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BOUNDARY ELEMENT APPROXIMATION OF STEKLOV EIGENVALUE PROBLEM FOR HELMHOLTZ EQUATION*1)2)

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Abstract

Steklov eigenvalue problem of Helmholtz equation is considered in the present paper. Steklov eigenvalue problem is reduced to a new variational formula on the boundary of a given domain, in which the self-adjoint property of the original differential operator is kept and the calculating of hyper-singular integral is avoided. A numerical example showing the efficiency of this method and an optimal error estimate are given.

Key words: Steklov eigenvalue problem, differential operator, error estimate, boundary element approximation.

1. Introduction

We consider the following Steklov eigenvalue problem:

Find nonzero u and number λ, such that

$$\begin{aligned} -\Delta u + u = 0, &\quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} = \lambda u, &\quad \text{on } \Gamma, \end{aligned} \tag{1.1}$$

where $\Omega \subset R^2$ is a bounded domain with sufficient smooth boundary Γ, $\frac{\partial}{\partial n}$ is the outward normal derivative on Γ.

Courant and Hilbert[1] studied the following eigenvalue problem:

$$\Delta u = 0, \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} = \lambda u, \quad \text{on } \Gamma, \tag{1.2}$$

which was reduced to the eigenvalue problem of an integral equation by using the Green's function of $\Delta u = 0$ with Nuemann boundary condition. From Fredholm theorem, we know that (1) the problem (1.2) has infinite number of eigenvalues, which are all real numbers, (2) suppose that $u_n(x)$, $u_m(x)$ are two eigenvalues of the problem (1.2) corresponding two different eigenvalues λ_n and λ_m, then

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$$\int_\Gamma u_n(x)u_m(x)ds_x = 0, \tag{1.3}$$

i.e. the trace of $u_n(x)$ and $u_m(x)$ on Γ are orthogonal on the space of $L^2(\Gamma)$.

Moreover, Courant and Hilbert[1] pointed out that analogous considerations held for the general self-adjoint second order elliptic differential equation, so for the problem (1.1).

But it is difficult to obtain the numerical solution of the problem (1.1), or (1.2) by the integral formula given by Courant and Hilbert. The reason is that for only a few of special domains, the Green's function is known. Bramble and Osborn[2] developed a finite element method for the Steklov eigenvalue problem and the optimal error estimate was given. Han, Guan and He discussed the boundary element approximation of the problem (1.2) [9] and the error estimate was given in [10] by Han and Guan. In this paper, a equivalent variational formula on the boundary Γ for the problem (1.1) is proposed, using the fundamental solution of $-\Delta u + u = 0$. Then the boundary finite element approximation of the problem (1.1) was obtain. A numerical example shows that the new method is very efficient.

2. A New Variational Formula on the Boundary Γ of Problem (1.1) and Its Boundary Element Approximation

The fundamental solution of equation $-\Delta u + u = 0$ in Ω is the modified Bessel function of zero order $K_0(|x-y|)$, which is given by

$$K_0(r) = \frac{\pi i}{2} H_0^{(1)}(ir) = \sum_{n=0}^{\infty} a_n r^{2n} \log\frac{1}{r} + \sum_{n=1}^{\infty} b_n r^{2n},\ a_0 = 1, \tag{2.1}$$

with a_n, b_n $(n = 1, 2, \cdots)$ unique determined nearby $r = 0$ and, we have

$$K_0(r) = \sqrt{\frac{\pi}{2r}} e^{-r} + \cdots, \tag{2.2}$$

at infinity. So $\lim_{r\to+\infty} K_0(r) = 0$. $K_0(r)$ satisfies the following differential equation

$$\frac{d^2K_0(r)}{dr^2} + \frac{1}{r}\frac{dK_0(r)}{dr} - K_0(r) = 0, \quad r \neq 0. \tag{2.3}$$

By using Green's formula it is obtained:

$$u(x) = -\frac{1}{2\pi}\int_\Gamma u(y)\frac{\partial K_0(|x-y|)}{\partial n_y} ds_y + \frac{1}{2\pi}\int_\Gamma p(y)K_0(|x-y|)ds_y, \quad \forall x \in \Omega, \tag{2.4}$$

where $u(x)$ is any solution of equation $-\Delta u + u = 0$, $p(y) = \frac{\partial u(y)}{\partial n_y}\Big|_\Gamma$, and n_y denotes the outward unit normal to Γ at point y. The formula (2.4) shows that every function u satisfying $-\Delta u + u = 0$ in Ω and continuously differentiable on $\Omega + \Gamma$ can be represented as the potential of a distribution on the boundary Γ consisting of a single-layer of density

$p(y) = \dfrac{\partial u(y)}{\partial n_y}$ and a double-layer of density $-u(x)$. From the continuity of the single-layer potential and the discontinuity of the double-layer potential on Γ [3],[4], the first relationship between $u|_\Gamma$ and p is obtained:

$$\frac{1}{2}u(x) = -\frac{1}{2\pi}\int_\Gamma u(y)\frac{\partial K_0(|x-y|)}{\partial n_y}ds_y + \frac{1}{2\pi}\int_\Gamma p(y)K_0(|x-y|)ds_y, \quad \forall x \in \Gamma. \quad (2.5)$$

Furthermore, by using the properties of the derivatives of the single and the double layer potentials [5], it is obtained that

$$\frac{1}{2}p(x) = -\frac{1}{2\pi}\int_\Gamma u(y)\frac{\partial^2 K_0(|x-y|)}{\partial n_x \partial n_y}ds_y + \frac{1}{2\pi}\int_\Gamma p(y)\frac{\partial K_0(|x-y|)}{\partial n_x}ds_y, \quad \forall x \in \Gamma, \quad (2.6)$$

where

$$\begin{aligned}\int_\Gamma u(y)\frac{\partial^2 K_0(|x-y|)}{\partial n_x \partial n_y}ds_y =& \frac{d}{ds_x}\int_\Gamma \frac{du(y)}{ds_y}K_0(|x-y|)ds_y \\ &- \int_\Gamma u(y)K_0(|x-y|)\cos(n_x,n_y)ds_y. \end{aligned} \quad (2.7)$$

Hence on the boundary Γ we have

$$\begin{aligned}\frac{\partial u(x)}{\partial n_x}\Big|_\Gamma =& p(x) = \frac{1}{2}p(x) - \frac{1}{2\pi}\frac{d}{ds_x}\int_\Gamma \frac{du(y)}{ds_y}K_0(|x-y|)ds_y \\ &+ \frac{1}{2\pi}\int_\Gamma u(y)K_0(|x-y|)\cos(n_x,n_y)ds_y + \frac{1}{2\pi}\int_\Gamma p(y)\frac{\partial K_0(|x-y|)}{\partial n_x}ds_y.\end{aligned}$$

Then the boundary condition in the problem (1.1) is rewritten as follows

$$\begin{aligned}&\frac{1}{2}p(x) - \frac{1}{2\pi}\frac{d}{ds_x}\int_\Gamma \frac{du(y)}{ds_y}K_0(|x-y|)ds_y + \frac{1}{2\pi}\int_\Gamma u(y)K_0(|x-y|) \\ &\quad\cdot \cos(n_x,n_y)ds_y + \frac{1}{2\pi}\int_\Gamma p(y)\frac{\partial K_0(|x-y|)}{\partial n_x}ds_y \\ &=\lambda u(x), \quad x \in \Gamma.\end{aligned} \quad (2.8)$$

In fact the equalities (2.5) and (2.8) hold for any solution u, $u \in H^1(\Omega)$, satisfying the Helmholtz equation in weak sense. In this paper, $H^\alpha(\Omega)$ denotes the Sobolev space on the domain Ω with norm $\|\cdot\|_{\alpha,\Omega}$ and $H^\beta(\Gamma)$ denotes the Sobolev space on the boundary Γ with norm $\|\cdot\|_{\beta,\Gamma}$ as usual [6]

Let $V = H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$. $\|(v,q)\|_V = (\|v\|^2_{\frac{1}{2},\Gamma} + \|q\|^2_{-\frac{1}{2},\Gamma})^{\frac{1}{2}}$.

By multiplying (2.8) by a function $v \in H^{\frac{1}{2}}(\Gamma)$ and (2.5) by a function $q \in H^{-\frac{1}{2}}(\Gamma)$, and by integrating over Γ, the following equivalent variational form of (1.1) is derived:

Find nonzero $(u,p) \in V$ and number λ, such that

$$\begin{aligned}&a_0(\dot{u},\dot{v}) + a_1(u,v) - b(p,v) = \lambda\int_\Gamma uv ds, \quad \forall v \in H^{\frac{1}{2}}(\Gamma), \\ &a_0(p,q) + b(q,u) = 0, \quad \forall q \in H^{-\frac{1}{2}}(\Gamma),\end{aligned} \quad (2.9)$$

where

$$
\begin{aligned}
a_0(p,q) &= \frac{1}{2\pi}\int_\Gamma\int_\Gamma p(y)q(x)K_0(|x-y|)ds_x ds_y,\\
a_1(u,v) &= \frac{1}{2\pi}\int_\Gamma\int_\Gamma u(y)v(x)K_0(|x-y|)\cos(n_x,n_y)ds_x ds_y,\\
b(p,v) &= -\frac{1}{2\pi}\int_\Gamma\int_\Gamma p(y)v(x)\frac{\partial K_0(|x-y|)}{\partial n_x}ds_x ds_y - \frac{1}{2}\int_\Gamma v(x)p(x)ds_x,\\
\dot{u}(y) &= \frac{du(y)}{ds_y}\Big|_\Gamma,\\
\dot{v}(y) &= \frac{dv(y)}{ds_y}\Big|_\Gamma.
\end{aligned}
$$

Let $A(u,p;v,q) = a_0(\dot{u},\dot{v}) + a_1(u,v) - b(p,v) + b(q,u) + a_0(p,q)$.

Then the problem (2.9) can be rewritten as follows:

Find nonzero $(u,p)\in V$ and number λ, such that

$$
A(u,p;v,q) = \lambda\int_\Gamma uvds, \quad \forall (v,q)\in V. \tag{2.10}
$$

For the bilinear $a_0(p,q)$, $a_1(p,q)$ and $b(q,v)$, the following lemma holds[5,7,8]

Lemma 2.1. (1) *$a_0(p,q)$ is a bounded bilinear form on $H^{-\frac{1}{2}}(\Gamma)\times H^{-\frac{1}{2}}(\Gamma)$ and $H^{-\frac{1}{2}}(\Gamma)$ –elliptic, i.e. there are two constants $M_0>0$, $\alpha_0>0$, such that $|a_0(p,q)|\le M_0\|p\|_{-\frac{1}{2},\Gamma}$, $\|q\|_{-\frac{1}{2},\Gamma}$, $\forall p,q\in H^{-\frac{1}{2}}(\Gamma)$, $a_0(q,q)\ge\alpha_0\|q\|^2_{-\frac{1}{2},\Gamma}$, $\forall q\in H^{-\frac{1}{2}(\Gamma)}$.*

(2) *Suppose $p\in H^{-\frac{1}{2}}(\Gamma)\cap H^{-\frac{1}{2}+t}(\Gamma)$, $(0\le t\le 1)$, then a constant $M_t>0$ must exist such that $|a_0(p,q)|\le M_t\|p\|_{-\frac{1}{2}+t,\Gamma}\ \|q\|_{-\frac{1}{2}-t,\Gamma}$, $\forall q\in H^{-\frac{1}{2}}(\Gamma)$.*

(3) *$b(q,v)$ is a bounded bilinear form on $H^{-\frac{1}{2}}(\Gamma)\times H^{\frac{1}{2}}(\Gamma)$, and $a_1(p,q)$ is a bounded bilinear form on $H^{\frac{1}{2}}(\Gamma)\times H^{\frac{1}{2}}(\Gamma)$, i.e. there are two constants M_1, M_2, such that $|a_0(q,v)|\le M_1\|v\|_{-\frac{1}{2},\Gamma}\|q\|_{-\frac{1}{2},\Gamma}$, $\forall q\in H^{-\frac{1}{2}}(\Gamma)$, $v\in H^{-\frac{1}{2}}(\Gamma)$, $|a_1(p,q)|\le M_2\|p\|_{\frac{1}{2},\Gamma}\ \|q\|_{\frac{1}{2},\Gamma}$, $\forall p,q\in H^{\frac{1}{2}}(\Gamma)$.*

From lemma 2.1, another lemma can be obtained.

Lemma 2.2. $A(u,p;v,q)$ is a bounded bilinear form on $V\times V$, namely there is a constant $M>0$, such that $|A(u,p;v,q)|\le M\|(u,p)\|_V\|(v,q)\|_V$.

To prove the coerciveness of $A(u,p;v,q)$, define the linear operator K_1 from $H^{-\frac{1}{2}}(\Gamma)$ to $H^{\frac{1}{2}}(\Gamma)$, and K_2 from $H^{\frac{1}{2}}(\Gamma)$ to $H^{\frac{1}{2}}(\Gamma)$ by

$$
K_1 q = \frac{1}{2\pi}\int_\Gamma q(y)K_0(|x-y|)ds_y, \quad \forall q\in H^{-\frac{1}{2}}(\Gamma). \tag{2.11}
$$

$$
K_2 v = -\frac{1}{2\pi}\int_\Gamma v(y)\frac{\partial K_0(|x-y|)}{\partial n_y}ds_y + \frac{1}{2}v(y), \quad \forall v\in H^{\frac{1}{2}}(\Gamma). \tag{2.12}
$$

From [7], [5] we know that there are two constants $d_1, d_2 > 0$, such that

$$\|K_1 q\|_{\frac{1}{2},\Gamma} \geq d_1 \|q\|_{-\frac{1}{2},\Gamma}, \quad \forall q \in H^{-\frac{1}{2}}(\Gamma). \tag{2.13}$$

$$\|K_2 v\|_{\frac{1}{2},\Gamma} \geq d_2 \|v\|_{\frac{1}{2},\Gamma}, \quad \forall v \in H^{\frac{1}{2}}(\Gamma). \tag{2.14}$$

Lemma 2.3. *There exists a constant $\mu > 0$, such that*

$$A(v,q;v,q) \geq \mu \|(v,q)\|_V^2, \quad \forall (v,q) \in V. \tag{2.15}$$

Proof. Let

$$\begin{aligned}
u_1(x) &= \frac{1}{2\pi}\int_\Gamma q(y) K_0(|x-y|) ds_y, \quad x \in \Omega,\\
u_2(x) &= \frac{1}{2\pi}\int_\Gamma q(y) K_0(|x-y|) ds_y, \quad x \in \Omega^c,\\
u_3(x) &= -\frac{1}{2\pi}\int_\Gamma v(y) \frac{\partial K_0(|x-y|)}{\partial n_y} ds_y, \quad x \in \Omega,\\
u_4(x) &= -\frac{1}{2\pi}\int_\Gamma v(y) \frac{\partial K_0(|x-y|)}{\partial n_y} ds_y, \quad x \in \Omega^c,
\end{aligned}$$

where $\Omega^c = R^2 \backslash \overline{\Omega}$.

From the boundary behaviour of the double layar potential, the following equations hold

$$\begin{aligned}
&u_1(x)|_\Gamma = u_2(x)|_\Gamma,\\
&\frac{\partial u_1(x)}{\partial n_x}|_\Gamma = \frac{1}{2\pi}\int_\Gamma q(y)\frac{\partial K_0(|x-y|)}{\partial n_x} ds_y + \frac{1}{2}q(x),\\
&\frac{\partial u_2(x)}{\partial n_x}\Big|_\Gamma = \frac{1}{2\pi}\int_\Gamma q(y)\frac{\partial K_0(|x-y|)}{\partial n_x} ds_y - \frac{1}{2}q(x),\\
&u_3(x)|_\Gamma = -\frac{1}{2\pi}\int_\Gamma v(y)\frac{\partial K_0(|x-y|)}{\partial n_y} ds_y + \frac{1}{2}v(x),\\
&u_4(x)|_\Gamma = -\frac{1}{2\pi}\int_\Gamma v(y)\frac{\partial K_0(|x-y|)}{\partial n_y} ds_y - \frac{1}{2}v(x),\\
&\frac{\partial u_3(x)}{\partial n_x}\Big|_\Gamma = \frac{\partial u_4(x)}{\partial n_x}|_\Gamma,
\end{aligned}$$

which is followed by

$$\frac{\partial u_1(x)}{\partial n_x}\Big|_\Gamma - \frac{\partial u_2(x)}{\partial n_x}\Big|_\Gamma = q(x), \quad u_3(x)|_\Gamma - u_4(x)|_\Gamma = v(x),$$

on the other hand when $|x| \to +\infty$, we have

$$u_i(x) = o\Big(\frac{1}{|x|^2}\Big), \quad (i=2,4), \qquad \frac{\partial u_i(x)}{\partial n_x} = o\Big(\frac{1}{|x|^2}\Big), \quad (i=2,4).$$

An application of the Green's formula yields

$$\int_\Omega (\bigtriangledown u_1 \bigtriangledown u_1 + {u_1}^2) dx + \int_\Omega (\bigtriangledown u_3 \bigtriangledown u_3 + {u_3}^2) dx$$

$$+\int_{\Omega^c}(\nabla u_2 \nabla u_2 + {u_2}^2)dx + \int_{\Omega^c}(\nabla u_4 \nabla u_4 + {u_4}^2)dx$$
$$= \int_\Gamma \Big(\frac{\partial u_1}{\partial n_x} - \frac{\partial u_2}{\partial n_x}\Big)u_1(x)ds_x + \int_\Gamma \frac{\partial u_3}{\partial n_x}(u_3(x) - u_4(x))ds_x = A(v,q;v,q).$$

Hence the follows hold

$$A(v,q;v,q) \geq \|u_1\|^2_{1,\Omega} + \|u_3\|^2_{1,\Omega}. \tag{2.16}$$

From the trace theorem and inequalities (2.16), (2.13), and (2.14) there are two constants $c_1 > 0$, $c_2 > 0$, such that

$$\|u_1\|_{1,\Omega} \geq c_1\|u_1\|_{\frac{1}{2},\Gamma} = c_1\|K_1 q\|_{\frac{1}{2},\Gamma} \geq c_1 d_1 \|q\|_{-\frac{1}{2},\Gamma}, \tag{2,17}$$
$$\|u_3\|_{1,\Omega} \geq c_2\|u_3\|_{\frac{1}{2},\Gamma} = c_2\|K_2 v\|_{\frac{1}{2},\Gamma} \geq c_2 d_2 \|v\|_{-\frac{1}{2},\Gamma}, \tag{2.18}$$

so the inequality (2.15) immediately follows with $\mu = \max(c_1^2 d_1^2,\, c_2^2 d_2^2)$, $A(v,q;v,q) \geq \mu\|(v,q)\|^2_V$, $(v,q) \in V$. From lemma 2.2 and 2.3, the eigenvalue problem (2.9) on the boundary Γ is equivalent to the Steklov eigenvalue problem (1.1).

Now suppose that the boundary Γ of the domain Ω is represented as $x_1 = x_1(s)$, $x_2 = x_2(s)$, $0 \leq s \leq L$, and $x_j(0) = x_j(L)$, $j = 1,2$. Furthermore, Γ is divided into segments $\{T\}$ by the points $x^i = (x_1(s_i), x_2(s_i))$, $i = 0,1,2,\cdots,N$, with $0 = s_0 < s_1 < \cdots < s_N = L$. Define $h = \max\limits_{1\leq i\leq N}|s_i - s_{i-1}|$, and this partition of Γ is denoted as J_h. Let

$$S_h = \{v_h|\ v_h \in C^0(\Gamma), v_h|_T \ \text{ is a linear function}, \quad \forall T \in J_h\}.$$
$$M_h = \{q_h|\ q_h|_T \ \text{ is a constant}, \quad \forall T \in J_h\}.$$

It is obviously that S_h is a subspace of $H^{\frac{1}{2}}(\Gamma)$ with dimension N and M_h is a subspace of $H^{-\frac{1}{2}}(\Gamma)$ with dimension N. S_h and M_h are two regular finite element space in sense of Babuška and Aziz[8], which satisfy the following approximation properties:

$$\inf_{v_h\in S_h}\|u - v_h\|_{\frac{1}{2},\Gamma} \leq ch^t\|u\|_{\frac{1}{2}+t,\Gamma}, \quad \forall u \in H^{\frac{1}{2}+t}(\Gamma), \tag{2.19}$$
$$\inf_{q_h\in M_h}\|p - q_h\|_{-\frac{1}{2},\Gamma} \leq ch^t\|p\|_{-\frac{1}{2}+t,\Gamma}, \quad \forall p \in H^{-\frac{1}{2}+t}(\Gamma), \tag{2.20}$$

where $0 \leq t \leq 1$.

Now we consider the discrete problem of (2.9)

Find nonzero $(u_h, p_h) \in S_h \times M_h$ and number λ_h, such that

$$a_0(\dot{u}_h, \dot{v}_h) + a_1(u_h, v_h) - b(p_h, v_h) = \lambda_h \int_\Gamma u_h v_h ds, \quad \forall v_h \in S_h,$$
$$a_0(p_h, q_h) + b(q_h, u_h) = 0, \quad \forall q_h \in M_h. \tag{2.21}$$

Assuming that the base functions of space S_h and the space M_h are given, the problem (2.21) can be reduced to a matrix eigenvalue problem. By solving it, the approximation solution of original problem (1.1) can be obtained.

3. The Error Estimate of the Boundary Element Approximation

In this section, the error estimate of the boundary element approximation will be discussed by the general approximation theory of eigenvalues for a certain class of the compact operators [2]. Hence we identify the boundary eigenvalue problem (2.9) with the eigenvalues of a compact operator. Let $\mu = \lambda + 1$, then the problem (2.9) can be rewritten as follows:

$$\text{Find nonzero } (u,p) \in V \text{ and number } \lambda, \text{ such that}$$
$$a_0(\dot{u},\dot{v}) + \langle u,v\rangle + a_1(u,v) - b(p,v) = \mu\langle u,v\rangle, \quad \forall v \in H^{\frac{1}{2}}(\Gamma),$$
$$a_0(p,q) + b(q,u) = 0, \quad \forall q \in H^{-\frac{1}{2}}(\Gamma), \tag{3.1}$$

where $\langle u,v\rangle = \int_\Gamma uvds$.

For any given $u \in H^{\frac{1}{2}}(\Gamma)$, $b(q,u)$ is a bounded linear functional on $H^{-\frac{1}{2}}(\Gamma)$. From the lemma 2.1, the following variational problem

$$\text{Find } p \in H^{-\frac{1}{2}}(\Gamma), \text{ such that}$$
$$a_0(p,q) + b(q,u) = 0, \quad \forall q \in H^{-\frac{1}{2}}(\Gamma), \tag{3.2}$$

has a unique solution p. Let $Pu = p$, then we obtain the bounded operator $P : H^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$, and there is a constant C, such that

$$\|Pu\|_{-\frac{1}{2},\Gamma} \le C\|u\|_{\frac{1}{2},\Gamma}, \quad \forall u \in H^{\frac{1}{2}}(\Gamma). \tag{3.3}$$

In fact, for any given $u \in H^{\frac{1}{2}}(\Gamma)$, it can be extended to the total domain Ω and $u \in H^1(\Omega)$ is the weak solution of Helmholtz equation $-\Delta u + u = 0$. Then $Pu = \frac{\partial u}{\partial n}\Big|_\Gamma$, and the operator P can be extented[7] to P: $H^{s+1}(\Gamma) \to H^s(\Gamma)$, $s \ge -\frac{1}{2}$, and

$$\|Pu\|_{s,\Gamma} \le C\|u\|_{s+1,\Gamma}. \tag{3.4}$$

By using $p = Pu$, the unknown function p can be eliminated in the problem (3.1). We note that $a_0(Pu,Pv) = -b(Pu,v)$, $\forall v \in H^{\frac{1}{2}}(\Gamma)$. Then the problem (3.1) is reduced to

$$\text{Find nonzero } u \in H^{\frac{1}{2}}(\Gamma) \text{ and number } \mu, \text{ such that}$$
$$E(u,v) = \mu\langle u,v\rangle, \quad \forall v \in H^{\frac{1}{2}}(\Gamma). \tag{3.5}$$

where $E(u,v) = a_0(\dot{u},\dot{v}) + a_1(u,v) + a_0(Pu,Pv) + \langle u,v\rangle$. From the lemma 2.1, 2.2 and 2.3, the following lemma is obtained.

Lemma 3.1. *$E(u,v)$ is a bounded bilinear form on $H^{\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$ and $H^{\frac{1}{2}}(\Gamma)$-elliptic, i.e. there exist two constants $M_3 > 0$, $\alpha_3 > 0$, such that*

$$|E(u,v)| \le M_3\|u\|_{\frac{1}{2},\Gamma}\|v\|_{\frac{1}{2},\Gamma}, \quad \forall u,v \in H^{\frac{1}{2}}(\Gamma)$$

$$E(v,v) \geq \alpha_3 \|v\|^2_{\frac{1}{2},\Gamma}, \quad \forall v \in H^{\frac{1}{2}}(\Gamma).$$

For any given $g \in H^{-\frac{1}{2}}(\Gamma)$, consider the following variational problem

$$\text{Find nonzero } u \in H^{\frac{1}{2}}(\Gamma), \text{ such that}$$
$$E(u,v) = \langle g, v\rangle, \quad \forall v \in H^{\frac{1}{2}}(\Gamma). \tag{3.6}$$

From the lemma 3.1, and Lax-Milgram theorem, the problem (3.7) has a unique solution $u \in H^{\frac{1}{2}}(\Gamma)$. Let $Tg = u$, we obtain a bounded operator T: $H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ In fact, for any given $g \in H^{-\frac{1}{2}}(\Gamma)$, $Tg = u$ is the restriction on Γ of the solution of the following boundary problem: $-\Delta u + u = 0$, in Ω, $\dfrac{\partial u}{\partial n} + u = g$, on G. Hence we know that[7] T: $H^s(\Gamma) \to H^{s+1}(\Gamma)$, $s \geq -\dfrac{1}{2}$, and

$$\|Tg\|_{s+1,\Gamma} \leq C\|u\|_{s+\frac{3}{2},\Omega} \leq C\|g\|_{s,\Gamma}. \tag{3.7}$$

Suppose $\dfrac{1}{\mu}$ is a nonzero eigenvalue of T on $H^{-\frac{1}{2}}(\Gamma)$, i.e. there is a nonzero $g \in H^{-\frac{1}{2}}(\Gamma)$, such that $Tg = \dfrac{1}{\mu}g$. Then $E(Tg,v) = \langle g,v\rangle = \mu\langle Tg,v\rangle$, $\forall v \in H^{\frac{1}{2}}(\Gamma)$, and Tg is nonzero. Thus μ is an eigenvalue of the problem (3.5) with Tg, the corresponding eigenfunction. Conversely, suppose μ is a nonzero eigenvalue of the problem (3.5) with the corresponding eigenfunction u, namely $E(u,v) = \mu\langle u,v\rangle$, $\forall v \in H^{\frac{1}{2}}(\Gamma)$. Then $Tu = \dfrac{1}{\mu}u$, $\dfrac{1}{\mu}$ is an eigenvalue of T with the corresponding eigenfunction u. Therefore the eigenvalues of the problem (3.1) are the reciprocals of the eigenvalues of the compact operator T.

Similarly, in the approximation problem (2.21), let $\mu_h = \lambda_h + 1$, then

$$\text{Find nonzero } (u_h, p_h) \in V_h \text{ and number } \mu_h, \text{ such that}$$
$$a_0(\dot{u}_h, \dot{v}_h) + \langle u_h, v_h\rangle + a_1(u_h, v_h) - b(p_h, v_h) = \mu_h\langle u_h, v_h\rangle, \quad \forall v_h \in S_h$$
$$a_0(p_h, q_h) + b(q_h, u_h) = 0, \quad \forall q_h \in M_h. \tag{3.8}$$

For any given $u \in H^{\frac{1}{2}}(\Gamma)$, consider the following variational problem

$$\text{Find nonzero } p_h \in M_h, \text{ such that}$$
$$a_0(p_h, q) + b(q, u) = 0, \quad \forall q \in M_h. \tag{3.9}$$

By the lemma 2.1, the problem (3.9) has a unique solution p_h. Let $P_h u = p_h$, then we get an operator P_h: $H^{s+1}(\Gamma) \to M_h$, $s \geq -\dfrac{1}{2}$, and

$$\|P_h u\|_{-\frac{1}{2},\Gamma} \leq C\|u\|_{\frac{1}{2},\Gamma}. \tag{3.10}$$

Obviously, $P_h u$ is the boundary element approximation of Pu in space M_h, and the following error estimate holds[7]

$$\|Pu - P_h u\|_{-\frac{1}{2},\Gamma} \leq Ch^s\|Pu\|_{s-\frac{1}{2}}. \tag{3.11}$$

It is noted that $a_0(P_h u, P_h v) = -b(P_h u, v)$, $\forall v \in H^{\frac{1}{2}}(\Gamma)$. Then the eigenvalue problem (3.8) is reduced to

$$\text{Find nonzero } u_h \in S_h \text{ and number } \mu_h \text{, such that}$$
$$E_h(u_h, v) = \mu_h \langle u_h, v\rangle, \quad \forall v \in S_h, \tag{3.12}$$

where $E_h(u,v) = a_0(\dot{u}, \dot{v}) + \langle u, v\rangle + a_1(u,v) + a_0(P_h u, P_h v)$. It is straight forward to check the following lemma.

Lemma 3.2. *There exist two positive constants M_4, α_4, independent of h, such that*

$$|E_h(u,v)| \le M_4 \|u\|_{\frac{1}{2},\Gamma} \|v\|_{\frac{1}{2},\Gamma}, \quad \forall u, v \in H^{\frac{1}{2}}(\Gamma). \tag{3.13}$$

$$E_h(v,v) \ge \alpha_4 \|v\|^2_{\frac{1}{2},\Gamma}, \quad \forall v \in H^{\frac{1}{2}}(\Gamma). \tag{3.14}$$

Hence for any given $g \in H^{-\frac{1}{2}}(\Gamma)$, the variational problem

$$\text{Find } u_h \in S_h \text{, such that}$$
$$E_h(u_h, v) = \langle g, v\rangle, \quad \forall v \in S_h, \tag{3.15}$$

has a unique solution u_h. Let $T_h g = u_h$, a bounded operator is obtained T_h: $H^{-\frac{1}{2}}(\Gamma) \to S_h \subset H^{\frac{1}{2}}(\Gamma)$, and $\|T_h g\|_{\frac{1}{2},\Gamma} \le C\|g\|_{-\frac{1}{2},\Gamma}$. Similarly, the eigenvalues of the problem (3.12) are the reciprocals of the eigenvalues of the operator T_h. Thus the eigenvalues of the problem (3.1) can be compared with the boundary element approximations by comparing the eigenvalues of the compact operator T with the approximate operator T_h. In order to obtain the eigenvalue estimates of T and T_h, we need to estimate the error $T - T_h$.

Lemma 3.3. For any $g \in H^{-\frac{1}{2}}(\Gamma)$, it holds that

$$E_h((T - T_h)g, v) = -a_0((P - P_h)Tg, Pv), \quad \forall v \in S_h. \tag{3.16}$$

Proof. From the definition of T and T_h, we derive $E(Tg, v) = \langle g, v\rangle$, $\forall v \in S_h$. $E_h(T_h g, v) = \langle g, v\rangle$, $\forall v \in S_h$. Hence for any $v \in S_h$ it is obtained that

$$\begin{aligned} 0 &= E(Tg, v) - E_h(T_h g, v) = E_h(Tg - T_h g, v) + a_0(PTg, Pv) - a_0(P_h Tg, P_h v) \\ &= E_h((T - T_h)g, v) + a_0((P - P_h)Tg, Pv) + a_0(P_h Tg, (P - P_h)v). \end{aligned}$$

On the other hand, by the definitions of the operator P and P_h we obtain $a_0(Pv, q) + b(q, v) = 0$, $\forall v \in S_h$, $q \in M_h$. $a_0(P_h v, q) + b(q, v) = 0$, $\forall v \in S_h$, $q \in M_h$. Thus $a_0((P - P_h)v, q) = 0$, $\forall v \in S_h, q \in M_h$. From the symmetry of the bilinear form $a_0(p, q)$ and $P_h Tg \in M_h$ the following equality is derived $a_0(P_h Tg, (P - P_h)v) = 0$, $\forall g \in H^{-\frac{1}{2}}(\Gamma), v \in S_h$. Then the lemma 3.3 is proved.

Lemma 3.4. *For any $g \in H^{-\frac{1}{2}+t}(\Gamma)$, $(0 \le t \le 1)$, there exists a constant C such that*

$$\|Tg - T_h g\|_{\frac{1}{2},\Gamma} \le Ch^t \|g\|_{-\frac{1}{2}+t,\Gamma}. \tag{3.17}$$

Proof. From the lemma 3.2, it holds that

$$\begin{aligned}|Tg-T_hg\|^2_{\frac{1}{2},\Gamma} &\le \frac{1}{\alpha_4}E_h((T-T_h)g,(T-T_h)g)\\ &=\frac{1}{\alpha_4}\{E_h((T-T_h)g,Tg-\chi)+E_h((T-T_h)g,\chi-T_hg)\}\\ &=\frac{1}{\alpha_4}\{E_h((T-T_h)g,Tg-\chi)-a_0((P-P_h)Tg,P(\chi-T_hg))\},\quad \forall\chi\in S_h.\end{aligned}$$

The last equality is from the lemma 3.3. Furthermore from the lemma 3.2 and lemma 2.1, it is obtained that

$$|E_h((T-T_h)g,Tg-\chi)|\le M_4\|(T-T_h)g\|_{\frac{1}{2},\Gamma}\|Tg-\chi\|_{\frac{1}{2},\Gamma},$$

and

$$\begin{aligned}&|a_0((P-P_h)Tg,P(\chi-T_hg))|\le|a_0((P-P_h)Tg,P(Tg-\chi))|\\ &\qquad+|a_0((P-P_h)Tg,P(T-T_h)g)|\\ &\qquad\le C\|(P-P_h)Tg\|_{-\frac{1}{2},\Gamma}\{\|P(Tg-\chi)\|_{-\frac{1}{2},\Gamma}+\|P(T-T_h)g\|_{-\frac{1}{2},\Gamma}\}.\end{aligned}$$

On the other hand, the inequalities (3.11), (3.4) and (3.7) lead to

$$\begin{aligned}&\|(P-P_h)Tg\|_{-\frac{1}{2},\Gamma}\le Ch^t\|PTg\|_{-\frac{1}{2}+t,\Gamma}\le Ch^t\|g\|_{-\frac{1}{2}+t,\Gamma}.\\ &\|P(Tg-\chi)\|_{-\frac{1}{2},\Gamma}\le C\|Tg-\chi\|_{\frac{1}{2},\Gamma}.\\ &\|P(T-T_h)g\|_{-\frac{1}{2},\Gamma}\le C\|(T-T_h)g\|_{\frac{1}{2},\Gamma}.\end{aligned}$$

Hence the following inequality is got

$$\begin{aligned}\|(T-T_h)g\|^2_{\frac{1}{2},\Gamma}\le &C\{\|(T-T_h)g\|_{\frac{1}{2},\Gamma}[\|Tg-\chi\|_{\frac{1}{2},\Gamma}+h^t\|g\|_{-\frac{1}{2}+t,\Gamma}]\\ &+h^t\|g\|_{-\frac{1}{2}+t,\Gamma}\|Tg-\chi\|_{\frac{1}{2},\Gamma}\},\quad\forall\chi\in S_h.\end{aligned}$$

Finally by the estimate

$$\inf_{\chi\in S_h}\|Tg-\chi\|_{\frac{1}{2},\Gamma}\le Ch^t\|Tg\|_{\frac{1}{2}+t,\Gamma}\le Ch^t\|g\|_{-\frac{1}{2}+t,\Gamma}.$$

The inequality (3.17) is proved.

Lemma 3.5. *For any $g\in H^{-\frac{1}{2}+t}(\Gamma)$, $\psi\in H^{-\frac{1}{2}+t}(\Gamma)$, $(0\le s,t\le 1)$, the following inequality holds*

$$|\langle(T-T_h)g,\psi\rangle|\le Ch^{t+s}\|g\|_{-\frac{1}{2}+t,\Gamma}\|\psi\|_{-\frac{1}{2}+s,\Gamma}. \tag{3.18}$$

Proof. From the definition of the operator T it is known that $\langle Tg,\psi\rangle=E(Tg,T\psi)$. $\langle T_hg,\psi\rangle=E(T_hg,T\psi)$. Hence it is derived that

$$\begin{aligned}\langle(T-T_h)g,\psi\rangle&=E((T-T_h)g,T\psi)\\ &=E_h((T-T_h)g,T\psi)+a_0(P(T-T_h)g,PT\psi)-a_0(P_h(T-T_h)g,P_hT\psi)\end{aligned}$$

$$
\begin{aligned}
&= E_h((T-T_h)g, T\psi) + a_0(P(T-T_h)g, (P-P_h)T\psi) \\
&= E_h((T-T_h)g, (T-T_h)\psi) + a_0(P(T-T_h)g, (P-P_h)T\psi) \\
&\quad - a_0((P-P_h)Tg, PT_h\psi) \\
&= E_h((T-T_h)g, (T-T_h)\psi) + a_0(P(T-T_h)g, (P-P_h)T\psi) \\
&\quad + a_0((P-P_h)Tg, P(T-T_h)\psi) - a_0((P-P_h)Tg, PT\psi) \\
&= E_h((T-T_h)g, (T-T_h)\psi) + a_0(P(T-T_h)g, (P-P_h)T\psi) \\
&\quad + a_0((P-P_h)Tg, P(T-T_h)\psi) - a_0((P-P_h)Tg, (P-P_h)T\psi).
\end{aligned}
$$

By the lemma 3.2 and 3.4, it holds that

$$
\begin{aligned}
|E_h((T-T_h)g, (T-T_h)\psi)| &\le M_4 \|(T-T_h)g\|_{\frac{1}{2},\Gamma} \|(T-T_h)\psi\|_{\frac{1}{2},\Gamma} \\
&\le Ch^{t+s} \|g\|_{-\frac{1}{2}+t,\Gamma} \|\psi\|_{-\frac{1}{2}+s,\Gamma}.
\end{aligned}
$$

Furthermore from the inequalities (3.4), (3.17), (3.11) and (3.7) lead to

$$
\begin{aligned}
|a_0(P(T-T_h)g, (P-P_h)T\psi)| &\le Ch^{s+t} \|g\|_{-\frac{1}{2}+t,\Gamma} \|\psi\|_{-\frac{1}{2}+s,\Gamma}. \\
|a_0((P-P_h)Tg, P(T-T_h)\psi)| &\le Ch^{s+t} \|g\|_{-\frac{1}{2}+t,\Gamma} \|\psi\|_{-\frac{1}{2}+s,\Gamma}. \\
|a_0((P-P_h)Tg, (P-P_h)T\psi)| &\le Ch^{s+t} \|g\|_{-\frac{1}{2}+t,\Gamma} \|\psi\|_{-\frac{1}{2}+s,\Gamma}.
\end{aligned}
$$

Hence the inequality (3.18) follows immediately.

Let $\|T-T_h\|_{-t,s} = \sup\limits_{\psi \in H^s(\Gamma)} \sup\limits_{g \in H^t(\Gamma)} \dfrac{\langle (T-T_h)g, \psi\rangle}{\|g\|_{t,\Gamma} \|\psi\|_{s,\Gamma}}$, $\forall s, t \ge 0$. In the inequality (3.18), taking $t = \dfrac{1}{2}$, $s = \dfrac{1}{2}$ it is obtained that

$$
\|T-T_h\|_{0,0} \le Ch. \tag{3.19}
$$

From the well known eigenvalue convergence result [3], if $\nu^1, \nu^2, \cdots$, are the nonzero eigenvalues of T ordered by decreasing magnitude, taking account of algebraic multiplicities, then for each h there is an ordering (again counting according to algebraic multiplicities) of the eigenvalues of T_h, $\nu^1(h), \nu^2(h), \cdots$, such that $\lim\limits_{h\to 0} \nu^j(h) = \nu^j$ for each j. Hence for the Steklov eigenvalue problem (1.1), the following theorem holds that

Theorem 3.1. *If $\lambda^1, \lambda^2, \cdots$, are the eigenvalues of the Steklov eigenvalue problem* (1.1) *ordered by increasing magnitude, taking account of algebraic multiplicities, then for each h there is an ordering* (*again counting according to algebraic multiplicities*) *of the eigenvalues of the eigenvalue problem* (2.21), *$\lambda^1(h), \lambda^2(h), \cdots$, such that $\lim\limits_{h\to 0} \lambda^j(h) = \lambda^j$ for each j.*

Let λ be an eigenvalue of the Steklov eigenvalue problem (1.1) with algebraic multiplicity m. From the theorem 3.1 there are m eigenvalues $\lambda^1(h), \lambda^2(h), \cdots, \lambda^m(h)$ of the problem (2.21) such that $\lim\limits_{h\to 0} \lambda^j(h) = \lambda$. Furthermore $\nu = \dfrac{1}{\lambda+1}$ is an eigenvalue of the compact operators T with algebraic multiplicity m, and there are m eigenvalues

$\nu^1(h) = \dfrac{1}{\lambda^1(h)+1}, \nu^2(h) = \dfrac{1}{\lambda^2(h)+1}, \cdots, \nu^m(h) = \dfrac{1}{\lambda^m(h)+1}$ of the operator T_h, such that $\lim\limits_{h\to 0} \nu^j(h) = \nu$, $j = 1, 2, \cdots, m$. Taking $t = \dfrac{1}{2}$, $s = 1$; $t = 1$, $s = \dfrac{1}{2}$ and $t = 1$, $s = 1$ in the inequality (3.18) respectively, it follows that

$$\|T - T_h\|_{0,\frac{1}{2}} \le Ch^{\frac{3}{2}}; \quad \|T - T_h\|_{-\frac{1}{2},0} \le Ch^{\frac{3}{2}}; \quad \|T - T_h\|_{-\frac{1}{2},\frac{1}{2}} \le Ch^2.$$

An application of the theorem 3.2 in [2] yields the following error estimate

$$\Big|\nu - \frac{1}{m}\sum_{j=1}^{m} \nu^j(h)\Big| \le Ch^2.$$

Thus, for the Steklov eigenvalue problem the optimal error of the approximation can be estimated by:

Theorem 3.2. *The following error estimate holds*

$$|\lambda - \tilde{\lambda}(h)| \le Ch^2, \tag{3.20}$$

where $\tilde{\lambda}(h) = \Big[\dfrac{1}{m}\sum\limits_{j=1}^{m} \dfrac{1}{\lambda^j(h)+1}\Big]^{-1} - 1.$

4. Numerical example

Assume that the boundary Γ of domain Ω is the unit circle, then the solution of equation $-\Delta u + u = 0$ can be represented by polar coordinates (r, θ):

$$u(r,\theta) = a_0 I_0(r) + \sum_{n=1}^{\infty} I_n(r)(a_n \cos(n\theta) + b_n \sin(n\theta)),$$

where $I_n(r)$ is the modified Bessel function of n order. The solution of (1.1) can be found exactly. The eigenvalues of (1.1), $\lambda_0, \lambda_1, \cdots, \lambda_n, \cdots$, are derived as

$$\lambda_0 = \frac{\sum\limits_{k=0}^{\infty} \dfrac{1}{k!(k+1)!}\Big(\dfrac{1}{2}\Big)^{2k+1}}{\sum\limits_{k=0}^{\infty} \dfrac{1}{k!k!}\Big(\dfrac{1}{2}\Big)^{2k}}, \quad \lambda_n = \frac{\sum\limits_{k=0}^{\infty} \dfrac{2k+n}{k!(k+n)!}\Big(\dfrac{1}{2}\Big)^{2k}}{\sum\limits_{k=0}^{\infty} \dfrac{1}{k!(k+n)!}\Big(\dfrac{1}{2}\Big)^{2k}}, \quad (n = 1, 2, \cdots).$$

Table 1

N	No.	λ	$\lambda(h)$	$\|\lambda - \lambda(h)\|/\lambda$
4	1	0.4463900	0.4463853	1.0E-5
	2	1.2401937	1.1663211	6.0E-2
	3	1.2401937	1.1663211	6.0E-2
	4	2.1633061	1.1798572	4.6E-1

Table 2

N	No.	λ	$\lambda(h)$	$\|\lambda - \lambda(h)\|/\lambda$
8	1	0.4463900	0.4463892	1.8E-6
	2	1.2401937	1.2361035	3.3E-3
	3	1.2401937	1.2361035	3.3E-3
	4	2.1633061	2.0663995	4.5E-2
	5	2.1633061	2.0663995	4.5E-2
	6	3.1234693	2.4059694	3.2E-1
	7	3.1234693	2.4059694	3.2E-1
	8	4.0991784	4.4638917	4.1E-1

Table 3

N	No.	λ	$\lambda(h)$	$\lvert\lambda-\lambda(h)\rvert/\lambda$
16	1	0.4463900	0.4463897	5.6E-7
	2	1.2401937	1.2398677	2.6E-4
	3	1.2401937	1.2398677	2.6E-4
	4	2.1633061	2.1576004	2.6E-3
	5	2.1633061	2.1576004	2.6E-3
	6	3.1234693	3.0831201	1.3E-2
	7	3.1234693	3.0831201	1.3E-2
	8	4.0991784	3.9235421	4.3E-2
	9	4.0991784	3.9235421	4.3E-2

Table 4

N	No.	λ	$\lambda(h)$	$\lvert\lambda-\lambda(h)\rvert/\lambda$
16	10	5.0828424	4.1776521	1.8E-1
	11	5.0828424	4.4066215	1.3E-1
	12	6.0711122	4.4066215	2.7E-1
	13	6.0711122	4.5227156	2.6E-1
	14	7.0622843	4.5227156	3.6E-1
	15	7.0622843	4.6836211	3.4E-1
	16	8.0554020	4.6836211	4.2E-1

Table 5

N	No.	λ	$\lambda(h)$	$\lvert\lambda-\lambda(h)\rvert/\lambda$
32	1	0.4463900	0.4463898	3.6E-7
	2	1.2401937	1.2401639	2.4E-5
	3	1.2401937	1.2401639	2.4E-5
	4	2.1633061	2.1629144	1.8E-4
	5	2.1633061	2.1629144	1.8E-4
	6	3.1234693	3.1210453	7.8E-4
	7	3.1234693	3.1210453	7.8E-4
	8	4.0991784	4.0894507	2.4E-3
	9	4.0991784	4.0894507	2.4E-3
	10	5.0828424	5.0531427	5.8E-3
	11	5.0828424	5.0531427	5.8E-3
	12	6.0711122	5.9956369	1.2E-2
	13	6.0711122	5.9956369	1.2E-2
	14	7.0622843	6.8940502	2.4E-2
	15	7.0622843	6.8940502	2.4E-2
	16	8.0554020	7.7159208	4.2E-2
	17	8.0554020	7.7159208	4.2E-2
	18	9.0498868	8.3082573	8.2E-2
	19	9.0498868	8.4176671	7.0E-2
	20	10.045369	8.4176671	1.6E-1
	21	10.045369	8.4394114	1.6E-1
	22	11.041600	8.4394114	2.4E-1
	23	11.041600	8.7598205	2.1E-1
	24	12.038409	8.7598205	2.7E-1
	25	12.038409	8.9466070	2.6E-1
	26	13.035672	8.9466070	3.1E-1
	27	13.035672	9.1003002	3.0E-1
	28	14.033299	9.1003002	3.5E-1
	29	14.033299	9.2500676	3.4E-1
	30	15.031211	9.2500676	3.8E-1
	31	15.031211	9.2953988	3.8E-1
	32	16.029388	9.2953988	4.2E-1

Table 6

N	No.	λ	$\lambda(h)$	$\lvert\lambda-\lambda(h)\rvert/\lambda$
64	1	0.4463900	0.4463898	3.8E-7
	2	1.2401937	1.2401959	1.8E-6
	3	1.2401937	1.2401959	1.8E-6
	4	2.1633061	2.1632757	1.4E-5
	5	2.1633061	2.1632757	1.4E-5
	6	3.1234693	3.1233061	5.2E-5
	7	3.1234693	3.1233061	5.2E-5
	8	4.0991784	4.0985680	1.5E-4
	9	4.0991784	4.0985680	1.5E-4
	10	5.0828424	5.0810625	3.5E-4
	11	5.0828424	5.0810625	3.5E-4
	12	6.0711122	6.0667543	7.2E-4
	13	6.0711122	6.0667543	7.2E-4
	14	7.0622843	7.0528876	1.3E-3
	15	7.0622843	7.0528876	1.3E-3
	16	8.0554020	8.0369855	2.3E-3
	17	8.0554020	8.0369855	2.3E-3
	18	9.0498868	9.0163677	3.7E-3
	19	9.0498868	9.0163677	3.7E-3
	20	10.045369	9.9878516	5.7E-3
	21	10.045369	9.9878516	5.7E-3
	22	11.041600	10.947525	8.5E-3
	23	11.041600	10.947525	8.5E-3
	24	12.038409	11.890545	1.2E-2
	25	12.038409	11.890545	1.2E-2
	26	13.035672	12.810962	1.7E-2
	27	13.035672	12.810962	1.7E-2
	28	14.033299	13.701548	2.4E-2
	29	14.033299	13.701548	2.4E-2
	30	15.031211	14.553675	3.2E-2
	31	15.031211	14.553675	3.2E-2
	32	16.029388	15.357241	4.2E-2

In this example, Γ is divided into N segments with equal arc length by the nodes $x^i = (x_1(s_i), x_2(s_i))$, $i = 0, 1, \cdots, N$. The matrix eigenvalue problem which is derived from the discrete problem (2.21) is solved approximately by numerical method. The numerical results are arranged in the following table 1 to table 6. As can be seen from the table entries, the convergence of $\lambda(h)$ to λ is quadratic. It shows that the approximate method in this paper is very efficient, especially for the first few eigenvalues. In the following tables, λ_i and $\lambda_i(h)$ denote the exact eigenvalue of (1.1) and its approximation for $i = 0, 1, 2, \cdots$.

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