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SPECTRAL AND PSEUDOSPECTRAL APPROXIMATIONS IN TIME FOR PARABOLIC EQUATIONS*1)

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Abstract

In this paper, spectral and pseudospectral methods are applied to both time and space variables for parabolic equations. Spectral and pseudospectral schemes are given, and error estimates are obtained for approximate solutions.

Key words: Spectral approximation, pseudospectral approximation, parabolic equation, error estimate

1. Introduction

In recent years, it has been shown that spectral methods are very useful to solve partical differential equations. Spectral methods, in which the approximate solution is a polynomial of high degree, are known to be very accurate when the solution to be approximated is very smooth (see [2] for details). Using spectral methods to time-dependent partical differential equations, a standard scheme is done in space only, while finite difference is done in time (the same to finite element method, too). Hence, no matter how smooth the exact solution is, in general, the error order in time can not be raised. The error in time decide the global error of the approximate solution. Many efforts have been made on the discretization in time, for instance, in [6] and [7] discontinuous Galerkin method in time is studied for parabolic equations. Recently, I. Babuska and T. Janik[3] discussed the p-version of finite element method in time for parabolic equations. In [4] and [5] H.T. Ezer has proposed spectral methods in time using polynomial approximation of the evolution operator in Chebyshev least-squares sense for parabolic equations and hyperbolic equations. In this paper, for convenience we use the spectral methods in both space and time variables. If we use the finite element method in space, some parallel conclusions can also been obtained.

2. Variational Principle

Let $I=(-1,1)$, $D=[0,2\pi]$, $Q=D\times I$. For convenience we consider the following model problem

$$u_t - u_{xx} + u = f(x,t), \quad \text{in } Q \tag{2.1}$$
$$u(x,t) = u(x+2\pi,t), \tag{2.2}$$
$$u(x,-1) = g(x). \quad \text{in } D \tag{2.3}$$

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Remark 1. If $f \in H^{k,\frac{k}{2}}(Q)$, $g \in H_p^{k+2}(D)$, from regularity of solutions of parabolic equations, the solution $u(x,t)$ of (2.1)–(2.3) is in $H^{k+2,\frac{k}{2}+1}(Q)$.

Let $(\cdot,\cdot)$ and $\|\cdot\|$ denote respectively the inner product and the norm in $L^2(I)$, $H_p^m(D)$ denote the m order periodic Sobolev space with the norm $\|u\|_m$, $X = L^2(I; H_p^1(D))$ with the norm $\|u\|_X = \Big(\int_I \|u\|_1^2 dt\Big)^{\frac{1}{2}}$, $C_p^0 = \{v \in C^\infty(Q) \,|\, v(x,t) \in C_p^\infty(D),\ \forall t \in I,\ v(x,1) = 0\}$, and Y denote the complete space of C_p^0 with respect to the norm $\|v\|_Y = \Big(\int_I (\|\!|v_t|\!\|^2 + \|v\|_1^2) dt\Big)^{\frac{1}{2}}$, where

$$\|\!|v_t|\!\| = \sup_{z \in H_p^1(D)} \frac{|\int_D v_t z dx|}{\|z\|_1}.$$

Let us define on $X \times Y$ the bilinear form

$$B(u,v) = \int_I \int_D (-u\overline{v}_t + u_x \overline{v}_x + u\overline{v}) dx dt, \quad \forall u \in X, \quad v \in Y.$$

Let $F \in Y'$. We consider the following variational problem P: find u_0 in X such that

$$B(u_0, v) = F(v), \quad \forall v \in Y. \tag{2.4}$$

It is similar to problem P in [3] in proof, we obtain theorem 1 for the problem P.

Theorem 1. *Problem P has a unique solution u_0 in X and there exists a constant C independent of u_0 and F such that*

$$\|u_0\|_X \le C \|\!|F|\!\|_{Y'}.$$

Proof. Let $\lambda_j^2 = j^2 + 1$, $u_j = \frac{1}{\sqrt{2\pi}} e^{ijx}$, $j = 0, \pm 1, \pm 2, \cdots$, then λ_j^2, u_j respectively denote eigenvalue and eigenvector of an operator $A = -\frac{d^2}{dx^2} + I$, and $\text{span}\{u_j\} \subset H_p^1(D)$ is dense in $H_p^1(D)$. Let $u \in X$, $v \in Y$, then u and v can be written in the form

$$u = \sum_{j=-\infty}^{\infty} \alpha_j(t) u_j, \quad v = \sum_{j=-\infty}^{\infty} \beta_j(t) u_j,$$

with

$$\|u\|_X = \Big(\int_I \sum_{j=-\infty}^{\infty} \lambda_j^2 |\alpha_j(t)|^2 dt\Big)^{\frac{1}{2}},$$

$$\|v\|_Y = \Big(\int_I \sum_{j=-\infty}^{\infty} (\lambda_j^{-2} \mid \beta_j'(t) \mid^2 + \lambda_j^2 \mid \beta_j(t) \mid^2) dt\Big)^{\frac{1}{2}},$$

and $B(u,v)$ can also be written as follows

$$B(u,v) = \int_I \Big(\sum_{j=-\infty}^{\infty} (-\alpha_j \overline{\beta}_j' + \lambda_j^2 \alpha_j \overline{\beta}_j)\Big) dt = \int_I \Big(\sum_{j=-\infty}^{\infty} \lambda_j \alpha_j (-\lambda_j^{-1} \overline{\beta}_j' + \lambda_j \overline{\beta}_j)\Big) dt.$$

By Schwarz inequality we have

$$\begin{aligned} | B(u,v) | &\le \int_I \Big(\sum_{j=-\infty}^{\infty} \lambda_j^2 \mid \alpha_j \mid^2 \Big)^{\frac{1}{2}} \Big(\sum_{j=-\infty}^{\infty} \mid -\lambda_j^{-1}\beta_j' + \lambda_j\beta_j \mid^2 \Big)^{\frac{1}{2}} dt \\ &\le \Big(\int_I \sum_{j=-\infty}^{\infty} \lambda_j^2 \mid \alpha_j \mid^2 dt \Big)^{\frac{1}{2}} \Big(\int_I \sum_{j=-\infty}^{\infty} \mid -\lambda_j^{-1}\beta_j' + \lambda_j\beta_j \mid^2 dt \Big)^{\frac{1}{2}} \\ &\le \sqrt{2} \|u\|_X \, \|v\|_Y. \end{aligned} \tag{2.5}$$

Let T denote a mapping from X into Y such that for any $u = \sum_{j=-\infty}^{\infty} \alpha_j(t)u_j \in X$, $v = Tu = \sum_{j=-\infty}^{\infty} \beta_j(t)u_j \in Y$, $\beta_j(t)$ satisfy

$$-\lambda_j^{-1}\beta_j'(t) + \lambda_j\beta_j(t) = \lambda_j\alpha_j(t), \quad \beta_j(1) = 0, j = 0, \pm 1, \pm 2, \cdots.$$

Then

$$\|Tu\|_Y^2 = \int_I \sum_{j=-\infty}^{\infty} (\lambda_j^{-2} \mid \beta_j' \mid^2 + \lambda_j^2 \mid \beta_j \mid^2) dt \le \int_I \sum_{j=-\infty}^{\infty} \mid -\lambda_j^{-1}\beta_j' + \lambda_j\beta_j \mid^2 dt \le \|u\|_X^2.$$

Hence T is a continuous linear operator, and

$$\inf_{\substack{u\in X \\ \|u\|_X=1}} \sup_{\substack{v\in Y \\ \|v\|_Y\le 1}} | B(u,v) | \ge \inf_{\substack{u\in X \\ \|u\|_X=1}} | B(u,Tu) | = 1.$$

Similarly,

$$\inf_{\substack{v\in Y \\ \|v\|_Y=1}} \sup_{\substack{u\in X \\ \|u\|_X\le 1}} | B(u,v) | \ge \frac{\sqrt{2}}{2}.$$

Using now theorem 5.2.1 in [1], theorem 1 is proved.

Remark 2. If $F(v) = \int_D g(x)\overline{v}(x,-1)dx + \int_I \int_D f(x,t)\overline{v}(x,t)dxdt$, then the solution u_0 of the variational problem P is a weak solution of (2.1)-(2.3).

3. Spectral Approximation of Variational Problem P.

Let V_N denote the set of polynomials of degree N, $S_M = \text{span}\Big\{ u_k = \frac{1}{\sqrt{2\pi}} e^{ikx}, \mid k \mid \le M \Big\}$. $\overset{\circ)}{V}_N = \{p \in V_N; p(1) = 0\}$. We denote by $\widetilde{P}_M$: $L^2(D) \to S_M$ the orthogonal project operator on S_M in $L^2(D)$ and P_N: $L^2(I) \to V_N$ the orthogonal project operator on V_N in $L^2(I)$. Set $W = V_{N-1} \times S_M$, $U = \overset{\circ)}{V}_N \times S_M$. Obviously, $W \subset X$, $U \subset Y$.

We construct the following spectral scheme for the variational problem P: find u_p in W such that

$$B(u_p, v) = F(v), \quad \forall v \in U. \tag{3.1}$$

Let $u \in W$ and $v \in U$, then

$$u = \sum_{j=-M}^{M} \alpha_j(t) u_j, \quad v = \sum_{j=-M}^{M} \beta_j(t) u_j,$$

where $\alpha_j(t) \in V_{N-1}$, $\beta_j(t) \in \overset{\circ)}{V}_N$, and

$$B(u,v) = \int_I \sum_{j=-M}^{M} \lambda_j \alpha_j(t)(-\lambda_j^{-1}\overline{\beta}_j'(t) + \lambda_j \overline{\beta}_j(t))dt.$$

We define a linear operator T_p from W into U which satisfies that for any $u = \sum_{j=-M}^{M} \alpha_j(t)u_j \in W$, $v = T_p u = \sum_{j=-M}^{M} \beta_j(t)u_j \in U$, $\beta_j(t) \in \overset{\circ)}{V}_N$ satisfy

$$\int_I (-\lambda_j^{-1}\beta_j' + \lambda_j \beta_j) z dt = \lambda_j \int_I \alpha_j z dt, \quad \forall z \in V_{N-1}, \ | j | \leq M. \tag{3.2}$$

Obviously, the solution of the variational problem is existent. Taking $z = P_{N-1}(-\lambda_j^{-1} \overline{\beta}_j'(t) + \lambda_j \overline{\beta}_j(t))$ in (3.2), we have

$$\int_I (-\lambda_j^{-1}\beta_j' + \lambda_j\beta_j)P_{N-1}(-\lambda_j^{-1}\overline{\beta}_j' + \lambda_j\overline{\beta}_j)dt = \lambda_j \int_I \alpha_j P_{N-1}(-\lambda_j^{-1}\overline{\beta}_j' + \lambda_j\overline{\beta}_j)dt$$
$$= \lambda_j \int_I \alpha_j(-\lambda_j^{-1}\overline{\beta}_j' + \lambda_j\overline{\beta}_j)dt. \tag{3.3}$$

Taking $z = \lambda_j \overline{\alpha}_j$ in (3.2) again, we have

$$\lambda_j \int_I (-\lambda_j^{-1}\beta_j' + \lambda_j\beta_j)\overline{\alpha}_j dt = \int_I \lambda_j^2 \mid \alpha_j \mid^2 dt. \tag{3.4}$$

Therefore, combining (3.3) and (3.4), we obtain

$$\int_I \mid P_{N-1}(-\lambda_j^{-1}\beta_j' + \lambda_j\beta_j) \mid^2 dt = \int_I \lambda_j^2 \mid \alpha_j \mid^2 dt. \tag{3.5}$$

Let $\beta_j(t) = \sum_{i=0}^{N} c_{ji} L_i(t)$, where $L_i(t)$ is the ith Legendre polynomial. By $\beta_j(t) \in \overset{\circ)}{V}_N$, we have

$$\sum_{i=0}^{N} c_{ji} = 0.$$

Hence

$$c_{jN} = -\sum_{i=0}^{N-1} c_{ji}.$$

But

$$\int_I \mid \beta_j(t) \mid^2 dt = \sum_{i=0}^{N} \frac{2 \mid c_{ji} \mid^2}{2i+1},$$

$$\int_I \mid P_{N-1}\beta_j(t) \mid^2 dt = \sum_{i=0}^{N-1} \frac{2 \mid c_{ji} \mid^2}{2i+1}.$$

Thus from (3.5), we get

$$\int_I \mid -\lambda_j^{-1}\beta_j' + \lambda_j\beta_j \mid^2 dt = \int_I \lambda_j^2 \mid \alpha_j \mid^2 dt + \frac{2\lambda_j^2 \mid c_{jN} \mid^2}{2N+1}.$$

Because

$$\mid c_{jN} \mid^2 = \Big| \sum_{i=0}^{N-1} c_{ji} \Big|^2 \le \sum_{i=0}^{N-1} \frac{2 \mid c_{ji} \mid^2}{2i+1} \sum_{i=0}^{N-1} \frac{2i+1}{2} = \frac{N^2}{2} \int_I \mid P_{N-1}\beta_j \mid^2 dt$$

Hence

$$\begin{aligned}\int_I \mid -\lambda_j^{-1}\beta_j' + \lambda_j\beta_j \mid^2 dt &\le \int_I \lambda_j^2 \mid \alpha_j \mid^2 dt + \frac{\lambda_j^2 N^2}{2N+1} \int_I \mid P_{N-1}\beta_j \mid^2 dt \\ &\le \int_I \lambda_j^2 \mid \alpha_j \mid^2 dt + \frac{N^2}{2N+1} \int_I \mid P_{N-1}(-\lambda_j^{-1}\beta_j' + \lambda_j\beta_j) \mid^2 dt \\ &= \Big(1 + \frac{N^2}{2N+1}\Big) \int_I \lambda_j^2 \mid \alpha_j \mid^2 dt.\end{aligned}$$

Summing up for j from $-M$ to M for above inequality, we have

$$\int_I \sum_{j=-M}^{M} \mid -\lambda_j^{-1}\beta_j' + \lambda_j\beta_j \mid^2 dt \le \Big(1 + \frac{N^2}{2N+1}\Big) \int_I \sum_{j=-M}^{M} \lambda_j^2 \mid \alpha_j \mid^2 dt = \frac{(N+1)^2}{2N+1} \|u\|_X^2.$$

Thus

$$\begin{aligned}\|T_p u\|_Y^2 &= \int_I \sum_{j=-M}^{M} (\mid -\lambda_j^{-1}\beta_j' \mid^2 + \lambda_j^2 \mid \beta_j \mid^2) dt \le \int_I \sum_{j=-M}^{M} \mid -\lambda_j^{-1}\beta_j' + \lambda_j\beta_j \mid^2 dt \\ &\le \frac{(N+1)^2}{2N+1} \|u\|_X^2.\end{aligned}$$

Hence, we obtain

$$\inf_{\substack{u\in W \\ \|u\|_X=1}} \sup_{\substack{v\in U \\ \|v\|_Y\le 1}} \mid B(u,v) \mid \ge \inf_{\substack{u\in W \\ \|u\|_X=1}} \Big| B\Big(u, \frac{T_p u}{\|T_p u\|_Y}\Big)\Big| \ge \frac{\sqrt{2N+1}}{N+1}. \tag{3.6}$$

By (2.5) and (3.6),we have

$$\forall v \in U, \quad v \ne 0, \quad \sup_{u\in W} \mid B(u,v) \mid > 0. \tag{3.7}$$

Using theorem 6.2.1 in [1], we know that the spectral approximation of the problem P has a unique solution u_p and

$$\|u_0 - u_p\|_X \le \Big(1 + \frac{2(N+1)}{\sqrt{2N+1}}\Big) \inf_{v\in W} \|u_0 - v\|_X.$$

Finally, taking $v = P_{N-1}\ \widetilde{P}_M\ u_0$ in (3.8), using the estimates of P_{N-1} and $\widetilde{P}_M$ in [2], we obtain

Theorem 2. *If $f(x,t) \in H^{k,\frac{k}{2}}(Q)$, $g(x) \in H_p^{k+2}(D)$. Then there exists a unique solution u_p for spectral scheme and*

$$\|u_0 - u_p\|_X \le C(N^{\frac{1}{2}} M^{-(k+1)} + N^{-\frac{k}{2}})(\|f\|_{k,\frac{k}{2}} + \|g\|_{k+2}),$$

where C is a constant independent of N, M, f and g.

4. Pseudospectral Approximation of Variational Problem P.

In this section, first we consider problem (2.1)-(2.3) with homogeneous initial value $u(x,-1) = 0$. Let $\overset{\circ}{V}_N = \{v \in V_N,\ v(-1) = v(1) = 0\}$, $\overset{(\circ}{V}_{N-1} = \{v \in V_{N-1},\ v(-1) = 0\}$, $\overset{(\circ}{W} = S_M \times \overset{(\circ}{V}_{N-1}$, $\overset{\circ}{U} = S_M \times \overset{\circ}{V}_N$ and $x_j = jh$, $j = 0, 1, \cdots, 2M$, $h = \dfrac{2\pi}{2M+1}$. Then the following Gauss integration formula[2] holds

$$\int_D u(x)dx = h \sum_{j=0}^{2M} u(x_j), \quad \forall u \in S_M. \tag{4.1}$$

Let $\widetilde{I}_M$ be an interpolation operator from $C(D)$ to S_M such that

$$\widetilde{I}_M\ u(x_j) = u(x_j), \quad 0 \le j \le 2M.$$

Let t_i and $\omega_i (i = 0, 1, \cdots, N-1)$ denote notes and weights of the Gauss-Radau integration formula[8] respectively, then

$$\int_I p(t)dt = \sum_{i=0}^{N-1} \omega_i p(t_i), \quad \forall p \in V_{2N-2}, \tag{4.2}$$

where $\omega_i = \dfrac{1 - t_i}{N^2 L_{N-1}^2(t_i)}$, $t_i (i = 0, 1, 2, \cdots, N-1)$ are zeroes of the polynomial $L_N + L_{N-1}$. Let I_{N-1} be an interpolation operator from $C(\overline{I})$ to V_{N-1}, such that

$$I_{N-1} u(t_i) = u(t_i), \quad 0 \le i \le N-1.$$

Combining (4.1) and (4.2), we have the following Gauss integration formula on Q

$$\int_I \int_D p(x,t)dxdt = h \sum_{j=0}^{2M} \sum_{i=0}^{N-1} p(x_j, t_i)\omega_i, \quad \forall p \in V_{2N-2} \times S_M. \tag{4.3}$$

We construct the following pseudospectral scheme for the variational problem P: find u_c in $\overset{(\circ}{W}$ such that

$$u_{ct}(x_j,t_i) - u_{cxx}(x_j,t_i) + u_c(x_j,t_i) = f(x_j,t_i), \tag{4.4}$$

$$j = 0, 1, \cdots, 2M, \quad i = 1, \cdots, N-1.$$

We define now a discrete inner product and a norm as follows

$$(u, v)_{M,N} = h \sum_{j=0}^{2M} \sum_{i=0}^{N-1} u(x_j, t_i) v(x_j, t_i) \omega_i,$$
$$\|u\|_{M,N} = (u, u)_{M,N}^{\frac{1}{2}}, \quad \forall u, v \in C(\overline{Q}).$$

Let $v \in \overset{o}{U}$, then u_c and v can be written as

$$u_c = \sum_{j=-M}^{M} \alpha_j(t) u_j, \quad v = \sum_{j=-M}^{M} \beta_j(t) u_j,$$

where $\alpha_j(t) \in \overset{(o}{V}_{N-1}$, $\beta_j(t) \in \overset{o}{V}_N$, $|\, j \,| \leq M$. Using (4.1), we can write (4.4) equivalently as follows find $u_c \in \overset{(o}{W}$ such that

$$B_d(u_c, v) = F_d(v), \quad \forall v \in \overset{o}{U}\ .$$

where

$$B_d(u_c, v) = \sum_{j=-M}^{M} \sum_{i=0}^{N-1} \omega_i (\lambda_j \alpha_j(t_i)(-\lambda_j^{-1} \overline{\beta}_j'(t_i) + \lambda_j \overline{\beta}(t_i))),$$

and $F_d(v) = (f, v)_{M,N}$. We define a discrete norm on $C(\overline{I})$ by

$$\|\beta\|_N = \Big(\sum_{i=0}^{N-1} |\, \beta(t_i) \,|^2 \, \omega_i \Big)^{\frac{1}{2}}.$$

Lemma 1. *If* $\beta(t) \in \overset{o)}{V}_N$, *then*

$$\frac{\sqrt{4N-2}}{N+1} \|\beta\| \leq \|\beta\|_N \leq 2\|\beta\|.$$

Proof. Let $\beta(t) \in \overset{o)}{V}_N$, then

$$\beta(t) = \sum_{i=0}^{N} \sqrt{\frac{2i+1}{2}} a_i L_i(t),$$

with

$$\sum_{i=0}^{N} \sqrt{\frac{2i+1}{2}} a_i = 0, \tag{4.5}$$

Hence

$$\|\beta\|_N^2 = \sum_{i=0}^{N-1} |\, a_i \,|^2 + \frac{2N+1}{2} |\, a_N \,|^2 \sum_{i=0}^{N-1} L_N^2(t_i) \omega_i$$

$$+\sqrt{\frac{2N-1}{2}}\sqrt{\frac{2N+1}{2}}(\overline{a}_{N-1}a_N+a_{N-1}\overline{a}_N)\sum_{i=0}^{N-1}L_{N-1}(t_i)L_N(t_i)\omega_i$$

Due to t_i $(i=0,1,\cdots,N-1)$ are zeroes of $L_N(t)+L_{N-1}(t)$, it implies

$$\|\beta\|_N^2=\sum_{i=0}^{N-2}\mid a_i\mid^2+\mid\sqrt{\frac{2N+1}{2N-1}}a_N-a_{N-1}\mid^2 . \tag{4.6}$$

Therefore, from (4.5) we get

$$\sum_{i=0}^{N-2}\mid a_i\mid^2\geq\frac{2N-1}{(N-1)^2}\mid\sqrt{\frac{2N+1}{2N-1}}a_N+a_{N-1}\mid^2 .$$

Set $\varepsilon=\dfrac{(N-1)^2}{(N+1)^2-2}$, we obtain

$$\begin{aligned}\|\beta\|_N^2\geq&(1-\varepsilon)\sum_{i=0}^{N-2}\mid a_i\mid^2+\frac{\varepsilon}{(N-1)^2}\mid\sqrt{2N+1}a_N+\sqrt{2N-1}a_{N-1}\mid^2\\&+\frac{1}{2N-1}\mid\sqrt{2N+1}a_N-\sqrt{2N-1}a_{N-1}\mid^2\\\geq&(1-\varepsilon)\sum_{i=0}^{N-2}\mid a_i\mid^2+\frac{2\varepsilon}{(N-1)^2}((2N+1)\mid a_N\mid^2+(2N-1)\mid a_{N-1}\mid^2)\\\geq&(1-\varepsilon)\sum_{i=0}^{N-2}\mid a_i\mid^2+\frac{2\varepsilon(2N-1)}{(N-1)_2}(\mid a_N\mid^2+\mid a_{N-1}\mid^2)\geq\frac{4N-2}{(N+1)^2}\|\beta\|^2.\end{aligned}$$

Therefore, we have

$$\frac{\sqrt{4N-2}}{N+1}\|\beta\|\leq\|\beta\|_N.$$

Finally, (4.6) implies

$$\begin{aligned}\|\beta\|_N\leq&\Big(\sum_{i=0}^{N-1}\mid a_i\mid^2+\frac{2N+1}{2N-1}(\mid a_{N-1}\mid^2+\mid a_N\mid^2)\Big)^{\frac{1}{2}}\\\leq&\Big(1+\frac{2N+1}{2N-1}\Big)\Big(\sum_{i=0}^{N}\mid a_i\mid^2\Big)^{\frac{1}{2}}\leq 2\|\beta\|.\end{aligned}$$

Remark 3. The power of N can not be improved in the estimate of Lemma 1. In fact, consider the function

$$\beta(t)=\sum_{i=0}^{N-2}\frac{2i+1}{2}L_i(t)-\frac{(N-1)^2}{4}(L_{N-1}(t)+L_N(t)),$$

for which one has

$$\frac{4N^2-1}{(N^3+2N^2+N-1)}\|\beta\|^2=\|\beta\|_N^2.$$

By Lemma 1 and (4.3), we obtain immediately.

Lemma 2. *For any* $u \in \overset{o}{U}$, *we have*

$$\frac{\sqrt{4N-2}}{N+1}\Big(\int_Q |u|^2 dxdt\Big)^{\frac{1}{2}} \le \|u\|_{M,N} \le 2\Big(\int_Q |u|^2 dxdt\Big)^{\frac{1}{2}}.$$

Lemma 3. *If* $\beta \in \overset{o}{V}_N$, $d > 0$, $\lambda > 0$, *then*

$$\min\Big(d, \frac{\sqrt{4N-2}}{N+1}\Big)\| -\lambda^{-1}\beta' + \lambda\beta\| \le \| -d\lambda^{-1}\beta' + \lambda\beta\|_N.$$

Proof. Let $\beta(t) = \sum_{i=0}^{N}\sqrt{\frac{2i+1}{2}}a_i L_i(t)$, by the definition of the discrete inner product and Lemma 1, we have

$$\begin{aligned}
\| -d\lambda^{-1}\beta' + \lambda\beta\|_N^2 &= d^2\lambda^{-2}\|\beta'\|^2 - 2dRe\sum_{i=0}^{N-1}\beta'(t_i)\overline{\beta}(t_i)\omega_i + \lambda^2\sum_{i=0}^{N-1} \mid \beta(t_i)\mid^2 \omega_i \\
&= d^2\lambda^{-2}\|\beta'\|^2 - 2dRe\sum_{i=0}^{N-1}\sum_{k=0}^{N}\sqrt{\frac{2k+1}{2}}\overline{a}_k L_k(t_i)\beta'(t_i)\omega_i + \lambda^2\|\beta\|_N^2 \\
&= d^2\lambda^{-2}\|\beta'\|^2 - 2dRe\int_I \beta' P_{N-1}\overline{\beta}dt \\
&\quad - 2dRe\sum_{i=0}^{N-1}\sqrt{\frac{2N+1}{2}}\overline{a}_N L_N(t_i)\beta'(t_i)\omega_i + \lambda^2\|\beta\|_N^2
\end{aligned}$$

since t_i $(i = 0, 1, \cdots, N-1)$ are the zeroes of the polynomial $L_N(t)+L_{N-1}(t)$, it follows that

$$\begin{aligned}
\| -d\lambda^{-1}\beta' + \lambda\beta\|_N^2 &= d^2\lambda^{-2}\|\beta'\|^2 - 2dRe\int_I \beta'\overline{\beta}dt \\
&\quad + 2dRe\sum_{i=0}^{N-1}\sqrt{\frac{2N+1}{2}}\overline{a}_N L_{N-1}(t_i)\beta'(t_i)\omega_i + \lambda^2\|\beta\|_N^2 \\
&= d^2\lambda^{-2}\|\beta'\|^2 - 2dRe\int_I \beta'\overline{\beta}dt \\
&\quad + d(2N+1) \mid a_N \mid^2 \sum_{i=0}^{N-1} L_{N-1}(t_i)L_N'(t_i)\omega_i + \lambda^2\|\beta\|_N^2 \\
&= d^2\lambda^{-2}\|\beta'\|^2 + 2d(2N+1) \mid a_N \mid^2 + \lambda^2\|\beta\|_N^2,
\end{aligned}$$

by Lemma 1, we have

$$\begin{aligned}
\| -d\lambda^{-1}\beta' + \lambda\beta\|_N &\ge (d^2\lambda^{-2}\|\beta'\|^2 + \frac{4N-2}{(N+1)^2}\lambda^2\|\beta\|^2)^{\frac{1}{2}} \\
&\ge \min\Big(d, \frac{\sqrt{4N-2}}{N+1}\Big)\| -\lambda^{-1}\beta' + \lambda\beta\|,
\end{aligned}$$

this completes the proof of Lemma 3.

For any $u\in\overset{(o}{W}$, $v\in\overset{o}{U}$, $u(x,t)=\sum\limits_{j=-M}^{M}\alpha_j(t)u_j$, $v(x,t)=\sum\limits_{j=-M}^{M}\beta_j(t)u_j$, then by Schwarz inequality, we have

$$\begin{aligned}
\mid B_d(u,v)\mid &= \Big|\sum_{i=0}^{N-1}\omega_i\sum_{j=-M}^{M}\lambda_j\alpha_j(t_i)[-\lambda_j^{-1}\overline{\beta}_j'(t_i)+\lambda_j\overline{\beta}_j(t_i)]\Big| \\
&\le \sum_{i=0}^{N-1}\Big(\sum_{j=-M}^{M}\lambda_j^2\mid\alpha_j(t_i)\mid^2\omega_i\Big)^{\frac12}\Big(\sum_{j=-M}^{M}\mid-\lambda_j^{-1}\beta_j'(t_i)+\lambda_j\beta_j(t_i)\mid^2\omega_i\Big)^{\frac12} \\
&\le \Big(\sum_{i=0}^{N-1}\sum_{j=-M}^{M}\lambda_j^2\mid\alpha_j(t_i)\mid^2\omega_i\Big)^{\frac12}\Big(\sum_{i=0}^{N-1}\sum_{j=-M}^{M}\mid-\lambda_j^{-1}\beta_j'(t_i)+\lambda_j\beta_j(t_i)\mid^2\omega_i\Big)^{\frac12} \\
&\le 2\sqrt{2}\|u\|_X\|v\|_Y. \qquad (4.7)
\end{aligned}$$

We define a linear operator G: $\overset{o}{V}_N\rightarrow\overset{(o}{V}_{N-1}$ by

$$\forall\beta\in\overset{o}{V}_N,\quad (G\beta,z)=(\beta',z),\forall z\in\overset{(o}{V}_{N-1}\ .$$

Then taking $z=G\beta$, we have

$$\begin{aligned}
(G\beta,G\beta) &= (\beta',G\beta)=(\beta'-\beta'(-1),G\beta)+\beta'(-1)(1,G\beta) \\
&= (\beta'-\beta'(-1),\beta)+\beta'(-1)(1,G\beta)=(\beta',\beta')+\beta'(-1)(1,G\beta) \\
&\ge \|\beta'\|^2-\sqrt{2}\|\beta'\|_{L^\infty(I)}\|G\beta\| \\
&\ge \|\beta'\|^2-\frac{N}{\sqrt{2}}\|\beta'\|\,\|G\beta\| \qquad \text{(by inverse inequality[2])} \\
&\ge \|\beta'\|^2-\frac12\|\beta'\|^2-\frac{N^2}{4}\|G\beta\|^2,
\end{aligned}$$

hence, we obtain

$$\|G\beta\|^2\ge\frac{2}{4+N^2}\|\beta'\|^2. \qquad (4.8)$$

We define again a linear operator T_c which maps $\overset{(o}{W}$ into $\overset{o}{U}$, by for any $u=\sum\limits_{j=-M}^{M}\alpha_j(t)u_j$ $\in\overset{(o}{W}$, $v=T_cu=\sum\limits_{j=-M}^{M}\beta_j(t)u_j$, $\beta_j(t)\in\overset{o}{V}_N$ satisfy

$$\begin{aligned}
&\sum_{i=0}^{N-1}(-\lambda_j^{-1}\beta_j'(t_i)+\lambda_j\beta_j(t_i))\psi(t_i)\omega_i \\
&= \lambda_j\sum_{i=0}^{N-1}\alpha_j(t_i)\psi(t_i)\omega_i,\quad \forall\psi\in\overset{(o}{V}_{N-1},\quad j=0,\pm1,\cdots,\pm M. \qquad (4.9)
\end{aligned}$$

Taking $\psi = \lambda_j \overline{\alpha}_j$ in (4.9), we have

$$\lambda_j \sum_{i=0}^{N-1} \overline{\alpha}_j(t_i)(-\lambda_j^{-1}\beta_j'(t_i) + \lambda_j \beta_j(t_i))\omega_i = \lambda_j^2 \|\alpha_j\|^2 \tag{4.10}$$

Taking $\psi = -\lambda_j^{-1} G\overline{\beta}_j' + \lambda_j P_c \overline{\beta}_j$ in (4.9) again, by (4.10), we obtain

$$\begin{aligned}(-\lambda_j^{-1}\beta_j' + \lambda_j P_c\beta_j, -\lambda_j^{-1}G\beta_j + \lambda_j P_c \beta_j) &= \lambda_j(\alpha_j, -\lambda_j^{-1}G\beta_j + \lambda_j P_c\beta_j) \\ &= \lambda_j(\alpha_j, -\lambda_j^{-1}\beta_j' + \lambda_j P_c \beta_j) = \lambda_j^2 \|\alpha_j\|^2. \end{aligned} \tag{4.11}$$

Let $\beta_j(t) = \sum_{i=0}^{N} a_{ij} L_i(t)$. Because

$$\begin{aligned}-Re(P_c\beta_j, \beta_j') &= -Re(P_{N-1}\beta_j + a_{Nj}P_c L_N(t), \beta_j') = -Re(\beta_j, \beta_j') - Re a_{Nj}(L_N, \beta_j')_N \\ &= Re a_{Nj}(L_{N-1}, \beta_j')_N = \mid a_{Nj} \mid^2 (L_{N-1}, L_N') = 2 \mid a_{Nj} \mid^2 \geq 0,\end{aligned}$$

hence

$$\begin{aligned}&(-\lambda_j^{-1}\beta_j' + \lambda_j P_c\beta_j, -\lambda_j^{-1}G\beta_j + \lambda_j P_c\beta_j) \\ &= \lambda_j^{-2}(\beta_j', G\beta_j) - (P_c\beta_j, G\beta_j) + (-\lambda_j^{-1}\beta_j' + \lambda_j P_c\beta_j, \lambda_j P_c \beta_j) \\ &= \lambda_j^{-2}(G\beta_j, G\beta_j) - (P_c\beta_j, \beta_j') + (-\lambda_j^{-1}\beta_j' + \lambda_j P_c\beta_j, \lambda_j P_c \beta_j) \\ &\geq \frac{2}{\lambda_j^2(4+N^2)}(\beta_j', \beta_j') - (P_c\beta_j, \beta_j') + (-\lambda_j^{-1}\beta_j' + \lambda_j P_c\beta_j, \lambda_j P_c\beta_j) \\ &\geq \Big(-\frac{\sqrt{2}}{\lambda_j\sqrt{4+N^2}}\beta_j' + \lambda_j P_c\beta_j, -\frac{\sqrt{2}}{\lambda_j\sqrt{4+N^2}}\beta_j' + \lambda_j P_c\beta_j\Big) \\ &= \Big\| -\frac{\sqrt{2}}{\lambda_j\sqrt{4+N^2}}\beta_j' + \lambda_j P_c\beta_j\Big\|^2 = \Big\| -\frac{\sqrt{2}}{\lambda_j\sqrt{4+N^2}}\beta_j' + \lambda_j \beta_j\Big\|_N^2 \\ &\geq \frac{2}{4+N^2}\| -\lambda_j^{-1}\beta_j' + \lambda_j\beta_j\|^2 \qquad \text{(by Lemma 3)}\end{aligned} \tag{4.12}$$

Combining (4.10), (4.11) and (4.12), we obtain

$$\frac{2}{4+N^2}\| -\lambda_j^{-1}\beta_j' + \lambda_j\beta_j\|^2 \leq \lambda_j^2\|\alpha_j\|^2,$$

Summing up for j form $-M$ to M in above inequality, we obtain

$$\frac{2}{4+N^2}\int_I \sum_{j=-M}^{M} \mid -\lambda_j^{-1}\beta_j' + \lambda_j\beta_j \mid^2 dt \leq \int_I \sum_{j=-M}^{M} \lambda_j^2 \mid \alpha_j \mid^2 dt$$

This inequality implies

$$\|T_c u\|_Y \leq \sqrt{\frac{4+N^2}{2}}\|u\|_X.$$

Finally, we get

$$\inf_{\substack{u\in \overset{(o}{W} \\ \|u\|_X=1}} \sup_{\substack{v\in \overset{o}{U} \\ \|v\|_Y\leq 1}} \mid B(u,v) \mid \geq \inf_{\substack{u\in \overset{(o}{W} \\ \|u\|_X=1}} \mid B_d\Big(u, \frac{T_c u}{\|T_c u\|_Y}\Big) \mid \geq \sqrt{\frac{2}{4+N^2}}. \tag{4.13}$$

From (4.7) and (4.13),we have

$$\forall v \in \overset{o}{U}, \quad v \neq 0, \sup_{u\in \overset{(o}{W}} \mid B_d(u,v) \mid > 0.$$

By theorem 5.2.1 in [1], we know that the solution of pseudospectral scheme is existent and unique. From now on, we estimate the error of the pseudospectral scheme. For any $\widetilde{u}\in \overset{(o}{W}$, $v \in \overset{o}{U}$, by (2.4), we have

$$\begin{aligned} |B_d(\widetilde{u} - u_c, v)| &= |B_d(\widetilde{u}, v) - B(u_0, v) + F(v) - F_d(v)| \\ &\leq \Big(\sup_{z\in \overset{o}{U}} \frac{\mid B_d(\widetilde{u}, z) - B(u_0, z) \mid}{\|z\|_Y} + \sup_{z\in \overset{o}{U}} \frac{\mid F(z) - F_d(z) \mid}{\|z\|_Y}\Big)\|v\|_Y \end{aligned} \tag{4.14}$$

Taking $v = T_c(\widetilde{u} - u_c)$ in (4.14), by definition of T_c and (4.13), we have

$$\begin{aligned} \| \widetilde{u} - u_c\|_X^2 &= B_d(\widetilde{u} - u_c, T_c(\widetilde{u} - u_c)) \\ &\leq \Big(\sup_{z\in \overset{o}{U}} \frac{\mid B_d(\widetilde{u}, z) - B(u_0, z) \mid}{\|z\|_Y} + \sup_{z\in \overset{o}{U}} \frac{\mid F(z) - F_d(z) \mid}{\|z\|_Y}\Big)\|T_c(\widetilde{u} - u_c)\|_Y \\ &\leq \sqrt{\frac{4+N^2}{2}}\Big(\sup_{z\in \overset{o}{U}} \frac{\mid B_d(\widetilde{u}, z) - B(u_0, z) \mid}{\|z\|_Y} + \sup_{z\in \overset{o}{U}} \frac{\mid F(z) - F_d(z) \mid}{\|z\|_Y}\Big)\| \widetilde{u} - u_c\|_X \end{aligned}$$

Then by the triangular inequality

$$\begin{aligned} \|u_0 - u_c\|_X \leq & \sqrt{\frac{4+N^2}{2}}\Big(\inf_{\widetilde{u}\in \overset{(o}{W}} \sup_{z\in \overset{o}{U}} \frac{\mid B_d(\widetilde{u}, z) - B(u_0, z) \mid}{\|z\|_Y} + \sup_{z\in \overset{o}{U}} \frac{\mid F(z) - F_d(z) \mid}{\|z\|_Y}\Big) \\ & + \inf_{\widetilde{u}\in \overset{(o}{W}} \|u_0 - \widetilde{u}\|_X \end{aligned} \tag{4.15}$$

Taking $\widetilde{u} = I_{N-2} \widetilde{P}_M u_0$ in (4.15), thanks to (4.1), we have

$$B_d(I_{N-2} \widetilde{P}_M u_0, z) = B(I_{N-2} \widetilde{P}_M u_0, z), \quad \forall z \in \overset{o}{U},$$

hence, from (2.5), we obtain

$$\inf_{\widetilde{u}\in \overset{(o}{W}} \sup_{z\in \overset{o}{U}} \frac{\mid B_d(\widetilde{u}, z) - B(u_0, z) \mid}{\|z\|_Y} \leq \sqrt{2}\|u_0 - I_{N-2} \widetilde{P}_M u_0\|_X.$$

Finally, we estimate the last term in (4.15). By definition of F and F_d, thanks to (4.2) and Lemma 2, we have

$$\begin{aligned}
|F(z)-F_d(z)| &\le \Big|\int_Q (f\overline{z}dxdt - h\sum_{j=-M}^{M}\sum_{i=0}^{N-1} f(x_j,t_i)\overline{z}(x_j,t_i)\omega_i\Big| \\
&\le \Big|\int_Q (f-P_{N-2}\widetilde{P}_M\, f)\overline{z}dxdt\Big| + \Big|(P_{N-2}\widetilde{P}_M\, f - I_{N-1}\widetilde{I}_M\, f, z)_{M,N}\,\Big| \\
&\le \|f-P_{N-2}\widetilde{P}_M\, f\|_{L^2(Q)}\|z\|_{L^2(Q)} \\
&\quad + \|P_{N-2}\widetilde{P}_M\, f - I_{N-1}\widetilde{I}_M\, f\|_{M,N}\|z\|_{M,N} \\
&\le (\|f-P_{N-2}\widetilde{P}_M\, f\|_{L^2(Q)} + 2\|P_{N-2}\widetilde{P}_M\, f - I_{N-1}\widetilde{I}_M\, f\|_{L^2(Q)})\|z\|_Y
\end{aligned}$$

hence

$$\begin{aligned}
\sup_{z\in\overset{o}{U}} \frac{|\,F(z)-F_d(z)\,|}{\|z\|_Y} &\le \|f-P_{N-2}\widetilde{P}_M\, f\|_{L^2(Q)} + 2\|P_{N-2}\widetilde{P}_M\, f - I_{N-1}\widetilde{I}_M\, f\|_{L^2(Q)} \\
&\le 3\|f-P_{N-2}\widetilde{P}_M\, f\|_{L^2(Q)} + 2\|f - I_{N-1}\widetilde{I}_M\, f\|_{L^2(Q)}.
\end{aligned}$$

By the error estimates of the project operator and the interpolation operator in [2], we obtain

Theorem 3. *If conditions of theorem 2 are satisfted, and $k>\frac{3}{2}$, u_0 and u_c are solutions of problem P and pseudospectral scheme respectively. Then we have the following error estimate*

$$\|u_0-u_c\|_X \le C(NM^{-(k+1)} + N^{-\frac{k-1}{2}})\|f\|_{k,\frac{k}{2}},$$

where constant C is independent of N, M and f.

Remark 4. When we consider problem (2.1)-(2.3)with inhomogenous initial value $g(x)$, and $g(x)\in H_p^{k+2}(D)$, we construct the following pseudospectral scheme: find $u_c\in W$, such that

$$\begin{aligned}
&u_{ct}(x_j,t_i) - u_{cxx}(x_j,t_i) + u_c(x_j,t_i) = f(x_j,t_i), \quad j=0,1,\cdots,2M, \quad i=1,2,\cdots,N-1,\\
&u_c(x_j,-1) = g(x_j), \quad j=0,1,\cdots,2M.
\end{aligned}$$

It is similar to proof in Theorem 3, we have

$$\|u_0-u_c\|_X \le C(NM^{-(k+1)} + N^{-\frac{k-2}{2}})(\|f\|_{k,\frac{k}{2}} + \|g\|_{k+2}).$$

Remark 5. It can be seen from theorem 2 and theorem 3, the orders of convergence are equal in bath time and space when $M=N^{\frac{1}{2}}$ and $M=N^{\frac{k}{2(k+1)}}$ respectively. If the considered problem is in the domain $[T_0,T]\times D$, the collocation points in time are $(T-T_0)(t_i-1)/2+T$, $i=0,1,\cdots,N-1$, then the results of Theorem 3 also are valid.

5. Numerical Results

In this section, we consider the pseodospectral scheme (4.4), the text function is the exact solution of problem p, i.e. $u_0 = e^{1-t} - e^{-2(t+1)} \cos x$.

Let $N = 9$, $M = 4$; $N = 16$, $M = 4$ and $N = 18$, $M = 4$, $x(j) = \frac{2j\pi}{1+2M}$, $j = 0, 1, \cdots, 2M$. The computed results are listed in table 1, 2 and 3 on $t = 1$.

Table 1 $N = 9$, $M = 4$

$x(j)$	u_c	u_0	$\frac{(u_c-u_0)10^8}{u_0}$
$x(0)$	0.9816843076884547	0.9816843611112658	-5.441953970145995
$x(1)$	0.9859693667184228	0.9859694066071113	-4.045631456319635
$x(2)$	0.9968195241458999	0.9968195226841657	0.1466398023793764
$x(3)$	1.0091578722055620	1.0091578194443670	5.2282401949339110
$x(4)$	1.0172111586409040	1.0172110707087230	8.6444380524420870
$x(5)$	1.0172111637552210	1.0172110707087230	9.1472163958157720
$x(6)$	1.0091578824173790	1.0091578194443670	6.2401550450218620
$x(7)$	0.9968195369963077	0.9968195226841656	1.4357806705605410
$x(8)$	0.9859693743660365	0.9859694066071112	-3.269987332553165

The Table 1–3 show that the error between the approximate solution u_c and the exact solution u_0 will monotone decrease with increasing of N when M do not changed. From above tables we also can see that the error are very small when the exact solution of the model problem is very smooth. I believe that this algorithm can be used to more complicated problems.

Table 2 $N = 12$, $M = 4$

$x(j)$	u_c	u_0	$\frac{(u_c-u_0)10^9}{u_0}$
$x(0)$	0.9816843611112644	0.9816843611112658	-0.000001357124227
$x(1)$	0.9859694030150431	0.9859694066071113	-3.643184235101693
$x(2)$	0.9968195172242356	0.9968195226841657	-5.477350764502334
$x(3)$	1.0091578146863220	1.0091578194443670	-4.714866946805532
$x(4)$	1.0172110688408600	1.0172110707087230	-1.836258555475101
$x(5)$	1.0172110725765890	1.0172110707087230	1.8362616115020740
$x(6)$	1.0091578242024130	1.0091578194443670	4.7148684870128270
$x(7)$	0.9968195281440949	0.9968195226841656	5.4773698734900710
$x(8)$	0.9859693743660365	0.9859694066071112	3.6431814200472470

Table 3 $N = 18$, $M = 4$

$x(j)$	u_c	u_0	$\frac{(u_c-u_0)10^9}{u_0}$
$x(0)$	0.9816843611112627	0.9816843611112658	-0.000003166623196
$x(1)$	0.9859694035333920	0.9859694066071113	-3.117459079869445
$x(2)$	0.9968195179881529	0.9968195226841657	-4.710996016808643
$x(3)$	1.0091578153279600	1.0091578194443670	-4.079052213462996
$x(4)$	1.0172110690864240	1.0172110707087230	-1.594849451072798
$x(5)$	1.0172110723310150	1.0172110707087230	1.5948431207312120
$x(6)$	1.0091578235607690	1.0091578194443670	4.0790471527818880
$x(7)$	0.9968195273801735	0.9968195226841656	4.7109911162411910
$x(8)$	0.9859694096808249	0.9859694066071112	3.1174534497605540

References

[1] I. Babuska, A.K. Aziz, Survey lectures on the mathematical foundations of the finite element method in the mathematical foundations of the finite element method with applications to partial differential equations ed, A.K. Aziz, Academic press, New York, London, (1977), 3–363.

[2] C. Canuto, M.Y. Hussaini, A. Qarteroni, T.A. Zang, Spectral methods in fluid dynamics, Springer, Ser. Comput. Phys., Springer-Verlag, New York, 1988.

[3] I. Babuska, T. Janik, The $h-p$ Version of the finite element method for parabolic equations. Part 1.The p-Version in time, *Numerical Method For Partial Differential Equations,* **5** (1989), 363–399.
[4] H. TAL-Ezer, Spectral methods in time for parabolic problems, *SIAM J.Numer. Anal.,* **26** (1989), 1–11.
[5] H.TAL-Ezer, Spectral methods in time for hyperbolic problems, *SIAM J.Numer. Anal.,* **23** (1986), 11–26.
[6] P. Jamet, Galerkin-type approximation which are discontinuous in time for parabolic equations in a variable domain, *SIAM J, Nnmer. Anal.,* **15** (1978), 912–928.
[7] K. Eriksson, C. Johnson, V. Thomee, Time discretization of parabolic problems by discontinuous Galerkin method, R.A.I.R.O., *MAN* **19** (1985), 611–643.
[8] P.J. Davis, P. Rabinowitz, Methods for Nnmerical Integration, Academic Press, Orlando, 1985.