

$$\ll p, q \gg = \int_\Omega \nabla_{x,y}p \cdot \nabla_{x,y}\bar{q}\; dxdy,$$

$$((p,q)) = \int_\Omega p\bar{q}\ dxdy,$$
$$(u,v) = \int_0^\pi u\bar{v}\ dy.$$

We use the standard notations of the Sobolev spaces[2], and introduce the following function spaces $\mathcal{V}_s$, $\mathcal{V}_p$ and various bilinear forms:

$$\begin{aligned}
&\mathcal{V}_s \equiv \{p : p \in H^1(\Omega), \quad p|_\Gamma = 0\},\\
&\mathcal{V}_p \equiv \{u : u \in H^2(S), \quad u|_{\partial S} = 0\},\\
&a_s(p,q) = \frac{1}{\rho_0}[\ll p,q \gg +m^2((p,q))],\\
&b_s(p,q) = \frac{1}{\rho_0 c^2}((p,q)),\\
&c_\theta(u,q) = (u, \gamma_S q),\\
&\bar{c}_\theta(p,v) = (\gamma_S p, v),\\
&a_p(u,v) = D[(u^{''}, v^{''}) + 2m^2(u^{'}, v^{'}) + m^4(u,v)],\\
&b_p(u,v) = \rho_1(u,v).
\end{aligned}$$

Then the eigenvalue problem (4) leads to the following compact form:

$$\begin{cases} a_s(p,q) = \omega^2[b_s(p,q) + c_\theta(u,q)], \\ a_p(u,v) - \bar{c}_\theta(p,v) = \omega^2 b_p(u,v). \end{cases} \tag{5}$$

The function spaces $\mathcal{V}_s$ and $\mathcal{V}_p$ are Hilbert spaces with inner products $a_s(p,q)$ and $a_p(u,v)$, respectively. The Riesz representation theorem implies that there exist bounded operators A_s (in $\mathcal{V}_s$) and A_p (in $\mathcal{V}_p$) such that

$$\begin{cases} b_s(p,q) = a_s(A_s p, q) & \text{for all } q \in \mathcal{V}_s, \\ b_p(u,v) = a_p(A_p u, v) & \text{for all } v \in \mathcal{V}_p. \end{cases} \tag{6}$$

By the theory of elliptic equations and Rellich's lemma, the operators A_s and A_p are compact. The Hermitian symmetricity of a_s and a_p implies that A_s and A_p are both self-adjoint. We define the operators $T : \mathcal{V}_s \to \mathcal{V}_p$ and $T^* : \mathcal{V}_p \to \mathcal{V}_s$ as follows:

$$\begin{cases} c_\theta(u,q) = a_s(T^* u, q) & \text{for all } q \in \mathcal{V}_s, \\ \bar{c}_\theta(p,v) = a_p(Tp, v) & \text{for all } v \in \mathcal{V}_p. \end{cases} \tag{7}$$

Then these operators are also compact. The operator T^* is the adjoint of T and the operator TT^* is self-adjoint in $\mathcal{V}_p$ because

$$\begin{aligned}
&a_s(T^*u, q) = c_\theta(u,q) = \overline{\bar{c}_\theta(q,u)} = \overline{a_p(Tq,u)} = a_p(u, Tq),\\
&a_p(TT^*u, v) = a_s(T^*u, T^*v) = a_p(u, TT^*v).
\end{aligned}$$

By using the operators A_s, A_p, T and T^*, the eigenvalue problem (5) is transformed into the form:

$$\begin{cases} a_s(p,q) = \omega^2[a_s(A_s p, q) + a_s(T^* u, q)] & \text{for all } q \in \mathcal{V}_s, \\ -a_p(Tp, v) + a_p(u,v) = \omega^2 a_p(A_p u, v) & \text{for all } v \in \mathcal{V}_p. \end{cases} \tag{8}$$

This is equivalent to the following eigenvalue problem expressed by operators:

$$\begin{cases} p = \omega^2(A_s p + T^* u), \\ u - Tp = \omega^2 A_p u. \end{cases} \tag{9}$$

We can write it in a matrix form:

$$\begin{bmatrix} I & 0 \\ -T & I \end{bmatrix} \begin{bmatrix} p \\ u \end{bmatrix} = \omega^2 \begin{bmatrix} A_s & T^* \\ 0 & A_p \end{bmatrix} \begin{bmatrix} p \\ u \end{bmatrix}. \tag{10}$$

This eigenvalue problem is not of a standard self-adjoint form. But we can transform it into the self-adjoint formulation. Since

$$\begin{bmatrix} I & 0 \\ -T & I \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ T & I \end{bmatrix},$$

we get

$$\begin{bmatrix} p \\ u \end{bmatrix} = \omega^2 \begin{bmatrix} I & 0 \\ T & I \end{bmatrix} \begin{bmatrix} A_s & T^* \\ 0 & A_p \end{bmatrix} \begin{bmatrix} p \\ u \end{bmatrix} = \omega^2 \begin{bmatrix} A_s & T^* \\ TA_s & TT^* + A_p \end{bmatrix} \begin{bmatrix} p \\ u \end{bmatrix}. \tag{11}$$

Multiplying an operator

$$\begin{bmatrix} A_s^{1/2} & 0 \\ 0 & I \end{bmatrix}$$

from the left, we obtain

$$\begin{bmatrix} A_s^{1/2} p \\ u \end{bmatrix} = \omega^2 \begin{bmatrix} A_s & A_s^{1/2} T^* \\ TA_s^{1/2} & TT^* + A_p \end{bmatrix} \begin{bmatrix} A_s^{1/2} p \\ u \end{bmatrix}. \tag{12}$$

We introduce a Hilbert space $\mathcal{V} \equiv \mathcal{V}_s \times \mathcal{V}_p$ with an inner product

$$\left(\begin{pmatrix} q_1 \\ u_1 \end{pmatrix}, \begin{pmatrix} q_2 \\ u_2 \end{pmatrix} \right)_{\mathcal{V}} \equiv a_s(q_1, q_2) + a_p(u_1, u_2).$$

We define a vector x and an operator A in $\mathcal{V}$ as follows:

$$x \equiv \begin{pmatrix} A_s^{1/2} p \\ u \end{pmatrix}, \qquad A \equiv \begin{bmatrix} A_s & A_s^{1/2} T^* \\ TA_s^{1/2} & TT^* + A_p \end{bmatrix}. \tag{13}$$

Then we have finally the following symmetric eigenvalue problem in $\mathcal{V}$:

$$Ax = \frac{1}{\omega^2} x. \tag{14}$$

Since the operator A_s is compact and self-adjoint, the operator $A_s^{1/2}$ is also compact and self-adjoint. It is clear that A is a compact and self-adjoint operator in $\mathcal{V}$. Summing up these results, we have the following theorem.

Theorem 3.1. *The eigenvalue problem* (4) *is reduced to the symmetric eigenvalue problem:*

$$Ax = \frac{1}{\omega^2} x \tag{15}$$

with a compact self-adjoint operator A in $\mathcal{V}$, and hence the spectrum of A consists of zero and discrete eigenvalues $\lambda_n \equiv \dfrac{1}{\omega_n^2}$, $n = 1, 2, 3, \cdots$, which are real and at most countable with the property: $\lim_{n \to \infty} \lambda_n = 0$.

Remark. The idea to symmetrize the eigenvalue problem was first introduced by Irons[3] for the related discretized matrix eigenvalue problem.

4. Finite Element Approximation

Let us introduce finite dimensional spaces $\mathcal{V}_{sh} \subset \mathcal{V}_s$ and $\mathcal{V}_{ph} \subset \mathcal{V}_p$, and consider the finite element approximation problem (16) for (5):

Find $p_h \in \mathcal{V}_{sh}$ and $u_h \in \mathcal{V}_{ph}$ such that

$$\begin{cases} a_s(p_h, q_h) = \omega_h^2[(b_s(p_h, q_h) + c_\theta(u_h, q_h))] & \text{for all } q_h \in \mathcal{V}_{sh}, \\ a_p(u_h, v_h) - \bar{c}_\theta(p_h, v_h) = \omega_h^2 b_p(u_h, v_h) & \text{for all } v_h \in \mathcal{V}_{ph}. \end{cases} \tag{16}$$

We define orthogonal projections $P_{sh} : \mathcal{V}_s \to \mathcal{V}_{sh}$ and $P_{ph} : \mathcal{V}_p \to \mathcal{V}_{ph}$ as follows:

$$\begin{cases} a_s(p, q_h) = a_s(P_{sh}p, q_h) & \text{for all } q_h \in \mathcal{V}_{sh}, \\ a_p(u, v_h) = a_p(P_{ph}u, v_h) & \text{for all } v_h \in \mathcal{V}_{ph}. \end{cases} \tag{17}$$

Then, (16) is changed into (18) by using (6), (7) and (17):

Find $p_h \in \mathcal{V}_{sh}$ and $u_h \in \mathcal{V}_{ph}$ such that

$$\begin{cases} a_s(p_h, q_h) = \omega_h^2[a_s(P_{sh}A_s p_h, q_h) + a_s(P_{sh}T^*P_{ph}u_h, q_h)] & \text{for all } q_h \in \mathcal{V}_{sh}, \\ a_p(u_h, v_h) - a_p(P_{ph}Tp_h, v_h) = \omega_h^2 a_p(P_{ph}A_pP_{ph}u_h, v_h) & \text{for all } v_h \in \mathcal{V}_{ph}. \end{cases} \tag{18}$$

This is equivalent to the equation:

$$\begin{cases} p_h = \omega_h^2(P_{sh}A_s p_h + P_{sh}T^*P_{ph}u_h), \\ u_h - P_{ph}Tp_h = \omega_h^2 P_{ph}A_pP_{ph}u_h, \end{cases} \tag{19}$$

which leads to the matrix form:

$$\begin{bmatrix} I & 0 \\ -P_{ph}T & I \end{bmatrix} \begin{bmatrix} p_h \\ u_h \end{bmatrix} = \omega_h^2 \begin{bmatrix} P_{sh}A_s & P_{sh}T^*P_{ph} \\ 0 & P_{ph}A_pP_{ph} \end{bmatrix} \begin{bmatrix} p_h \\ u_h \end{bmatrix}. \tag{20}$$

This eigenvalue problem is not of a standard self-adjoint form. But we can transform it into the self-adjoint formulation. Since

$$\begin{bmatrix} I & 0 \\ -P_{ph}T & I \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ P_{ph}T & I \end{bmatrix},$$

we get:

$$\begin{bmatrix} p_h \\ u_h \end{bmatrix} = \omega_h^2 \begin{bmatrix} P_{sh}A_s & P_{sh}T^*P_{ph} \\ P_{ph}TP_{sh}A_s & P_{ph}TP_{sh}T^*P_{ph} + P_{ph}A_pP_{ph} \end{bmatrix} \begin{bmatrix} p_h \\ u_h \end{bmatrix}. \tag{21}$$

Multiplying

$$\begin{bmatrix} A_s^{1/2} & 0 \\ 0 & I \end{bmatrix}$$

from the left, we obtain:

$$\begin{bmatrix} A_s^{1/2}p_h \\ u_h \end{bmatrix} = \omega_h^2 \begin{bmatrix} A_s^{1/2}P_{sh}A_s^{1/2} & A_s^{1/2}P_{sh}T^*P_{ph} \\ P_{ph}TP_{sh}A_s^{1/2} & P_{ph}TP_{sh}T^*P_{ph} + P_{ph}A_pP_{ph} \end{bmatrix} \begin{bmatrix} A_s^{1/2}p_h \\ u_h \end{bmatrix}. \tag{22}$$

We introduce a finite element approximation subspace $\mathcal{V}_h \equiv A_s^{1/2}\mathcal{V}_{sh} \times \mathcal{V}_{ph}$, and define a vector x_h and an operator A_h in $\mathcal{V}_h$ as follows:

$$x_h = \begin{bmatrix} A_s^{1/2} p_h \\ u_h \end{bmatrix},$$
$$A_h = \begin{bmatrix} A_s^{1/2} P_{sh} A_s^{1/2} & A_s^{1/2} P_{sh} T^* P_{ph} \\ P_{ph} T P_{sh} A_s^{1/2} & P_{ph} T P_{sh} T^* P_{ph} + P_{ph} A_p P_{ph} \end{bmatrix}. \tag{23}$$

It is clear that A_h is a compact self-adjoint operator in $\mathcal{V}(A_h : \mathcal{V} \to \mathcal{V})$ as well as in $\mathcal{V}_h(A_h : \mathcal{V}_h \to \mathcal{V}_h)$. The difference between the spectrum of A_h in $\mathcal{V}$ and the one in $\mathcal{V}_h$ consists of only zero.

The approximate problem (16) is reduced to the following eigenvalue problem in $\mathcal{V}_h$:

$$A_h x_h = \lambda_{nh} x_h \quad \Big(\lambda_{nh} = \frac{1}{\omega_h^2},\ \lambda_{1h} \geq \lambda_{2h} \geq \cdots \geq \lambda_{nh} \geq \lambda_{(n+1)h} \geq \cdots\Big), \tag{24}$$

where we enumerate the eigenvalues repeatedly according to their multiplicity.

5. Error Estimation of Eigenvalues

In the previous section, $\mathcal{V}_h = A_s^{1/2}\mathcal{V}_{sh} \times \mathcal{V}_{ph}$ was an abstract finite dimensional subspace of $\mathcal{V}$. In this section, we assume that the domain Ω is a polygon, and introduce the following condition.

Condition R. For $p \in \mathcal{V}_s$ and $u \in \mathcal{V}_p$, there exists a unique solution q of the following elliptic boundary value problem:

$$\begin{cases} -\nabla^2 q + m^2 q = \dfrac{p}{c^2} \quad \text{in} \quad \Omega, \\ q|_\Gamma = 0, \\ \dfrac{\partial q}{\partial x}|_S = \rho_0 u, \end{cases}$$

and the elliptic regularity estimate:

$$\|q\|_{H^2(\Omega)} \leq C\{\|p\|_{L^2(\Omega)} + \|u\|_{\mathcal{V}_p}\}$$

holds for some constant C which is independent of p and u.

Remark. When the domain Ω is a polygon and the angles between S and Γ is $\pi/2$, the above Condition R is satisfied (see Section 4 in Grisvard[4]).

We apply the finite element method with the spaces

$$\mathcal{V}_{sh} = \{q \in \mathcal{V}_s : q|_K \in P_1(K),\ \text{for } K \in \mathcal{T}_h\} \tag{25}$$

for a polygonal domain Ω and

$$\mathcal{V}_{ph} = \{v \in \mathcal{V}_p : v|_{K'} \in P_3(K'),\ \text{for } K' \in \mathcal{T}_h'(S)\} \tag{26}$$

for a straight line S, where $\mathcal{T}_h$ is a regular triangulation of Ω with maximum mesh size h, $\mathcal{T}_h^{'}(S)$ is a regular partition of S with maximum mesh size h and, for $r \in N$,

$$P_r(K) = \Big\{ q : q(x) = \sum_{0 \le i+j \le r} a_{ij} x_1^i x_2^j \ \text{ for } x \in K,\ a_{ij} \in R \Big\}, \tag{27}$$

and

$$P_r(K^{'}) = \Big\{ v : v(x) = \sum_{0 \le i \le r} a_i x^r \ \text{ for } x \in K^{'},\ a_i \in R \Big\}. \tag{28}$$

Then we have the following estimate for the error $|\lambda_n - \lambda_{nh}|$.

Theorem 5.1. *Let $\lambda_n \neq 0$ be the n-th eigenvalue of A in* (15) *and λ_{nh} be the corresponding approximate n-th eigenvalues of A_h in* (24) *where $\mathcal{V}_h$ is defined through* (25) *and* (26). *Then we have the estimate*

$$|\lambda_n - \lambda_{nh}| \le C_{\lambda_n} h^2$$

with positive constant C_{λ_n} which does not depend on h but may depend on λ_n.

Proof. At first we calculate a difference $A - A_h$ and define operators A_{ij} $(i,j = 1,2)$ as

$$\begin{aligned} A - A_h &= \begin{bmatrix} A_s - A_s^{1/2} P_{sh} A_s^{1/2} & A_s^{1/2} T^* - A_s^{1/2} P_{sh} T^* P_{ph} \\ T A_s^{1/2} - P_{ph} T P_{sh} A_s^{1/2} & TT^* + A_p - P_{ph} T P_{sh} T^* P_{ph} - P_{ph} A_p P_{ph} \end{bmatrix} \\ &\equiv \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. \end{aligned} \tag{29}$$

Then we can prove that

$$\lim_{h \downarrow 0} \|A - A_h\|_{\mathcal{V}} = 0. \tag{30}$$

In fact, since the operators A_s, $A_s^{1/2}$, T and T^* are compact, using the basic properties of the Ritz projections P_{sh} and P_{ph} (see [2]):

$$s - \lim_{h \downarrow 0} P_{sh} = I \ \text{ in } \mathcal{V}_s \quad \text{ and } \quad s - \lim_{h \downarrow 0} P_{ph} = I \ \text{ in } \mathcal{V}_p,$$

we have the following estimates for A_{ij} $(i,j = 1,2)$:

$$\begin{aligned} \|A_{11}\|_{\mathcal{V}_s} &= \|A_s - A_s^{1/2} P_{sh} A_s^{1/2}\|_{\mathcal{V}_s} = \|A_s^{1/2}(I - P_{sh}) A_s^{1/2}\|_{\mathcal{V}_s} \to 0 \quad \text{as } h \to 0; \\ \|A_{12}\|_{\mathcal{V}_p \to \mathcal{V}_s} &= \|A_s^{1/2} T^* - A_s^{1/2} P_{sh} T^* P_{ph}\|_{\mathcal{V}_p \to \mathcal{V}_s} = \|A_s^{1/2}(T^* - P_{sh} T^* P_{ph})\|_{\mathcal{V}_p \to \mathcal{V}_s} \\ &= \|A_s^{1/2}\{(I - P_{sh}) T^* + P_{sh} T^* (I - P_{ph})\}\|_{\mathcal{V}_p \to \mathcal{V}_s} \\ &\le \|A_s^{1/2}(I - P_{sh}) T^*\|_{\mathcal{V}_p \to \mathcal{V}_s} + \|A_s^{1/2} P_{sh} T^* (I - P_{ph})\|_{\mathcal{V}_p \to \mathcal{V}_s} \to 0 \quad \text{as } h \to 0; \\ \|A_{21}\|_{\mathcal{V}_s \to \mathcal{V}_p} &= \|T A_s^{1/2} - P_{ph} T P_{sh} A_s^{1/2}\|_{\mathcal{V}_s \to \mathcal{V}_p} \\ &= \|((I - P_{ph}) T + P_{ph} T (I - P_{sh})) A_s^{1/2}\|_{\mathcal{V}_s \to \mathcal{V}_p} \\ &\le \|(I - P_{ph}) T A_s^{1/2}\|_{\mathcal{V}_s \to \mathcal{V}_p} + \|P_{ph} T (I - P_{sh}) A_s^{1/2}\|_{\mathcal{V}_s \to \mathcal{V}_p} \to 0 \quad \text{as } h \to 0; \end{aligned}$$

and

$$\|A_{22}\|_{\mathcal{V}_p} = \|TT^* + A_p - P_{ph} T P_{sh} T^* P_{ph} - P_{ph} A_p P_{ph}\|_{\mathcal{V}_p}$$

$$\begin{aligned}
&\le \|TT^* - P_{ph}TP_{sh}T^*P_{ph}\|_{\mathcal{V}_p} + \|A_p - P_{ph}A_pP_{ph}\|_{\mathcal{V}_p} \\
&\le \|(T - P_{ph}TP_{sh})T^*\|_{\mathcal{V}_p} + \|P_{ph}TP_{sh}T^*(I - P_{ph})\|_{\mathcal{V}_p} + \|(I - P_{ph})A_p\|_{\mathcal{V}_p} \\
&\quad + \|P_{ph}A_p(I - P_{ph})\|_{\mathcal{V}_p} \to 0 \quad \text{as } h \to 0.
\end{aligned}$$

Hence we have

$$\|A - A_h\|_{\mathcal{V}} \le \|A_{11}\|_{\mathcal{V}_s} + \|A_{12}\|_{\mathcal{V}_p \to \mathcal{V}_s} + \|A_{21}\|_{\mathcal{V}_s \to \mathcal{V}_p} + \|A_{22}\|_{\mathcal{V}_p} \to 0 \quad \text{as } h \to 0.$$

From this result, the n-th eigenvalue of A_h converges to the corresponding n-th eigenvalue of A (see Babuška and Osborn[1]).

Let $E(\lambda)$ be the orthogonal projection in $\mathcal{V}$ onto the space of eigenvectors of A associated with an eigenvalue λ and let $x \in E(\lambda)\mathcal{V}$. Then from the expression of A in (13), the eigenvector x can be written as

$$x = \begin{pmatrix} A_s^{1/2}p \\ u \end{pmatrix} = (A_s^{1/2}p, u)^t, \quad \text{with} \ \ p \in \mathcal{V}_s \ \ \text{and} \ \ u \in \mathcal{V}_p.$$

For the later use, we note that

$$\begin{aligned}
((A - A_h)x, x)_{\mathcal{V}} &= \left(\begin{pmatrix} A_{11}A_s^{1/2}p + A_{12}u \\ A_{21}A_s^{1/2}p + A_{22}u \end{pmatrix}, \begin{pmatrix} A_s^{1/2}p \\ u \end{pmatrix} \right)_{\mathcal{V}} \\
&\equiv a_s(A_{11}A_s^{1/2}p, A_s^{1/2}p) + a_s(A_{12}u, A_s^{1/2}p) + a_p(A_{21}A_s^{1/2}p, u) + a_p(A_{22}u, u).
\end{aligned}$$

Before continuing our proof, we introduce some lemmas which will be used later. In the following, we use C as a general constant and C_λ as a general constant which depends on λ.

Lemma 5.1. *For every $x = (A_s^{1/2}p, u)^t \in E(\lambda)\mathcal{V}$, the following inequality holds:*

$$\|A_{11}A_s^{1/2}p\|_{\mathcal{V}_s} \le Ch\|x\|_{\mathcal{V}}.$$

Lemma 5.2. *For every $x = (A_s^{1/2}p, u)^t \in E(\lambda)\mathcal{V}$, the following inequality holds:*

$$\|A_{12}u\|_{\mathcal{V}_s} \le C_\lambda h\|x\|_{\mathcal{V}}.$$

Lemma 5.3. *For every $x = (A_s^{1/2}p, u)^t \in E(\lambda)\mathcal{V}$, the following inequalities hold:*

$$\|A_{21}A_s^{1/2}p\|_{\mathcal{V}_p} \le C_\lambda h\|x\|_{\mathcal{V}};$$

and

$$\|A_{22}u\|_{\mathcal{V}_p} \le C_\lambda h\|x\|_{\mathcal{V}}.$$

Lemma 5.4. *For $x_j = (A_s^{1/2}p_j, u_j)^t \in E(\lambda)\mathcal{V}$, $j = i$ or k, the following inequalities hold:*

$$\begin{aligned}
&|a_s(A_{11}A_s^{1/2}p_i, A_s^{1/2}p_k)| \le Ch^2\|x_i\|_{\mathcal{V}}\|x_k\|_{\mathcal{V}}; \\
&|a_s(A_{12}u_i, A_s^{1/2}p_k)| \le C_\lambda h^2\|x_i\|_{\mathcal{V}}\|x_k\|_{\mathcal{V}}; \\
&|a_p(A_{21}A_s^{1/2}p_i, u_k)| \le C_\lambda h^2\|x_i\|_{\mathcal{V}}\|x_k\|_{\mathcal{V}};
\end{aligned}$$

$$|a_p(A_{22}u_i, u_k)| \le C_\lambda h^2 \|x_i\|_{\mathcal{V}} \|x_k\|_{\mathcal{V}}.$$

Proof of Theorem 5.1. (continued) Let $\lambda \equiv \lambda_n$ and $\lambda_h \equiv \lambda_{nh}$ and $E(\lambda)\mathcal{V}$ be as above, and $\varphi_1, \cdots, \varphi_m$ be an orthonormal basis of $E(\lambda)\mathcal{V}$. Then due to Theorem 7.3 by Babuška and Osborn in [2], we have the estimate:

$$|\lambda - \lambda_h| \le C\Big\{\Big|\Big(\sum_{i,k=1}^{m}(A - A_h)\varphi_i, \varphi_k\Big)_{\mathcal{V}}\Big| + \|(A - A_h)|_{E(\lambda)\mathcal{V}}\|_{\mathcal{V}}^2\Big\}.$$

From Lemma 5.1 Lemma 5.2 and Lemma 5.3, we obtain

$$\begin{aligned}
\|(A - A_h)|_{E(\lambda)\mathcal{V}}\|_{\mathcal{V}} &\equiv \sup_{x \in E(\lambda)\mathcal{V}, \|x\|_{\mathcal{V}}=1} \|(A - A_h)x\|_{\mathcal{V}} \\
&\equiv \sup_{x \in E(\lambda)\mathcal{V}, \|x\|_{\mathcal{V}}=1} \{(\|A_{11}A_s^{1/2}p + A_{12}u\|_{\mathcal{V}_s})^2 + (\|A_{21}A_s^{1/2}p + A_{22}u\|_{\mathcal{V}_p})^2\}^{1/2} \\
&\le \sup_{x \in E(\lambda)\mathcal{V}, \|x\|_{\mathcal{V}}=1} (\|A_{11}A_s^{1/2}p\|_{\mathcal{V}_s} + \|A_{12}u\|_{\mathcal{V}_s} + \|A_{21}A_s^{1/2}p\|_{\mathcal{V}_p} + \|A_{22}u\|_{\mathcal{V}_p}) \\
&\le Ch + C_\lambda h + C_\lambda h + C_\lambda h \le C_\lambda h,
\end{aligned}$$

which implies

$$\|(A - A_h)|_{E(\lambda)\mathcal{V}}\|_{\mathcal{V}} \le C_\lambda h. \tag{31}$$

Using Lemma 5.4 and (31), we obtain our estimate with convergence order 2 as follows and complete our proof. Let $\varphi_i = (A_s^{1/2}p_i, u_i)^t$, $i = 1, 2, \cdots, m$, then we have

$$\begin{aligned}
|\lambda - \lambda_h| &\le C\Big\{\Big|\Big(\sum_{i,k=1}^{m}(A - A_h)\varphi_i, \varphi_k\Big)_{\mathcal{V}}\Big| + C_\lambda h^2\Big\} \\
&\le C\Big\{\sum_{i,k=1}^{m}(|a_s(A_{11}A_s^{1/2}p_i, A_s^{1/2}p_k)| + |a_s(A_{12}u_i, A_s^{1/2}p_k)| \\
&\quad + |a_p(A_{21}A_s^{1/2}p_i, u_k)| + |a_p(A_{22}u_i, u_k)|) + C_\lambda h^2\Big\} \le C_\lambda h^2.
\end{aligned}$$

6. Proof of Lemmas

Proof of Lemma 5.1. Note that, for every $p \in \mathcal{V}_s$, $q \equiv A_s p$ satisfies the following equation:

$$\begin{cases}
-\nabla^2 q + m^2 q = \dfrac{p}{c^2} \quad \text{in} \quad \Omega, \\
q|_\Gamma = 0, \\
\dfrac{\partial q}{\partial x}|_S = 0.
\end{cases}$$

From Condition R with $u = 0$, there exists a constant C such that $\|q\|_{H^2(\Omega)} \le C\|p\|_{L^2(\Omega)}$. On the other hand, by definition (17), $q_h \equiv P_{sh}q$ is the finite element approximation of q in $\mathcal{V}_{sh}$. Hence, we have the following estimate for the error $q - q_h$ in $H^1(\Omega)$ -norm(see [2]; Theorem 18.1):

$$\|(I - P_{sh})A_s p\|_{\mathcal{V}_s} \le Ch\|A_s p\|_{H^2} \le Ch\|p\|_{L_2} \le Cha_s(A_s p, p)^{1/2}$$

$$= Cha_s(A_s^{1/2}p, A_s^{1/2}p)^{1/2} = Ch\|A_s^{1/2}p\|_{\mathcal{V}_s} \leq Ch\|x\|_{\mathcal{V}}. \tag{32}$$

Then we get the estimate for $\|A_{11}A_s^{1/2}p\|_{\mathcal{V}_s}$ as follows:

$$\begin{aligned}\|A_{11}A_s^{1/2}p\|_{\mathcal{V}_s} &= \|(A_s - A_s^{1/2}P_{sh}A_s^{1/2})A_s^{1/2}p\|_{\mathcal{V}_s} = \|A_s^{1/2}(A_s^{1/2} - P_{sh}A_s^{1/2})A_s^{1/2}p\|_{\mathcal{V}_s}\\ &\leq \|A_s^{1/2}\|_{\mathcal{V}_s}\|(I - P_{sh})A_sp\|_{\mathcal{V}_s} \leq Ch\|x\|_{\mathcal{V}}.\end{aligned}$$

Proof of Lemma 5.2. Note that for every $u \in \mathcal{V}_p$, $q \equiv T^*u$ satisfies the following equation:

$$\begin{cases} -\nabla^2 q + m^2 q = 0 \quad \text{in} \quad \Omega,\\ q|_\Gamma = 0,\\ \dfrac{\partial q}{\partial x}|_S = \rho_0 u.\end{cases}$$

Then using the results of finite element approximation and Condition R with $p = 0$, we obtain the following estimate:

$$\|(I - P_{sh})T^*u\|_{\mathcal{V}_s} \leq Ch\|T^*u\|_{H^2} \leq Ch\|u\|_{\mathcal{V}_p} \leq Ch\|x\|_{\mathcal{V}}. \tag{33}$$

Hence we have the estimate for $\|A_{12}u\|_{\mathcal{V}_s}$ as follows:

$$\begin{aligned}\|A_{12}u\|_{\mathcal{V}_s} =&\|(A_s^{1/2}T^* - A_s^{1/2}P_{sh}T^*P_{ph})u\|_{\mathcal{V}_s} \leq \|A_s^{1/2}(I - P_{sh})T^*u\|_{\mathcal{V}_s}\\ &+ \|A_s^{1/2}P_{sh}T^*\|_{\mathcal{V}_p\to\mathcal{V}_s}\|(I - P_{ph})u\|_{\mathcal{V}_p} \leq \|A_s^{1/2}\|_{\mathcal{V}_s}\|(I - P_{sh})T^*u\|_{\mathcal{V}_s}\\ &+ \|A_s^{1/2}P_{sh}T^*\|_{\mathcal{V}_p\to\mathcal{V}_s}\|(I - P_{ph})(Tp + (1/\lambda)A_pu)\|_{\mathcal{V}_p},\end{aligned} \tag{34}$$

here to prove the second inequality, we used the second equation in (9). The operator T has the following relation to A_p and γ_S:

$$a_p(Tp, v) = \bar{c}_\theta(p, v) = (\gamma_S p, v) = \frac{1}{\rho_1}b_p(\gamma_S p, v) = \frac{1}{\rho_1}a_p(A_p\gamma_S p, v),$$

which implies

$$T = \frac{1}{\rho_1}A_p\gamma_S.$$

Note that for every $u \in \mathcal{V}_p$, $v \equiv A_pu$ satisfies the following equation:

$$\begin{cases} D(v'''' - 2m^2v'' + m^4 v) = \rho_1 u \quad \text{on} \quad S \ (S = (0, \pi)),\\ v(0) = v''(0) = 0,\\ v(\pi) = v''(\pi) = 0.\end{cases}$$

From the results of the Fourier analysis, there exists a constant C such that $\|A_pu\|_{H^4(S)} \leq C\|u\|_{L^2(S)}$. Since, by definition (17), $v_h \equiv P_{ph}v$ is the finite element approximation of v in $\mathcal{V}_{ph}$, we have the following estimates in the $H^2(S)-$norm(see [2]; Theorem 18.1):

$$\|(I - P_{ph})A_pu\|_{\mathcal{V}_p} \leq Ch^2\|A_pu\|_{H^4} \leq Ch^2\|u\|_{L^2} \leq Ch^2\|u\|_{\mathcal{V}_p} \leq Ch^2\|x\|_{\mathcal{V}}, \tag{35}$$

$$\begin{aligned}\|(I - P_{ph})Tp\|_{\mathcal{V}_p} &\leq Ch^2\|Tp\|_{H^4} \leq Ch^2\|\gamma_S p\|_{L^2} \leq Ch^2\|p\|_{\mathcal{V}_s}\\ &\leq Ch^2\frac{1}{\lambda}\{\|A_sp\|_{\mathcal{V}_s} + \|T^*u\|_{\mathcal{V}_s}\} \leq C_\lambda h^2\|x\|_{\mathcal{V}},\end{aligned} \tag{36}$$

where we used the first equation in (9) to estimate $\|p\|_{\mathcal{V}_s}$.

Applying these estimates (35) and (36) as well as (33) to (34), we have

$$\|A_{12}u\|_{\mathcal{V}_s} \le C_\lambda h\|x\|_{\mathcal{V}}.$$

Note that here we have proved the estimate

$$\|(I-P_{ph})u\|_{\mathcal{V}_p} \le C_\lambda h^2\|x\|_{\mathcal{V}} \tag{37}$$

which will be used later.

Proof of Lemma 5.3. We get the following estimates for $\|A_{21}A_s^{1/2}p\|_{\mathcal{V}_p}$:

$$\begin{aligned}
\|A_{21}A_s^{1/2}p\|_{\mathcal{V}_p} &= \|(T-P_{ph}TP_{sh})A_sp\|_{\mathcal{V}_p}\\
&\le \|(I-P_{ph})TA_sp\|_{\mathcal{V}_p} + \|P_{ph}T\|_{\mathcal{V}_p\to\mathcal{V}_{ph}}\|(I-P_{sh})A_sp\|_{\mathcal{V}_s}\\
&\le C_\lambda h^2\|A_sp\|_{\mathcal{V}_s} + Ch\|x\|_{\mathcal{V}} \le C_\lambda h^2\|x\|_{\mathcal{V}} + Ch\|x\|_{\mathcal{V}} \le C_\lambda h\|x\|_{\mathcal{V}}.
\end{aligned}$$

and

$$\begin{aligned}
\|A_{22}u\|_{\mathcal{V}_p} &= \|(TT^*+A_p-P_{ph}TP_{sh}T^*P_{ph}-P_{ph}A_pP_{ph})u\|_{\mathcal{V}_p}\\
&\le \|(TT^*-P_{ph}TP_{sh}T^*P_{ph})u\|_{\mathcal{V}_p} + \|(A_p-P_{ph}A_pP_{ph})u\|_{\mathcal{V}_p}\\
&\le \|(T-P_{ph}TP_{sh})T^*u\|_{\mathcal{V}_p} + \|P_{ph}TP_{sh}T^*(I-P_{ph})u\|_{\mathcal{V}_p}\\
&\quad + \|(I-P_{ph})A_pu\|_{\mathcal{V}_p} + \|P_{ph}A_p(I-P_{ph})u\|_{\mathcal{V}_p}\\
&\le \|(I-P_{ph})TT^*u\|_{\mathcal{V}_p} + \|P_{ph}T\|_{\mathcal{V}_s\to\mathcal{V}_{ph}}\|(I-P_{sh})T^*u\|_{\mathcal{V}_s}\\
&\quad + \|P_{ph}TP_{sh}T^*\|_{\mathcal{V}_p}\|(I-P_{ph})u\|_{\mathcal{V}_p}\\
&\quad + \|(I-P_{ph})A_pu\|_{\mathcal{V}_p} + \|P_{ph}A_p\|_{\mathcal{V}_p}\|(I-P_{ph})u\|_{\mathcal{V}_p}\\
&\le C_\lambda h^2\|x\|_{\mathcal{V}} + Ch\|x\|_{\mathcal{V}} + C_\lambda h^2\|x\|_{\mathcal{V}} + Ch^2\|x\|_{\mathcal{V}} + C_\lambda h^2\|x\|_{\mathcal{V}}\\
&\le C_\lambda(h^2+h)\|x\|_{\mathcal{V}} \le C_\lambda h\|x\|_{\mathcal{V}}.
\end{aligned}$$

Proof Lemma 5.4. From Lemma 5.1, Lemma 5.2 and Lemma 5.3, we get the following estimates. Let $x_j = (A_s^{1/2}p_j, u_j)^t \in E(\lambda)\mathcal{V}$, $j = i$ or k. Then we have

$$\begin{aligned}
|a_s(A_{11}A_s^{1/2}p_i, A_s^{1/2}p_k)| &= |a_s(A_s^{1/2}(I-P_{sh})A_sp_i, A_s^{1/2}p_k)|\\
&= |a_s(I-P_{sh})A_sp_i, (I-P_{sh})A_sp_k)|\\
&\le Ch^2\|A_s^{1/2}p_i\|_{\mathcal{V}_s}\|A_s^{1/2}p_k\|_{\mathcal{V}_s} \le Ch^2\|x_i\|_{\mathcal{V}}\|x_k\|_{\mathcal{V}}.
\end{aligned}$$

Next, we have

$$\begin{aligned}
|a_s(A_{12}u_i, A_s^{1/2}p_k)| &= |a_s((A_s^{1/2}T^* - A_s^{1/2}P_{sh}T^*P_{ph})u_i, A_s^{1/2}p_k)|\\
&= |a_s((I-P_{sh})T^*u_i, A_sp_k) + a_s(P_{sh}T^*(I-P_{ph})u_i, A_sp_k)|\\
&= |a_s((I-P_{sh})T^*u_i, (I-P_{sh})A_sp_k)\\
&\quad + a_p((I-P_{ph})u_i, (I-P_{ph})TP_{sh}A_sp_k)|\\
&\le Ch^2\|x_i\|_{\mathcal{V}}\|x_k\|_{\mathcal{V}} + C_\lambda h^4\|x_i\|_{\mathcal{V}}\|x_k\|_{\mathcal{V}} \le C_\lambda h^2\|x_i\|_{\mathcal{V}}\|x_k\|_{\mathcal{V}}.
\end{aligned}$$

Further, we have

$$
\begin{aligned}
|a_p(A_{21}A_s^{1/2}p_i, u_k)| &= |a_p((T - P_{ph}TP_{sh})A_s p_i, u_k)| = |a_p((I - P_{ph})TA_s p_i, (I - P_{ph})u_k) \\
&\quad + a_s((I - P_{sh})A_s p_i, (I - P_{sh})T^* P_{ph} u_k)| \\
&\le C_\lambda h^4 \|x_i\|_{\mathcal{V}} \|x_k\|_{\mathcal{V}} + Ch^2 \|x_i\|_{\mathcal{V}} \|x_k\|_{\mathcal{V}} \le C_\lambda h^2 \|x_i\|_{\mathcal{V}} \|x_k\|_{\mathcal{V}}.
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
|a_p(A_{22}u_i, u_k)| &= |a_p((TT^* - P_{ph}TP_{sh}T^*P_{ph})u_i, u_k) + a_p((A_p - P_{ph}A_pP_{ph})u_i, u_k)| \\
&= |a_p((I - P_{ph})TT^*u_i, u_k) + a_p(P_{ph}T(I - P_{sh})T^*u_i, u_k) \\
&\quad + a_p(P_{ph}TP_{sh}T^*(I - P_{ph})u_i, u_k) + a_p((I - P_{ph})A_p u_i, u_k) \\
&\quad + a_p(P_{ph}A_p(I - P_{ph})u_i, u_k)| = |a_p((I - P_{ph})TT^*u_i, (I - P_{ph})u_k) \\
&\quad + a_s((I - P_{sh})T^*u_i, (I - P_{sh})T^*P_{ph}u_k) \\
&\quad + a_p((I - P_{ph})u_i, (I - P_{ph})TP_{sh}T^*P_{ph}u_k) \\
&\quad + a_p((I - P_{ph})A_p u_i, (I - P_{ph})u_k) + a_p((I - P_{ph})u_i, (I - P_{ph})A_pP_{ph}u_k)| \\
&\le C_\lambda h^4 \|x_i\|_{\mathcal{V}} \|x_k\|_{\mathcal{V}} + Ch^2 \|x_i\|_{\mathcal{V}} \|x_k\|_{\mathcal{V}} + C_\lambda h^4 \|x_i\|_{\mathcal{V}} \|x_k\|_{\mathcal{V}} \\
&\quad + C_\lambda h^4 \|x_i\|_{\mathcal{V}} \|x_k\|_{\mathcal{V}} + C_\lambda h^4 \|x_i\|_{\mathcal{V}} \|x_k\|_{\mathcal{V}} \\
&\le C_\lambda h^2 \|x_i\|_{\mathcal{V}} \|x_k\|_{\mathcal{V}}.
\end{aligned}
$$

7. Numerical Results

We show some numerical results for the problem (3) where Ω_0 is a cube with side length π and the plate consists of a side of the cube (Fig.1 and 2). We apply the finite element method using $\mathcal{V}_{sh}$ and $\mathcal{V}_{ph}$ in (25), (26) of Section 5. We calculated the test problem by FORTRAN77 on SONY NWS-5000 with double precision until $n = 32$ of partitions in x and y directions. The approximate eigenvalues are solved by the QR method[6] and also by the inverse iteration method. The numerical results are compared with the exact eigenvalues (see Appendix) and their convergence order is a bit greater than 2 (Fig.3) which is consistent with the results of numerical analysis.

Table 1 Example ($\rho_0 = 5,\ \rho_1 = 50,\ D = 2,\ m = 1,\ n = 1,\ c = 2.5$.)

number of partitions	1st. eigenvalue λ_{1h}	$error = \lvert\lambda_1 - \lambda_{1h}\rvert$
4	6.5088995	0.1855771
8	6.6567066	0.0377700
16	6.6871783	0.0072983
32	6.6929724	0.0015042
exact	6.6944766	———

Fig. 3 Convergence Order

Appendix: Exact Solution for a Test Problem

To check the validity of numerical computation by the finite element method, a test problem is chosen for which the exact solution is known. Namely since the acoustic field is a cube and the boundary condition for the acoustic field is the Dirichlet condition, and the boundary condition for plate is the pinned condition, by the Fourier mode decomposition in y direction, the solution $p(x, y)$ to problem (3) can be written as

$$p(x, y) = B_n(x) \sin ny,$$

which satisfies boundary condition. Then the exact eigenvalue ω can be given by solving the following transcendental equations for ω^2:

$$\omega^2 = \begin{cases} (\rho_0\pi + \rho_1)c^4/D, \text{ with } & m^2 + n^2 - \omega^2/c^2 = 0 & \text{(at most one)}, \\ \dfrac{Dg(m^2+n^2)^2}{\rho_1 g + \rho_0 \tanh g\pi}, \text{ with } & m^2 + n^2 - \omega^2/c^2 = g^2 > 0 & (g: \text{ real}), \\ \dfrac{Dg(m^2+n^2)^2}{\rho_1 g + \rho_0 \tan g\pi}, \text{ with } & m^2 + n^2 - \omega^2/c^2 = -g^2 < 0 & (g: \text{ real}). \end{cases}$$

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