Journal of Computational Mathematics, Vol.15, No.3, 1997, 279–287.

SPLITTING A CONCAVE DOMAIN TO CONVEX SUBDOMAINS*[1)]

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Abstract

We will study the convergence property of Schwarz alternating method for concave region where the concave region is decomposed into convex subdomains. Optimality of regular preconditioner deduced from Schwarz alternating is also proved. It is shown that the convergent rate and the condition number are independent of the mesh size but dependent on the relative geometric position of subdomains. Special care is devoted to non-uniform meshes, exclusively, local properties like the shape regularity of the finite elements are utilized.

1. Introduction

Turning large scale problem to small scale subproblems and regularizing irregular problem are two main subjects of domain decomposition. In regularization, regularizing irregular region is of first importance. Irregularity often means concavity, for example, L-shaped, T-shaped and C-shaped domains are irregular domains. In this paper, we will study domain decomposition method for elliptic problems defined on irregular region.

Schwarz alternating method is the basis of almost all domain decomposition method developed. Other methods are variations of it in nature and it was originally designed to regularize concave domain. When the domain is regularized, various fast algorithms may be used.

For continuous problem, [1] has given a complete theory. When subdomains have uniform overlap, Schwarz method has been studied sufficiently for discrete problem. When the domain is concave, the subdomains will have not a uniform overlap. Schwarz method has not been understood clearly for discrete problem. [2] and [3] studied this problem in special cases. We will study this problem generally. We will show that the convergence rate is independent of the mesh size but dependent on the relative position of subdomains. Some optimal preconditioners derived from Schwarz method will be studied as well. Triangulation will not be supposed to be quasi-uniform but it should be local shape regular.

* Received May 6, 1995.
[1)]This work was supported by Research Grants Council of the Hong Kong UGC.

2. On Some Projection Operators

Let $\Omega \subset R^2$ be a concave polygonal region

$$Lu = -\sum_{i,j=1}^{2} \frac{\partial u}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) + cu$$

be an elliptic operator defined on it, here, $(a_{i,j})_{i,j=1,2}$ is symmetric positive definite and bounded from above and below on Ω, $c \geq 0$.

$$\begin{cases} a(u,v) = (f,v), \quad v \in H_0^1(\Omega) \\ u \in H_0^1(\Omega) \end{cases} \tag{2.1}$$

is the variational form of the boundary value problem, the bilinear form

$$a(u,v) = \int_\Omega \Big[\sum_{i,j=1}^{2} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + cu\dot{v}\Big].$$

For convenience we only discuss the homogeneous Dirichlet boundary value problem here. The norm in $H_0^1(\Omega)$ introduced by $a(\cdot,\cdot)$ is equivalent to the original one. $H_0^1(\Omega)$ will be treated as a Hilbert space with inner product $a(\cdot,\cdot)$ in the following.

(2.1) is discretized by finite element method. Triangulation and linear continuous element will be discussed. The triangulation is supposed to be local shape regular. The diameter of an element, if the element does not intersect with the boundary of Ω, does not exceed the product of a constant and the distance from the element to the boundary of Ω.

$S_0^h(\Omega)$ represents the finite element space.

The discrete form of (2.1) is

$$\begin{cases} a(u,v) = (f,v), \qquad v \in S_0^h(\Omega) \\ u \in S_0^h(\Omega) \end{cases} \tag{2.2}$$

Ω_1 and Ω_2 are two convex subdomains of Ω, $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 \neq \emptyset$. The boundaries of Ω_1 and Ω_2 coincide with the finite element triangulation. $\Gamma_1 = \partial\Omega_1 \cap \Omega_2$,$\Gamma_2 = \partial\Omega_2 \cap \Omega_1$, $\bar{\Gamma}_1 \cap \bar{\Gamma}_2$ is the concave point. Here, we suppose Γ_1 and Γ_2 are straight lines and the angle between Γ_1 and Γ_2 is θ.

$$S_0^h(\Omega_1) = S_0^h(\Omega) \cap H_0^1(\Omega_1), \quad S_0^h(\Omega_2) = S_0^h(\Omega) \cap H_0^1(\Omega_2). \tag{2.3}$$

Figure 1

Figure 2

We use P_1 and P_2 to represent the orthogonal projections from $S_0^h(\Omega)$ to $S_0^h(\Omega_1)$ and $S_0^h(\Omega_2)$ under the inner product $a(\cdot,\cdot)$. The upper and lower bounds estimation of

$$\frac{a(P_1u + P_2u, u)}{a(u,u)} \tag{2.4}$$

is very important in analysis of convergence of additive and multiplicative Schwarz methods, and is crucial in estimation of many preconditioners([1], [3], [4], [5]).

Obviously,

$$a(P_1u + P_2u, u) = a(P_1u, P_1u) + a(P_2u, P_2u) \le 2a(u,u),$$

we have got the upper bound estimation of (2.4), we need the following lemmas for the lower bound estimation.

Lemma 2.1. ([1]) *If there exists a constant C, so that for any $u \in S_0^h(\Omega)$ there exist $u_1 \in S_0^h(\Omega_1)$ and $u_2 \in S_0^h(\Omega_2)$, $u = u_1 + u_2$, and*

$$||u_1||^2 + ||u_2||^2 \le C||u||^2,$$

we have

$$a(u,u) \le Ca(P_1u + P_2u, u).$$

We will estimate the lower bound of (2.4) by the help of lemma 2.1. To do this, we need a proper decomposition of function in $S_0^h(\Omega)$. Unit decomposition belong to the open cover $\{\Omega_1, \Omega_2\}$ on Ω is very important to us. To construct the unit decomposition, we need

Lemma 2.2. *On a cone-shaped domain (Figure 2), where $\theta \in [0, \frac{\pi}{2}]$), there exists a differentiable function ψ, so that $0 \le \psi \le 1$, $\psi|_{\Gamma_1} = 0$, $\psi|_{\Gamma_2} = 1$, and*

$$|\nabla\psi(X)|^2 \le \rho^2(\theta)\frac{1}{(d(X))^2}, \tag{2.5}$$

here,

$$\rho^2(\theta) = (1 + \mathrm{tg}^2\theta)\Big(1 + \frac{1}{\mathrm{tg}^2\theta}\Big), \tag{2.6}$$

$d(X)$ is the distance from X to the boundary of the cone.

Proof. Let

$$\psi(x,y) = \frac{y}{x\mathrm{tg}\,\theta},$$

obviously,

$$\psi|_{\Gamma_1} = 0, \psi|_{\Gamma_2} = 1, \frac{\partial\psi}{\partial x} = -\frac{y}{x^2\mathrm{tg}\,\theta}, \frac{\partial\psi}{\partial y} = \frac{1}{x\mathrm{tg}\,\theta},$$

then

$$|\nabla\psi|^2 = \Big|\frac{\partial\psi}{\partial x}|^2 + \Big|\frac{\partial\psi}{\partial y}\Big|^2 = \frac{1}{x^2\mathrm{tg}^2\theta} + \frac{y^2}{x^4\mathrm{tg}^2\theta}.$$

Since $y < x\mathrm{tg}\,\theta$ on the cone, we have

$$|\nabla\psi|^2 \le \frac{1}{x^2}\Big(1 + \frac{1}{\mathrm{tg}^2\theta}\Big).$$

From $d^2(X) = x^2 + y^2 \le (1 + \operatorname{tg}^2\theta)x^2$ we see $\dfrac{1}{x^2} \le \dfrac{1}{d^2(X)}(1 + \operatorname{tg}^2\theta)$, therefore

$$|\nabla\psi|^2 \le (1 + \operatorname{tg}^2\theta)\Big(1 + \frac{1}{\operatorname{tg}^2\theta}\Big)\frac{1}{d^2(X)}.$$

It should be noted that in (2.5), $\rho^2(\theta) = (1 + \operatorname{tg}^2\theta)\Big(1 + \dfrac{1}{\operatorname{tg}^2\theta}\Big)$ will get its limit value at $\theta = \dfrac{\pi}{4}$, when $\theta \to 0$ or $\theta \to \dfrac{\pi}{2}$,$\rho^2(\theta)$ will tend to infinity.

Lemma 3.2. *Let $\Omega \subset R^2$ be a polygonal concave domain as in Figure 1 and it is decomposed into overlapping subdomains $\Omega + \Omega_1 \cup \Omega_2$, θ is the angle between Γ_1 and Γ_2. The triangulation satisfies above assumptions, then, there exists a constant C independent of the triangulation and subdomain selection so that for any $u \in S_0^h(\Omega)$, there exist $u_1 \in S_0^h(\Omega_1)$ and $u_2 \in S_0^h(\Omega_2)$ so that*

$$u = u_1 + u_2$$

and

$$||u_i||^2 + ||u_2||^2 \le C(1 + \sigma^2(\theta))||u||^2.$$

Here,

$$\sigma^2(\theta) = \begin{cases} \rho^2(\theta), & 0 < \theta \le \dfrac{\pi}{4} \\ \rho^2\Big(\dfrac{\pi}{4}\Big) = 4, & \theta > \dfrac{\pi}{4} \end{cases}. \tag{2.7}$$

Proof. We construct the unit decomposition firstly. If $\theta \le \dfrac{\pi}{4}$ (Figure 1), let

$$\psi_1(X) = \begin{cases} 1, & X \in \Omega - \Omega_2 \\ \psi(X), & X \in \Omega_1 \cap \Omega_2 \\ 0, & X \in \Omega - \Omega_1 \end{cases}$$

if $\theta > \dfrac{\pi}{4}$, we select a subdomain Ω_1' of Ω_1 so that the angle between the boundary Γ_1' of Ω_1' and Γ_2 is $\dfrac{\pi}{4}$, let

$$\psi_1(X) = \begin{cases} 1, & X \in \Omega - \Omega_2 \\ \psi(X), & X \in \Omega_1' \cap \Omega_2 \\ 0, & X \in \Omega - \Omega_1' \end{cases}$$

let $\psi_2(X) = 1 - \psi_1(X)$, $(X \in \Omega)$.

Obviously, $0 \le \psi_1, \psi_2 \le 1$, $\psi_1(X) + \psi_2(X) = 1$, $Supp(\psi_1) \subset \Omega_1$, $Supp(\psi_2) \subset \Omega_2$ and from lemma 2.2

$$|\nabla\psi_i(X)|^2 \le \sigma^2(\theta)\frac{1}{(d(X))^2}, \quad (i = 1, 2) \tag{2.8}$$

For any $u \in S_0^h(\Omega)$, we have $u = \psi_1 u + \psi_2 u$ and $\psi_1 u \in C^0(\Omega_1), \psi_2 u \in C^0(\Omega_2)$. We use I_1 and I_2 to represent the interpolation operators from $C^0(\Omega_1)$ and $C^0(\Omega_2)$ to

$S_0^h(\Omega_1)$ and $S_0^h(\Omega_2)$ let $u_1 = I_1(\psi_1 u)$, $u_2 = I_2(\psi_2 u)$. It is obvious that $u = u_1 + u_2$, this is the decomposition we need,. To use lemma 2.1, we need to estimate $\|u_1\|^2$ and $\|u_2\|^2$. In what follows C will always be a constant independent of the finite element triangulation and subdomain selection.

By Poincaré inequality

$$\|u\|^2 = \|I_1(\psi_1 u)\|^2 \le C|I_1(\psi_1 u)|^2_{1,\Omega} = C\sum_{T\in\Omega} |I_1(\psi_1 u)|^2_{1,T}$$

T is element of the finite element triangulation, we use h and $|T|$ to represent the diameter and area of T.

If T intersects with $\partial\Omega$, $I_1(\psi_1 u)$ will be zero somewhere on ∂T and

$$\begin{aligned}|I_1(\psi_1 u)|^2_{1,T} &\le Ch^{-2}\|I_1(\psi_1 u)\|^2_{L^2(T)} \le Ch^{-2}|T|\max_T |I_1(\psi_1 u)|^2\\ &\le Ch_2|T|\max_T |u|^2 \le Ch^{-2}||u||^2_{L^2(T)} \le C|u|^2_{1,T}. \end{aligned} \tag{2.9}$$

If T does not intersect with $\partial\Omega$, let d be the distance between T and $\partial\Omega$, we have

$$\begin{aligned}|I_1(\psi_1 u)|^2_{1,T} &\le C(|I_1(\psi_1 - \bar\psi_1)u|^2_{1,T} + |I_1(\bar\psi_1 u)|^2_{1,T})\\ &\le C(|u_1|^2_{1,T} + h^{-2}||I_1(\psi_1 - \bar\psi_1)u||^2_{L^2(T)})\\ &\le C(|u_1|^2_{1,T} + h^{-2}|T|\max_T|\psi_1 - \bar\psi_1|^2 \max_T |u|^2)\\ &\le C(|u_1|^2_{1,T} + h^{-2}|T|h^2\frac{\sigma^2(\theta)}{d^2}\Big(\min_T |u|^2 + h^2\frac{|u|^2_{1,T}}{|T|}\Big)\\ &\le C\Big(|u_1|^2_{1,T} + \sigma^2(\theta)\Big(|T|\frac{1}{d^2}\min_T |u|^2 + \frac{h^2}{d^2}|u|^2_{1,T}\Big)\Big),\end{aligned}$$

here, $\bar\psi$ is the average value of ψ on T, by the assumption on the triangulation, $\dfrac{h}{d} \le C$, let X_T be the minimum point of $|u|^2$ on T, we have

$$|I_1(\psi_1 u)|^2_{1,T} \le C\Big((1+\sigma^2(\theta))|u|^2_{1,T} + \sigma^2(\theta)\frac{u^2(X_T)}{d^2(X_T)}|T|\Big) \tag{2.10}$$

Summing (2.10) up with respect to all elements except those intersect with $\partial\Omega$ (let Ω_0 be such a subdomain of Ω that is composed of all elements which does not intersect $\partial\Omega$), we get

$$|I_1(\psi_1 u)|^2_{1,\Omega_0} \le C\Big((1+\sigma^2(\theta))|u|^2_{1,\Omega_0} + \sigma^2(\theta)\int_{\Omega_0}\frac{u^2}{d^2}\Big) \tag{2.11}$$

By (2.9) and (2.11) we get

$$|I_1(\psi_1 u)|^2_{1,\Omega} \le C\Big((1+\sigma^2(\theta))|u|^2_{1,\Omega} + \sigma^2(\theta)\int_{\Omega}\frac{u^2}{d^2}\Big)$$

By a classical inequality $\displaystyle\int_\Omega \frac{u^2}{d^2} \le C|u|^2_{1,\Omega}$ ([1]) we get

$$|I_1(\psi_1 u)|^2_{1,\Omega} \le C((1+\sigma^2(\theta))|u|^2_{1,\Omega}$$

From the Poincaré inequality we obtain

$$||I_1(\psi_1 u)||^2_{1,\Omega} \leq C((1+\sigma^2(\theta))||u||^2_{1,\Omega} \tag{2.12}$$

Hence

$$||u_1||^2 + ||u_2||^2 = ||u_1||^2 + ||u-u_1||^2 \leq C(||u_1||^2 + ||u||^2) \leq (1+\sigma^2(\theta))||u||^2.$$

The lemma has been proved.

From lemma 2.1 and lemma 2.3 we obtain

Theorem 2.1. *There exists a constant C independent of the finite element triangulation and subdomain selection so that*

$$C\frac{1}{1+\sigma^2(\theta)} \leq \frac{a(P_1u+P_2u,u)}{a(u,u)} \leq 2 \tag{2.13}$$

here, $\sigma^2(\theta)$ was defined by (2.6) *and* (2.7).

3. Convergence of Schwarz Method on Irregular Domain

Let $\Omega \subset R^2$ be a concave polygonal domain. Ω is decomposed into overlapping subdomains Ω_1 and Ω_2, as shown in Figure 1. The finite element triangulation satisfies the above assumption.

Let $S_0^h(\Omega_1)^\perp$ and $S_0^h(\Omega_2)^\perp$ be the orthogonal complements of $S_0^h(\Omega_1)$ and $S_0^h(\Omega_1)$ in $S_0^h(\Omega)$ under inner product $a(\cdot,\cdot)$. P_1^0 and P_2^0 are the orthogonal projections from $S_0^h(\Omega)$ to $S_0^h(\Omega_1)^\perp$ and $S_0^h(\Omega_2)^\perp$. The convergence factor of Schwarz alternating method (Multiplicative) will be $\max\{||P_1^0P_2^0||, ||P_2^0P_1^0||\}$, the convergence factor of the additive Schwarz method ([6]) is $\left\|\frac{P_1^0+P_2^0}{2}\right\|$, we have

Theorem 3.1.

$$||P_1^0P_2^0|| = ||P_2^0P_1^0||,$$

and

$$||P_1^0P_2^0|| = \left(2\left\|\frac{P_1^0+P_2^0}{2}\right\| - 1\right)^2. \tag{3.1}$$

Theorem 3.2. *There exists a constant C independent of the finite element triangulation and subdomain selection so that*

$$||P_1^0P_2^0|| \leq 1 - \frac{1}{C(1+\sigma^2(\theta))}. \tag{3.2}$$

This theorem can be proved by the method of [1] and theorem 2.1.

Theorem 3.3. *There exists a constant C independent of the finite element triangulation and subdomain selection so that*

$$\left\|\frac{P_1^0+P_2^0}{2}\right\| \leq \sqrt{1 - \frac{1}{2C(1+\sigma^2(\theta))}}. \tag{3.3}$$

Figure 3 Figure 4 Figure 5

If Γ_1 and Γ_2 in Figure 1 are not straight lines, we may select two straight lines Γ_1' and Γ_2', and use θ' to replace θ, conclusions of lemma 2.3, theorem 2.1, 3.2, and theorem 3.3 are still true(Figure 3)

If the domain and subdomains are as shown in Figure 4, we may inscribe a rhomb in $\Omega_1 \cap \Omega_2$, all above results will still be true if θ is replaced by $\theta' = \min\{\theta_1, \theta_2\}$.

L-shaped, T-shaped and C-shaped domains are special cases of above analysis, see Figure 5.

4. Preconditioner of Capacitance Matrix on Interface

There is another kind of domain decomposition method where the domain is decomposed into non overlapping subdomains by interfaces. The problem may be attributed to the capacitance equation on the interface, this small scale problem is often solved by preconditioned iterative method.

Let Γ_1 be the interface, Ω is decomposed into Ω_1 and $\Omega - \Omega_1$. We use $\hat{\Omega}$ to represent the set of finite element node points in Ω, $\hat{\Omega}_1 = \Omega_1 \cap \hat{\Omega}$, $\hat{\Omega}_2 = \Omega_2 \cap \hat{\Omega}$, $\hat{\Gamma}_1 = \Gamma_1 \cap \hat{\Omega}$, $\hat{\Gamma}_2 = \Gamma_2 \cap \hat{\Omega}$.

The stiffness matrix of (2.3) is

$$A = (a(\phi_i, \phi_j)_{i,j \in \hat{\Omega}})$$

ϕ_i, ϕ_j are the usual finite element basis functions.

For $i \in \hat{\Gamma}_1$, let $\tilde{\phi}_i$ be such a function , $\tilde{\phi}_i|_{\Gamma_1} = \phi_i|_{\Gamma_1}, \tilde{\phi}_i|_{\partial\Omega} = 0$ and it is discrete harmonic in Ω_1 and $\Omega - \Omega_1$. The capacitance matrix will be

$$C = (a(\tilde{\phi}_i, \tilde{\phi}_j)_{i,j \in \hat{\Gamma}_1})$$

For $i \in \hat{\Gamma}_1$, let $\hat{\phi}_i$ be such a function , $\hat{\phi}_i|_{\Gamma_1} = \phi_i|_{\Gamma_1}, \hat{\phi}_i|_{\partial\Omega_2} = 0$ and it is discrete harmonic in $\Omega_1 \cap \Omega_2$ and $\Omega_2 - \Omega_1$. The preconditioner for the capacitance matrix is defined by

$$\hat{C} = (a(\hat{\phi}_i, \hat{\phi}_j)_{i,j \in \hat{\Gamma}_1})$$

Theorem 4.1. *There exists a constants C independent of the finite element triangulation and subdomain selection so that*

$$\text{Cond }(\hat{C}^{-1}C) \leq C(1+\sigma^2(\theta)).$$

Proof. The condition number $\text{Cond}(\hat{C}^{-1}C)$ can be estimated by the ratio of the upper and lower bounds of the generalized Rayleigh quotient

$$\frac{(C\hat{C}^{-1}Cx,x)}{(Cx,x)}. \tag{4.1}$$

It can be proved that([7])

$$\frac{(C\hat{C}^{-1}Cx,x)}{(Cx,x)} = \frac{a(P_2\tilde{u},\tilde{u})}{a(\tilde{u},\tilde{u})} \tag{4.2}$$

here, $x = \tilde{u}|_{\hat{\Gamma}_1}$, $\tilde{u}$ is a finite element function and it is discrete harmonic in $\Omega - \Omega_1$ and Ω_1. It is obvious that $P_1\tilde{u} = 0$ and so

$$\frac{(C\hat{C}^{-1}Cx,x)}{(Cx,x)} = \frac{a(P_1\tilde{u}+P_2\tilde{u},\tilde{u})}{a(\tilde{u},\tilde{u})}.$$

From theorem 2.1 we get the conclusion of theorem 4.1, by (4.2) the upper bound of (4.1) is 1.

The resolution of $\hat{C}x = d$ is equivalent to the resolution of

$$A_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} d \\ 0 \end{pmatrix} \tag{4.3}$$

here, $A_2 = (a(\phi_i,\phi_j)_{i,j\in\hat{\Omega}_2})$, (4.3) is a homogeneous boundary value Dirichlet problem on Ω_2.

There is a great freedom in the selection of Ω_2, for a given problem, Ω_2 should be selected so that the subproblem (4.3) can be solved as easily as possible, for example, when the domain Ω is a L-shaped , T-shaped or C-shaped domain, Ω_2 may be selected to be a rectangular[5] and various fast algorithm, FFT for example, can be used to solve (4.3).

5. Numerical Experiment

We take the Poinsson equation on the L-shaped domain and 5 points difference scheme as an example. The subdomain is selected as shown in Figure 5, in the Figure, j,k,l, and n are the number of node points in the interior of the corresponding interval.

The relation between the convergence factor α of Schwarz alternating method and $\text{tg}\,\theta$ is shown in table 1. When j,k, and n are fixed and l increased, $\text{tg }\theta\Big(=\dfrac{k+1}{l+1}\Big)$ will be decreased, and the convergence factor α will become larger, this is in keep with our theoretical analysis.

Table 2 shows that, when the subdomains are fixed, the convergence factor α is independent of the mesh size.

Table 1 Relation Between Convergence Factor α and $\mathrm{tg}\,\theta$

l	k	j	n	α	$\mathrm{tg}\,\theta$
5	0	20	20	0.141	1/6
10	0	20	20	0.327	1/11
20	0	20	20	0.549	1/21
30	0	20	20	0.638	1/31
40	0	20	20	0.692	1/41
11	1	20	20	0.134	1/6
21	1	20	20	0.291	1/11
41	1	20	20	0.492	1/21
61	1	41	41	0.629	1/31

Table 2 Concergence Factor is Independent of the Mesh Size

l	k	j	n	α	$\mathrm{tg}\,\theta$
2	0	2	2	0.00261	1/3
5	1	5	5	0.00234	1/3
11	3	11	11	0.00242	1/3
23	7	23	23	0.00271	1/3
47	15	47	47	0.00286	1/3

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