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SUBSTRUCTURE PRECONDITIONERS FOR NONCONFORMING PLATE ELEMENTS*1)

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Abstract

In this paper, we consider the problem of solving finite element equations of biharmonic Dirichlet problems. We divide the given domain into non-overlapping subdomains, construct a preconditioner for Morley element by substructuring on the basis of a function decomposition for discrete biharmonic functions. The function decomposition is introduced by partitioning these finite element functions into the low and high frequency components through the intergrid transfer operators between coarse mesh and fine mesh, and the conforming interpolation operators. The method leads to a preconditioned system with the condition number bounded by $C(1+\log^2 H/h)$ in the case with interior cross points, and by C in the case without interior cross points, where H is the subdomain size and h is the mesh size. These techniques are applicable to other nonconforming elements and are well suited to a parallel computation.

Key words: Substructure Preconditioner, biharmonic equation nonconforming plate element

1. Introduction

In this paper, we generalize the BPS algorithm [1] to nonconforming element approximations of the biharmonic equation. We construct a preconditioner for Morley element by substructuring on the basis of a function decomposition for discrete biharmonic functions. The function decomposition is introduced by partitioning discrete biharmonic functions into low and high frequency components through intergrid transfer operators between coarse and fine meshes and a conforming interpolation operator. The method leads to a preconditioned system with the condition number bounded by $C(1+\log^2 H/h)$ in the case with interior cross points, and by C in the case without interior cross points, where H is the subdomain size and h is the mesh size. These

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techniques are applicable to other nonconforming elements and are well suited to a parallel computation.

For conforming element discrete problems of a second order elliptic equation, Bramble et al [1] and Widlund [9] have obtained certain preconditioners which are easily inversed in parallel and can reduce the condition number of a discrete system from $O(h^{-2})$ to $O(1+\log^2 H/h)$. The main idea is the decomposition as $v=\Pi_H v+(v-\Pi_H v)$, where Π_H is the interpolation operator on coarse meshes, and an extension theorem. Gu and Hu [5] have obtained a similar result for Wilson nonconforming element which is with continuity at the vertices. Zhang [11] has constructed preconditioners for certain conforming plate elements on the basis of a space decomposition by adding certain vertex spaces. However, for Morley element, since the finite element spaces are not nested, and the functions have bad continuities, the space decomposition similar to those mentioned above does not hold.

We introduce a conforming interpolation operator for Morley element and related intergrid transfer operators, and then construct a function decomposition for discrete biharmonic functions to overcome these difficulties. Brenner [2] has introduced the conforming interpolation operator E_h by taking averages of the nodal parameters associated with the function and its first derivatives among the relevant elements, and taking zero as the nodal parameters associated with its second-order derivatives, in order to deal with an overlapping domain decomposition method. To be suited to a parallel computation in the substructure preconditioning, we modify Brenner's approach so that the nodal parameters of $E_h v_h$ depend only on those of v_h on the boundaries of substructures. On the other hand, Zhang [11] has defined an interpolation operator for certain conforming plate elements by setting the nodal parameters for second-order derivatives be zero. We use it to define the intergrid transfer operator I_H from fine meshes to coarse meshes. Then we generalize the BPS algorithms and Widlund theory of substructure preconditioning to nonconforming plate elements.

2. A Preconditioning Algorithm

Let Ω be a bounded polygonal domain in R^2. Consider the biharmonic problem in Ω with the clamped boundary conditions

$$\Delta^2 u = f \text{ in } \Omega,\ \ u=\partial_n u = 0 \text{ on } \partial\Omega. \tag{2.1}$$

The variational form of (2.1) is: Find $u\in H_0^2(\Omega)$ such that

$$a(u,v)=(f,v),\quad \forall v\in H_0^2(\Omega), \tag{2.2}$$

where

$$a(u,v)=\sum_{|\alpha|=2}\int_\Omega D^\alpha u D^\alpha v dx,\quad (f,v)=\int_\Omega f v dx.$$

Let J_h and J_H be quasi-uniform triangulations of Ω with h and H as mesh parameters respectively. Assume that J_h can be obtained by refining J_H, so that J_H and J_h form a two-level triangulations on Ω. Let $S^h(\Omega)$ be Morley element space [8] and $S_0^h(\Omega)$ be a subspace of $S^h(\Omega)$ with nodal parameters vanishing at boundary nodes. The Morley element discrete problem is: Find $u_h \in S_0^h(\Omega)$ such that

$$a_h(u_h, v_h) = (f, v_h), \quad \forall v \in S_0^h(\Omega), \tag{2.3}$$

where

$$a_h(u,v) = \sum_{T\in J_h}\sum_{|\alpha|=2}\int_T D^\alpha u D^\alpha v dx, \quad (f,v) = \int_\Omega f v dx.$$

Let $J_H = \{\Omega_k\}_{k=1}^N$. The vertices of J_H will be labeled by v_j (ordered in some way) and Γ_{ij} will denote the edge with endpoints v_i and v_j. $S_0^h(\Omega_j)$ will denote the subspace of $S_0^h(\Omega)$ consisting of functions with nodal parameters vanishing on $\bar{\Omega}\backslash\Omega_j$. In addition, $S^h(\Omega_j)$ will be the set of functions which are restrictions of those in $S_0^h(\Omega)$ to $\bar{\Omega}_j$. In what follows, c and C (with or without subscript) will denote generic positive constants which are independent of H, h and Ω_k.

We construct our preconditioner B through its corresponding bilinear form $B(\cdot,\cdot)$ defined on $S_0^h(\Omega)\times S_0^h(\Omega)$.

We decompose functions in $S_0^h(\Omega)$ as follows:

Write $w = w_P + w_H$, where $w_P \in S_0^h(\Omega_1)\oplus\cdots\oplus S_0^h(\Omega_N)$ and satisfies

$$a_h^k(w_P, \phi) = a_h^k(w, \phi), \quad \forall \phi \in S_0^h(\Omega_k), \text{ for each } k, \tag{2.4}$$

where

$$a_h^k(u,v) = \sum_{T\in J_h, T\subset\Omega_k}\sum_{|\alpha|=2}\int_T D^\alpha u D^\alpha v dx.$$

Notice that w_P is determined on Ω_k by the nodal parameters of w on Ω_k and that

$$a_h^k(w_H, \phi) = 0 \text{ for all } \phi \in S_0^h(\Omega_k). \tag{2.5}$$

Thus on each Ω_k, w is decomposed into a function w_P whose nodal parameters vanish on $\partial\Omega_k$ and a function $w_H \in S^h(\Omega_k)$ which satisfies the above homogeneous equations and has the same nodal parameters as w at $\bigcup_k \partial\Omega_k$. We shall refer to such a function w_H as "discrete a_h^k-biharmonic".

We note that the above decomposition is orthogonal with respect to the inner-product $a_h(\cdot,\cdot)$, and hence $a_h(w,w) = a_h(w_P,w_P) + a_h(w_H,w_H)$.

To define the bilinear form $B(\cdot,\cdot)$, we introduce a linear interpolation operator E_h, and an intergrid transfer operator I_H. The conforming relative of Morley element is Argyris quintic element. Let $AR^h(\Omega)$ be Argyris quintic element space associated with J_h, and $B^H(\Omega)$ be Bell element space associated with J_H [3].

For an arbitrary vertex p of J_h, we assign to it one of its adjacent edge midpoints e_p. If $p \in \bigcup\Gamma_{ij}$, we assign to it e_p which belongs to $\bigcup\Gamma_{ij}$. If $p \in \partial\Omega$, we assign to it e_p which belongs to $\partial\Omega$. Let e_p' be a vertex of J_h such that e_p is the midpoint of segment pe_p' (cf. Figure 2.1). For $v \in S_0^h(\Omega)$, we define $E_h v \in AR^h(\Omega)$ such that

$$\begin{aligned} &E_h v(p) = v(p), \quad \forall \text{ verties } p \\ &\partial_n E_h v(m) = \partial_n v(m), \quad \forall \text{ midpoint } m \end{aligned} \tag{2.6}$$

$$D^\alpha E_h v(p) = 0, \quad |\alpha| = 2;$$

and

$$\partial_x E_h v(p) = \partial_n v(e_p)\cos\beta + \frac{v(p) - v(e'_p)}{l_{pe'_p}}\sin\beta,$$
$$\partial_y E_h v(p) = \partial_n v(e_p)\sin\beta + \frac{v(e'_p) - v(p)}{l_{pe'_p}}\cos\beta; \tag{2.7}$$

where $n = (\cos\beta, \sin\beta)$, $s = (-\sin\beta, \cos\beta)$ are the unit normal and tangential vector respectively, $l_{pe'_p}$ is the length of the segment pe'_p (cf. Figure 2.1). We note that (2.6) is defined as Brenner [2] but (2.7) is different.

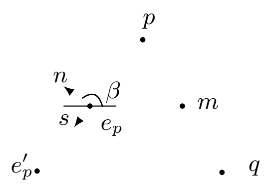

Figure 2.1

About the operator E_h we can prove the following proposition by the argument similar to that in [2].

Proposition 1. *For arbitrary $v \in S_0^h(\Omega)$, $T \in J_h$ we have*

$$\|v - E_h v\|_{L^2(T)} + h|v - E_h v|_{1,T} + h^2|E_h v|_{2,T} \le Ch^2 \sum_{\bar{T}'\cap\bar{T}\neq\phi} |v|_{2,T'}. \tag{2.8}$$

where and from now on $|v|_{i,T}^2 = |v|_{H^i(T)}^2$ and $T' \in J_h$.

From the definition of the conforming interpolation operator E_h we can see that nodal parameters of $E_h v_h$ on $\cup\Gamma_{ij}$ depend only on those of v_h on $\cup\Gamma_{ij}$. This property is important in our discussion.

The nodal interpolation operator $\Pi_h : B^H(\Omega) \longrightarrow S_0^h(\Omega)$ is defined by (cf.[11])

$$\begin{cases} \Pi_h v(p) = v(p), \text{ for arbitrary vertex } p \text{ of } J_h \\ \partial_n \Pi_h v(m) = \partial_n v(m), \text{ for all internal midpoint } m \text{ of} J_h. \end{cases} \tag{2.9}$$

The intergrid transfer operator $I_H : AR^h(\Omega) \longrightarrow B^H(\Omega)$ is defined by

$$\begin{cases} D^\alpha I_H v(p) = D^\alpha v(p), \text{ for arbitrary vertex } p \text{ of } J_h, \ |\alpha| \le 1 \\ D^\alpha I_H v(p) = 0, \text{ for } |\alpha| = 2 \end{cases} \tag{2.10}$$

The operator Π_h have the following property.

Proposition 2.

$$\|v - \Pi_h v\|_{L^2(T)} + h|v - \Pi_h v|_{1,T} + h^2|\Pi_h v|_{2,T} \le Ch^2|v|_{2,T}, \quad \forall\, T \in J_h, \tag{2.11}$$

for arbitrary $v \in B^H(\Omega)$.

Now we construct a preconditioner. Set the node set

$$V = \{p, e_p, e_p'; p \text{ is a vertex of } J_H\}$$

We decompose $w_H \in S^h(\Omega_k)$ into $w_H = w_E + w_V$, where $w_V \in S^h(\Omega_k)$ is a discrete a_h^k-biharmonic function such that the nodal parameters of w_V on $\cup\partial\Omega_k\backslash V$ are those of $\Pi_h I_H E_h w_H$, and the nodal parameters of w_V on V are those of w_H. Thus w_E is a discrete a_h^k-biharmonic function in Ω_k for each k, and the nodal parameters of $I_H E_h w_E$ vanish at all nodes of coarse meshes. In virtue of this decomposition, we now define the bilinear form $B(\cdot,\cdot)$ as follows

$$\begin{aligned} B(w,\phi) =& a_h(w_P,\phi_P) + \sum_{\Gamma_{ij}} \{\langle \partial_s \bar{w}_E, \partial_s \bar{\phi}_E\rangle_{H_{00}^{1/2}(\Gamma_{ij})} + \langle \partial_n \bar{w}_E, \partial_n \bar{\phi}_E\rangle_{H_{00}^{1/2}(\Gamma_{ij})}\} \\ &+ \sum_{\Gamma_{ij}}\{(w_V(v_i) - w_V(v_j) - D\bar{w}_V(v_i)(v_i - v_j)) \\ &\cdot (\phi_V(v_i) - \phi_V(v_j) - D\bar{\phi}_V(v_i)(v_i - v_j))H^{-2} \\ &+ (D\bar{w}_V(v_i) - D\bar{w}_V(v_j))(D\bar{\phi}_V(v_i) - D\bar{\phi}_V(v_j))\} \end{aligned} \tag{2.12}$$

where and from now on $\bar{v} = E_h v$, and $\langle\cdot,\cdot\rangle_{H_{00}^{1/2}(\Gamma_{ij})}$ means $H_{00}^{1/2}(\Gamma_{ij})$-inner product which is defined by

$$\begin{aligned} \langle v, w\rangle_{H_{00}^{1/2}(\Gamma_{ij})} =& \int_{\Gamma_{ij}}\int_{\Gamma_{ij}} \frac{(v(x)-v(y))(w(x)-w(y))}{|x-y|^2} ds(x)ds(y) \\ &+ \int_{\Gamma_{ij}} v(x)w(x)\Big(\frac{1}{|x-v_i|} + \frac{1}{|x-v_j|}\Big)ds(x), v, w \in H_{00}^{1/2}(\Gamma_{ij}). \end{aligned}$$

We shall demonstrate how the linear system $Bw = g$ can be solved efficiently.

Given g, the problem of solving $Bw = g$ reduces to finding the functions w_P and w_H. The function w_P restricted to Ω_k satisfies

$$a_h^k(w_P,\phi) = (g,\phi) \text{ for all } \phi \in S_0^h(\Omega_k). \tag{2.13}$$

Thus it can be obtained by solving in parallel the corresponding biharmonic Dirichlet problem (2.13) on each subdomain. With w_P known, we are left with the equation

$$\begin{aligned} &\sum_{\Gamma_{ij}} \{\langle \partial_s \bar{w}_E, \partial_s \bar{\phi}_E\rangle_{H_{00}^{1/2}(\Gamma_{ij})} + \langle \partial_n \bar{w}_E, \partial_n \bar{\phi}_E\rangle_{H_{00}^{1/2}(\Gamma_{ij})}\} \\ &\quad + \sum_{\Gamma_{ij}}\{(w_V(v_i) - w_V(v_j) - D\bar{w}_V(v_i)(v_i - v_j)) \\ &\quad \cdot (\phi_V(v_i) - \phi_V(v_j) - D\bar{\phi}_V(v_i)(v_i - v_j))H^{-2} \\ &\quad + (D\bar{w}_V(v_i) - D\bar{w}_V(v_j))(D\bar{\phi}_V(v_i) - D\bar{\phi}_V(v_j))\} \\ &= (g,\phi) - a_h(w_P,\phi). \end{aligned} \tag{2.14}$$

(The last equality holds since $a_h(w_P,\phi_H) = 0$). Notice that the value of $(g,\phi) - a_h(w_P,\phi)$ for each ϕ depends only on the nodal parameters of $\bar{\phi}$ on all Γ_{ij}. From the definition of the interpolation operator E_h, we see that the value of $(g,\phi)-a_h(w_P,\phi)$ for each ϕ depends only on the nodal parameters of ϕ on all Γ_{ij}. Thus (2.14) gives rise to a set of equations which can be treated as follows: for each Γ_{ij}, choose ϕ in a subspace of $S_0^h(\Omega)$ such that the nodal parameters of ϕ vanish in the all interior mesh points of every Ω_k and those of $\bar{\phi}$ vanish on all other Γ_{ij}. Thus, on this subspace, (2.14) decouples into independent problems of finding $\bar{w}_E \in AR_0^h(\Gamma_{ij})$, $I_H\bar{w}_E=0$ given by

$$\begin{aligned}\langle\partial_s\bar{w}_E,\partial_s\bar{\phi}\rangle_{H_{00}^{1/2}(\Gamma_{ij})}&+\langle\partial_n\bar{w}_E,\partial_n\bar{\phi}\rangle_{H_{00}^{1/2}(\Gamma_{ij})}\\ &=(g,\phi)-a_h(w_P,\phi),\ \forall\phi\in S_0^h(\Omega),\ I_H\bar{\phi}=0,\ \bar{\phi}\in AR_0^h(\Gamma_{ij})\end{aligned}\tag{2.15}$$

for each Γ_{ij}. Note that these are local problems with unknowns corresponding to the nodes on Γ_{ij} and may be solved in parallel.

Next we solve for $\bar{w}_V$ on the edges. We consider the subspace $\{\phi\in S_0^h(\Omega)$; nodal parameters of $\bar{\phi}$ on $\cup\partial\Omega_i\backslash V=$ those of $\Pi_h I_H\bar{\phi}$ on $\cup\partial\Omega_i\backslash V$, those of ϕ on all Ω_i vanish, then (2.14) reduces to

$$\begin{aligned}\sum_{\Gamma_{ij}}\{&(w_V(v_i)-w_V(v_j)-D\bar{w}_V(v_i)(v_i-v_j))\cdot(\phi_V(v_i)-\phi_V(v_j)-D\bar{\phi}_V(v_i)(v_i-v_j))H^{-2}\\ &+(D\bar{w}_V(v_i)-D\bar{w}_V(v_j))(D\bar{\phi}_V(v_i)-D\bar{\phi}_V(v_j))\}\\ =&(g,\phi)-a_h(w_P,\phi).\end{aligned}\tag{2.16}$$

The nodal parameters of $\bar{w}_V$ at nodes of $T\in J_H$ determine those of w_V on all edges Γ_{ij}, and hence $w_H=w_E+w_V$ is known on all edges Γ_{ij}.

The last step consists of determining w_H in each Ω_k so that

$$a_h^k(w_H,\phi)=0 \text{ for } \phi\in S_0^h(\Omega_k).\tag{2.17}$$

This problem is similar to (2.13), which can also be solved in parallel on each subdomain. Hence the solution of $Bw=g$ is determined by $w=w_P+w_H$.

We summarize the process by outlining the steps for obtaining the solution of

$$B(w,\phi)=(g,\phi) \text{ for all } \phi\in S_0^h(\Omega),$$

and hence for computing the action of B^{-1}.

Algorithm.

1. Find w_P by solving biharmonic Dirichlet problems on subdomains. The solution of each individual Dirichlet problem on subdomains may be done in parallel.

2. Find $\bar{w}_E$ on Γ_{ij} by solving one-dimensional equation on each Γ_{ij}, which may be done in parallel.

3. Find $\bar{w}_V$ on $\bigcup\Gamma_{ij}$ by solving a coarse mesh equation and then extending it to all edges Γ_{ij} by operator Π_h.

4. Find w_H by extending the nodal values of w_E+w_V on $\cup\Gamma_{ij}$ to all subdomains. As step 1, the solution may be done in parallel.

3. Estimates of the Condition Number

We have the following theorem.

Theorem 1. *There are positive constants* λ_0, λ_1 *and* C *such that*

$$\lambda_0 B(w,w) \le a_h(w,w) \le \lambda_1 B(w,w), \quad \forall w \in S_0^h(\Omega), \tag{3.1}$$

where $\lambda_1/\lambda_0 \le C(1+\log^2 H/h)$. *If all of the nodes of* Ω_k *lie on* $\partial\Omega$, *then* $\lambda_1/\lambda_0 \le C$.

It means that the condition number grows at most like $(1+\log^2 H/h)$ as h tends to zero so that the preconditioned iteration converges rapidly.

The theorem will be proved in the last of the section.

Set $\mathbf{Q} = \{v \in H^2(\Omega_i)$; there exist an $v_h \in AR_h(\Omega_i)$, $v|_{\partial\Omega_i} = v_h|_{\partial\Omega_i}$, $\partial_n v|_{\partial\Omega_i} = \partial_n v_h|_{\partial\Omega_i}\}$. Let $\Pi_h' : \mathbf{Q} \longrightarrow S^h(\Omega)$ be defined by (cf. Figure 2.1)

$$\begin{cases} \Pi_h' v(p) = v(p), \quad \forall p \text{ vertex of } T, T \in J_h,\ T \subset \Omega_i; \\ \partial_n \Pi_h' v(m) = \partial_n v(m), \text{ if } m \in \partial\Omega_i; \\ \partial_n \Pi_h' v(m) = \dfrac{1}{|e|}\displaystyle\int_e v(s)ds, \text{ if } m \in \Omega_i; \end{cases} \tag{3.2}$$

here m is the midpoint of an edge $e, e \subset \Omega_i$.

Proposition 3. For the operator Π_h' we have the following stability estimate:

$$|\Pi_h' v|_{2,T} \le C|v|_{2,T}, \quad \forall v \in \mathbf{Q}, \quad \forall T \in J_h. \tag{3.3}$$

For completeness we shall include a proof of Proposition 3 at the end of this section.

The derivation of the estimates in this section requires the use of various norms defined on the subdomain boundaries. Let Ω_i be a subdomain of J_h(as defined in section 2) and β_i be the set of indices jk with $\Gamma_{jk} \subset \partial\Omega_i$, hence $\partial\Omega_i = \cup_{jk\in\beta_i}\Gamma_{jk}$ or $jk \in \beta_i$. The Sobolev space of order one half on $\partial\Omega_i$ will be denoted $H^{1/2}(\partial\Omega_i)$ and is defined in [4,7]. Define the weight norm on $H^{1/2}(\partial\Omega_i)$ as [1] by

$$\|w\|_{1/2,\partial\Omega_i} = \Big(\int_{\partial\Omega_i}\int_{\partial\Omega_i} \frac{(w(x)-w(y))^2}{|x-y|^2} ds(x)ds(y) + H^{-1}|w|^2_{L^2(\partial\Omega_i)}\Big)^{1/2}, \tag{3.4}$$

where s is arc length along $\partial\Omega_i$. For a smooth function v on $\partial\Omega_i$ with support contained in one of the edges $\Gamma_{jk} \subset \partial\Omega_i$, define $\|v\|_{H_{00}^{1/2}(\Gamma_{jk})}$ by the square root of

$$\int_{\Gamma_{jk}}\int_{\Gamma_{jk}} \frac{(w(x)-w(y))^2}{|x-y|^2} ds(x)ds(y) + \int_{\Gamma_{jk}} \Big(\frac{(w(x))^2}{|x-v_j|} + \frac{(w(x))^2}{|x-v_k|}\Big) ds(x).$$

Similarly we define the weight norm of $H^{3/2}(\partial\Omega_i)$ as follows

$$\begin{aligned} \|u\|^2_{H^{3/2}(\partial\Omega_i)} =& H^2 \int_{\partial\Omega_i}\int_{\partial\Omega_i} \frac{(\partial_s w(x)-\partial_s w(y))^2}{|x-y|^2} ds(x)ds(y) \\ &+ H\int_{\partial\Omega_i} (\partial_s w(x))^2 ds(x) + \frac{1}{H}\int_{\partial\Omega_i} (w(x))^2 ds(x). \end{aligned}$$

and the weight norm of $H_{00}^{3/2}(\Gamma_{jk})$ as

$$\|u\|^2_{H_{00}^{3/2}(\Gamma_{jk})} = H^2 \int_{\Gamma_{jk}} \int_{\Gamma_{jk}} \frac{|\partial_s u(x) - \partial_s u(y)|^2}{|x-y|^2} ds(x) ds(y) + H^2 \int_{\Gamma_{jk}} |\partial_s u|^2 \Big(\frac{1}{|x-v_j|} + \frac{1}{|x-v_k|} \Big) ds(x).$$

We shall need several lemmas which will be used in the proof of the main theorem.

Lemma 1. *Let $w \in S^h(\Omega)$ be discrete a_h^i−biharmonic i.e.*

$$a_h^i(w,v) = 0, \ \forall v \in S_0^h(\Omega_i) \tag{3.5}$$

and $I_H \bar{w} = 0$, then

$$a_h^i(w,w) \le C \sum_{jk \in \beta_i} \{ \langle \partial_s \bar{w}, \partial_s \bar{w} \rangle_{H_{00}^{1/2}(\Gamma_{jk})} + \langle \partial_n \bar{w}, \ \partial_n \bar{w} \rangle_{H_{00}^{1/2}(\Gamma_{jk})} \}, \tag{3.6}$$

where $jk \in \beta_i$ means that Γ_{jk} is an edge of Ω_i, and $\bar{w} = E_h w$.

Proof. It is not difficult to see (by scaling Ω_i to unite size) that it suffices to prove (3.6) under the assumption that Ω_i is a standard element. Let Γ_{jk} be any edge of Ω_i and let $w_{jk} \in S_0^h(\Omega)$ be the discrete a_h^i−biharmonic function for Morley element such that

$$a_h^i(w_{jk}, v) = 0, \ \forall v \in S_0^h(\Omega_i), \ w_{jk} \in S_0^h(\Omega), \tag{3.7}$$

whose nodal parameters on Γ_{jk} are equal to those of $\Pi_h \bar{w}$ on Γ_{jk} and vanish on all the other edges of $\partial\Omega_i, i = 1, \cdots, N$. Clearly, since $I_H \bar{w} = 0$, $w|_{\Omega_i} = \sum_{jk \in \beta_i} w_{jk}|_{\Omega_i}$. It follows from the triangle inequality that

$$a_h^i(w,w) \le C \sum_{jk \in \beta_i} a_h^i(w_{jk}, w_{jk}). \tag{3.8}$$

For each w_{jk}, let $w^{jk}|_{\Omega_i} \in H^2(\Omega_i)$, $i = 1, \cdots, N$, be the biharmonic function such that

$$\begin{cases} a(w^{jk}, v) = 0, \ \forall v \in H_0^2(\Omega_i), \\ w^{jk} = \bar{w}, \ \partial_n w^{jk} = \partial_n \bar{w} \text{ on } \Gamma_{jk}, \\ w^{jk} = \partial_n w^{jk} = 0 \ \text{ on } \partial\Omega_i \backslash \Gamma_{jk}, \end{cases} \tag{3.9}$$

and $w^{jk} = \partial_n w^{jk} = 0$ on all the other edges of $\partial\Omega_i$, $i = 1, \cdots, N$. Since $w_{jk} \in S^h(\Omega)$ is a discrete a_h^i−biharmonic function and nodal parameters of $\Pi_h' w^{jk}$ are equal to those of w_{jk} on $\partial\Omega_i$, we obtain that

$$\sum_{jk \in \beta_i} a_h^i(w_{jk}, w_{jk}) \le C \sum_{jk \in \beta_i} a_h^i(\Pi_h' w^{jk}, \Pi_h' w^{jk}). \tag{3.10}$$

It follows from (3.8)–(3.10) and the stability estimate (3.3) that

$$a_h^i(w,w) \le \sum_{jk \in \beta_i} a_h^i(\Pi_h' w^{jk}, \Pi_h' w^{jk}) \le C \sum_{jk \in \beta_i} a_h^i(w^{jk}, w^{jk}). \tag{3.11}$$

Note that $D^\alpha w^{jk}(|\alpha| \le 1)$ vanish on $\partial\Omega_i \backslash \Gamma_{jk}$. Now using a well-known priori inequality [4,7], an inverse property and Poincaré inequality, we see that

$$\begin{aligned}\sum_{jk\in\beta_i} a_h^i(w^{jk}, w^{jk}) &\le C \sum_{jk\in\beta_i} (\|w^{jk}\|^2_{H^{3/2}(\partial\Omega_i)} + \|\partial_n w^{jk}\|^2_{H^{1/2}(\partial\Omega_i)}) \\ &\le C \sum_{jk\in\beta_i} (\|w^{jk}\|^2_{H_{00}^{3/2}(\Gamma_{jk})} + \|\partial_n w^{jk}\|^2_{H_{00}^{1/2}(\Gamma_{jk})}) \\ &\le C \sum_{jk\in\beta_i} \{\langle \partial_s \bar{w}, \partial_s \bar{w}\rangle_{H_{00}^{1/2}(\Gamma_{jk})} + \langle \partial_n \bar{w}, \partial_n \bar{w}\rangle_{H_{00}^{1/2}(\Gamma_{jk})}\}. \end{aligned} \tag{3.12}$$

(3.6) follows from (3.11)-(3.12). The proof is complete. #

Lemma 2.

$$\begin{aligned} a_h^i(w_V, w_V) \le & C \sum_{jk\in\beta_i} \{(w_V(v_j) - w_V(v_k) - D\bar{w}_V(v_j)(v_j - v_k))^2 H^{-2} \\ & + (D\bar{w}_V(v_j) - D\bar{w}_V(v_k)\}^2. \end{aligned} \tag{3.13}$$

Proof. Let $\tilde{w} \in S_0^h(\Omega)$ be the discrete biharmonic function such that

$$a_h^i(\tilde{w}, v) = 0,\ \forall v \in S_0^h(\Omega) \tag{3.14}$$

and the nodal parameters of $\tilde{w}|_{\partial\Omega_i}$ = those of $w_V - \Pi_h I_H E_h w_V$. Since $w_V - \tilde{w} \in S^h(\Omega)$ is also a discrete biharmonic function and the nodal parameters of $w_V - \tilde{w}$ on $\partial\Omega_i$ are equal to those of $\Pi_h I_H E_h w_V$ on $\partial\Omega_i$, we have

$$\begin{aligned} a_h^i(w_V, w_V) &\le 2a_h^i(\tilde{w}, \tilde{w}) + 2a_h^i(w_V - \tilde{w}, w_V - \tilde{w}) \\ &\le C(a_h^i(\tilde{w}, \tilde{w}) + a_h^i(\Pi_h I_H E_h w_V, \Pi_h I_H E_h w_V)). \end{aligned} \tag{3.15}$$

Set $g \in S^h(\Omega)$ such that the parameters of g on $\cup\Omega_i$ vanish, and those of g on $\cup\partial\Omega_i$ are equal to those of $w_V - \Pi_h I_H E_h w_V$ on $\cup\partial\Omega_i$. Then we have

$$a_h^i(\tilde{w}, \tilde{w}) \le a_h^i(g, g) \le C \sum_{\substack{\tau\in S \\ \tau\subset\Omega_i}} |g|^2_{2,\tau}, \tag{3.16}$$

where

$$S = \{T \in J_h;\ \text{there exists a vertex } p \text{ of} J_H \text{ such that } e_p \in \partial T\}. \tag{3.17}$$

To analyze the right-hand of (3.16), we first consider the case that for $\tau \in S$, $\partial\tau$ contains only one vertex p in (3.17). In this case, we have

$$\begin{aligned} |g|^2_{2,\tau} &\le C\Big(\frac{\partial g}{\partial n}(e_p) - \frac{\partial g^I}{\partial n}(e_p)\Big)^2 \\ &\le C[(\partial_n I_H E_h w_V(p) - \partial_n I_H E_h w_V(e_p))^2 + |\partial_n g^I(e_p)|^2] \\ &\le C\Big(|I_H \bar{w}_V|^2_{2,\tau} + \Big|\frac{\partial g^I}{\partial n}(e_p)\Big|^2\Big). \end{aligned} \tag{3.18}$$

With the definition of the operator E_h in (2.6) and (2.7), and that of the operator I_H in (2.10), we have

$$g(e'_p) = w_V(e'_p) - I_H\bar{w}_V(e'_p) = (\partial_{pe'_p} I_H\bar{w}_V(p)\cdot pe'_p + I_H\bar{w}_V(p)) - I_H\bar{w}_V(e'_p),$$
$$g(p) = g(w) = 0 \text{ (cf. Figure 2.1);}$$

then we can easily show that

$$|g(e'_p) - g(p)| \le Ch|I_H\bar{w}_V|_{2,\tau},$$

and hence

$$\Big|\frac{\partial g^I}{\partial n}(e_p)\Big| \le Ch|I_H\bar{w}_V|_{2,\tau}.$$

Therefore,

$$\begin{aligned}|g|^2_{2,\tau} &\le C\Big|\frac{\partial g}{\partial n}(e_p) - \frac{\partial g^I}{\partial n}(e_p)\Big|^2 \le C|I_H\bar{w}_V|^2_{2,\tau} + C\Big|\frac{\partial g^I}{\partial n}(e_p)\Big|^2\\ &\le C|I_H\bar{w}_V|^2_{2,\tau}.\end{aligned} \tag{3.19}$$

For other cases of $\tau \in S$, we can prove similarly that (3.19) holds.

Since the number of τ in (3.16) is not larger than 3, we obtain that

$$a^i_h(\tilde{w},\tilde{w}) \le Ca^i_h(I_H\bar{w}_V, I_H\bar{w}_V). \tag{3.20}$$

(3.13) follows from (3.15),(3.16), (3.19) and (2.11). This completes the proof of the lemma 2. #

Lemma 3. *Let $w \in S^h_0(\Omega)$. Then*

(i) if there exists $p_\alpha \in \bar{\Omega}_i$ such that $D^\alpha\bar{w}(p_\alpha) = 0$, $\forall|\alpha| \le 1$, then we have

$$\max_{\substack{e\in J_h\\ e\subset\Omega_i}} \|\nabla w\|^2_{L^\infty(e)} \le C \sum_{\bar{\Omega}_j\cap\bar{\Omega}_i\neq\phi} (1+\log(H/h))|w|^2_{2,h,\Omega_j}; \tag{3.21}$$

(ii)

$$\begin{aligned}&\sum_{jk\in\beta_i}\{(w(v_j)-w(v_k)-D\bar{w}(v_j)(v_j-v_k))^2H^{-2} + (D\bar{w}(v_j)-D\bar{w}(v_k))^2\}\\ &\qquad\le C(1+\log H/h)\sum_{\bar{\Omega}_j\cap\bar{\Omega}_i\neq 9\phi} a^j_h(w,w).\end{aligned} \tag{3.22}$$

The proof can be given by the arguments similar to those of lemma 3.5 in [1].

Lemma 4. *Let $w \in S^h(\Omega)$ satisfy $I_H\bar{w} = 0$ and let $w_L \in S^h(\Omega)$ be a discrete a^i_h-biharmonic function mentioned in Lemma 1 and the nodal parameters of $\bar{w}_L$ on $\cup\partial\Omega_i\backslash V$ are equal to those of $\Pi_h I_H\bar{w}_L$ on $\cup\partial\Omega_i\backslash V$. Then we have*

$$\begin{aligned}&\sum_{jk\in\beta_i}\{\langle\partial_s\bar{w},\partial_s\bar{w}\rangle_{H^{1/2}_{00}(\Gamma_{ij})} + \langle\partial_n\bar{w},\partial_n\bar{w}\rangle_{H^{1/2}_{00}(\Gamma_{ij})}\}\\ &\qquad\le C(1+\log^2 H/h)\sum_{\bar{\Omega}_j\cap\bar{\Omega}_i\neq\phi} a^j_h(w+w_L, w+w_L).\end{aligned} \tag{3.23}$$

Proof. We shall first prove (3.23) in the case that $w_L = 0$. Let Γ_{jk} be any edge of Ω_i. We have

$$\begin{aligned}\langle \partial_s \bar{w}, \partial_s \bar{w}\rangle_{H_{00}^{1/2}(\Gamma_{ij})} + \langle \partial_n \bar{w}, \partial_n \bar{w}\rangle_{H_{00}^{1/2}(\Gamma_{ij})} \leq\, & C(|\partial_x \bar{w}|^2_{1/2,\partial\Omega_i} + |\partial_y \bar{w}|^2_{1/2,\partial\Omega_i}) \\ & + \Big\{ \int_{\Gamma_{jk}} \Big(\frac{|\partial_x \bar{w}|^2}{|x - v_k|} + \frac{|\partial_x \bar{w}|^2}{|x - v_j|} + \frac{|\partial_y \bar{w}|^2}{|x - v_k|} + \frac{|\partial_y \bar{w}|^2}{|x - v_j|} \Big) ds \Big\} \\ \equiv & I + I(\bar{w}). \end{aligned} \tag{3.24}$$

By the argument similar to that of lemma 3.3 in [1] and (2.8) we can prove that

$$I \leq C(1 + \log H/h) a_h^i(\bar{w}, \bar{w}) \leq C \sum_{\bar{\Omega}_j \cap \bar{\Omega}_i \neq \phi} (1 + \log H/h) a_h^j(w, w). \tag{3.25}$$

Since $I_H \bar{w} = 0$, by the argument similar to (3.25) in [1], lemma 3 and (2.8) we have

$$\begin{aligned} I(\bar{w}) \leq & \int_{\Gamma_{jk}} \Big(\frac{|\partial_x \bar{w}|^2}{|x - v_k|} + \frac{|\partial_x \bar{w}|^2}{|x - v_j|} + \frac{|\partial_y \bar{w}|^2}{|x - v_k|} + \frac{|\partial_y \bar{w}|^2}{|x - v_j|} \Big) ds \\ \leq & C(1 + \log H/h)(|\partial_x \bar{w}|^2_{L^\infty(\Gamma_{jk})} + |\partial_y \bar{w}|^2_{L^\infty(\Gamma_{jk})}) \leq C(1 + \log^2 H/h) a_h^i(\bar{w}, \bar{w}) \\ \leq & C(1 + \log^2 H/h) \sum_{\bar{\Omega}_j \cap \bar{\Omega}_i \neq \phi} a_h^j(w, w). \end{aligned} \tag{3.26}$$

Hence (3.23) in the case that $w_L = 0$ follows from (3.24)–(3.26).

To prove (3.23) in the general case, let $w_\perp \in S^h(\Omega_i)$ be the function in $S^h(\Omega_i)$ which satisfies $I_H(\bar{w}_L - \bar{w}_\perp) = 0$ and $a_h^i(w_\perp, \phi) = 0$ for all $\phi \in S_0^h(\Omega)$ with $I_H \bar{\phi} = 0$. Note that $I_H(\bar{w} + \bar{w}_L - \bar{w}_\perp) = 0$, apply the special case of (3.23) proved above we can obtain that

$$\begin{aligned} & \langle \partial_s \bar{w}, \partial_s \bar{w}\rangle_{H_{00}^{1/2}(\Gamma_{ij})} + \langle \partial_n \bar{w}, \partial_n \bar{w}\rangle_{H_{00}^{1/2}(\Gamma_{ij})} \\ & \leq 2\langle \partial_s(\bar{w} + \bar{w}_L - \bar{w}_\perp), \partial_s(\bar{w} + \bar{w}_L - \bar{w}_\perp)\rangle_{H_{00}^{1/2}(\Gamma_{jk})} \\ & \quad + 2\langle \partial_n(\bar{w} + \bar{w}_L - \bar{w}_\perp), \partial_n(\bar{w} + \bar{w}_L - \bar{w}_\perp)\rangle_{H_{00}^{1/2}(\Gamma_{jk})} \\ & \quad + 2\langle \partial_s(\bar{w}_L - \bar{w}_\perp), \partial_s(\bar{w}_L - \bar{w}_\perp)\rangle_{H_{00}^{1/2}(\Gamma_{jk})} \\ & \quad + 2\langle \partial_n(\bar{w}_L - \bar{w}_\perp), \partial_n(\bar{w}_L - \bar{w}_\perp)\rangle_{H_{00}^{1/2}(\Gamma_{jk})} \\ & \leq C(1 + \log^2 H/h) \sum_{\bar{\Omega}_j \cap \bar{\Omega}_i \neq \phi} a_h^j(w + w_L - w_\perp, w + w_L - w_\perp) + \Lambda \\ & \leq C(1 + \log^2 H/h) \sum_{\bar{\Omega}_j \cap \bar{\Omega}_i \neq \phi} a_h^j(w + w_L, w + w_L) + \Lambda, \end{aligned} \tag{3.27}$$

where

$$\Lambda \equiv 2\langle \partial_s(\bar{w}_L - \bar{w}_\perp), \partial_s(\bar{w}_L - \bar{w}_\perp)\rangle_{H_{00}^{1/2}(\Gamma_{jk})} + 2\langle \partial_n(\bar{w}_L - \bar{w}_\perp), \partial_n(\bar{w}_L - \bar{w}_\perp)\rangle_{H_{00}^{1/2}(\Gamma_{jk})}.$$

It remains to prove that

$$\Lambda \leq C(1 + \log^2 H/h) \sum_{\bar{\Omega}_j \cap \bar{\Omega}_i \neq \phi} a_h^j(w + w_L, w + w_L). \tag{3.28}$$

Since $I_H(\bar{w}_L - \bar{w}_\perp) = 0$, applying the inequality (3.24) and the subsequent arguments gives

$$\Lambda \le C(1+\log H/h) \sum_{\bar{\Omega}_j \cap \bar{\Omega}_i \neq \phi} a_h^j(w_L - w_\perp, w_L - w_\perp) + I(\bar{w}_L - \bar{w}_\perp),$$

where $I(\cdot)$ is defined in (3.24). Since $w_\perp$ is orthogonal to $w_L - w_\perp$ with respect to the $a_h^i(\cdot,\cdot)$-inner products, we have

$$a_h^j(w_L - w_\perp, w_L - w_\perp) \le a_h^i(w_L, w_L),$$

and hence we obtain that

$$\Lambda \le C(1+\log H/h) \sum_{\bar{\Omega}_j \cap \bar{\Omega}_i \neq \phi} a_h^j(w_L, w_L) + I(\bar{w}_L - \bar{w}_\perp). \tag{3.29}$$

Since $w_L \in S^h(\Omega)$ is a discrete a^i-biharmonic function and $I_H \bar{w} = 0$, by the arguments similar to those in Lemma 2, and Lemma 3 we have

$$\begin{aligned}
\sum_{\bar{\Omega}_j \cap \bar{\Omega}_i \neq \phi} a_h^j(w_L, w_L) \le & C \sum_{\bar{\Omega}_j \cap \bar{\Omega}_i \neq \phi} a_h^j(\Pi_h I_H \bar{w}_L, \Pi_h I_H \bar{w}_L) \le C \sum_{\bar{\Omega}_l \cap \bar{\Omega}_i \neq \phi} a_h^l(I_H \bar{w}_L, I_H \bar{w}_L) \\
\le & C \sum_{\bar{\Omega}_l \cap \bar{\Omega}_i \neq \phi} \sum_{jk \in \beta_l} \{(w_L(v_j) - w_L(v_k) - D\bar{w}_L(v_j)(v_j - v_k))^2 H^{-2} \\
& + (D\bar{w}_L(v_j) - D\bar{w}_L(v_k))^2\} \\
\le & C(1+\log H/h) \sum_{\bar{\Omega}_j \cap \bar{\Omega}_i \neq \phi} a_j(w + w_L, w + w_L).
\end{aligned}$$

Combining this inequality with (3.29) yields

$$\Lambda \le C(1+\log H/h) \sum_{\bar{\Omega}_j \cap \bar{\Omega}_i \neq \phi} a_h^j(w + w_L, w + w_L) + I(\bar{w}_L - \bar{w}_\perp). \tag{3.30}$$

Therefore, in order to complete the proof of (3.28), it suffices to show that

$$I(\bar{w}_L - \bar{w}_\perp) \le C(1+\log^2 H/h) \sum_{\bar{\Omega}_j \cap \bar{\Omega}_i \neq \phi} a_h^j(w + w_L, w + w_L). \tag{3.31}$$

We write $I(\bar{w}_L - \bar{w}_\perp)$ as follows

$$I(\bar{w}_L - \bar{w}_\perp) = I_{11}(\bar{w}_L - \bar{w}_\perp) + I_{12}(\bar{w}_L - \bar{w}_\perp) + I_{21}(\bar{w}_L - \bar{w}_\perp) + I_{22}(\bar{w}_L - \bar{w}_\perp), \tag{3.32}$$

where

$$I_{11}(\bar{w}_L - \bar{w}_\perp) = \int_{\Gamma_{jk}} \frac{|\partial_x(\bar{w}_L - \bar{w}_\perp)|^2}{|x - v_j|} ds, \tag{3.33}$$

and the rest terms of the right-hand side of (3.32) are similar to that of (3.33).

By the argument similar to that of (3.26) and the fact that

$$a_h^i(w_\perp, w_\perp) \le a_h^i(w + w_L, w + w_L)$$

we have

$$\begin{aligned} I_{11}(\bar{w}_L - \bar{w}_\perp) \le & C\int_{\Gamma_{jk}} \frac{|\partial_x \bar{w}_\perp - \partial_x \bar{w}_\perp(v_j)|^2}{|x-v_j|} ds + C\int_{\Gamma_{jk}} \frac{|\partial_x \bar{w}_L - \partial_x \bar{w}_L(v_j)|^2}{|x-v_j|} ds \\ \le & C\int_{\Gamma_{jk}} \frac{|\partial_x \bar{w}_L - \partial_x \bar{w}_L(v_j)|^2}{|x-v_j|} ds + C(1+\log^2 H/h) \sum_{\bar{\Omega}_j \cap \bar{\Omega}_i \ne \phi} a_h^j(w_\perp, w_\perp) \\ \le & C\int_{\Gamma_{jk}} \frac{|\partial_x \bar{w}_L - \partial_x \bar{w}_L(v_j))|^2}{|x-v_j|} ds \\ & + C(1+\log^2 H/h) \sum_{\bar{\Omega}_j \cap \bar{\Omega}_i \ne \phi} a_h^j(w+w_L, w+w_L). \end{aligned} \tag{3.34}$$

By the triangle inequality we obtain that

$$\begin{aligned} \int_{\Gamma_{jk}} \frac{|\partial_x \bar{w}_L - \partial_x \bar{w}_L(v_j)|^2}{|x-v_j|} ds \le & 2\int_{\Gamma_{jk}} \frac{|\partial_x \bar{w}_L - \partial_x I_H \bar{w}_L|^2}{|x-v_j|} ds \\ & + 2\int_{\Gamma_{jk}} \frac{|\partial_x I_H \bar{w}_L - \partial_x I_H \bar{w}_L(v_k)|^2}{|x-v_j|} ds = I_1 + I_2. \end{aligned} \tag{3.35}$$

Without loss of generality, we assume that v_j is the origin and that Γ_{jk} is the line segment with $x_1 = 0$ and $x_2 \in [0, Y]$. Let the vertices of Γ_{jk} be $y_0, \cdots, y_{l+1}$ such that $0 = y_0 < y_1 < \cdots < y_{l+1} = Y$. Then

$$\begin{aligned} I_1 = & \sum_{m=0}^{l} \int_{y_m}^{y_{m+1}} \frac{|\partial_x \bar{w}_L - \partial_x I_H \bar{w}_L|^2}{y} ds \le C(1+\log H/h) a_h^i(\bar{w}_L - I_H \bar{w}_L, \bar{w}_L - I_H \bar{w}_L) \\ & + \sum_{m=1}^{l-1} \int_{y_m}^{y_{m+1}} \frac{|\partial_x \bar{w}_L - \partial_x I_H \bar{w}_L|^2}{y} ds \equiv J_1 + J_2. \end{aligned} \tag{3.36}$$

By the argument similar to (3.30), the fact that $I_H \bar{w} = 0$, and (3.22) in Lemma 3, we have

$$\begin{aligned} J_1 \le & C(1+\log H/h) \sum_{\bar{\Omega}_j \cap \bar{\Omega}_i \ne \phi} a_h^j(w_L, w_L) \\ & + C(1+\log H/h) a_h^i(I_H(\bar{w}_L + \bar{w}), I_H(\bar{w}_L + \bar{w})) \\ \le & C(1+\log H/h) \sum_{\bar{\Omega}_j \cap \bar{\Omega}_i \ne \phi} a_h^j(w_L, w_L). \end{aligned} \tag{3.37}$$

From the definition of w_L, we have

$$J_2 = \sum_{m=1}^{l-1} \int_{y_m}^{y_{m+1}} \frac{|\partial_x E_h \Pi_h I_H \bar{w}_L - \partial_x I_H \bar{w}_L|^2}{y} ds, \tag{3.38}$$

where Π_h is the restriction of the nodal interpolation operator for Morley element on Ω_i. We can easily show that

$$J_2 \le C a_h^i(I_H \bar{w}_L, I_H \bar{w}_L) \le C(1+\log H/h) \sum_{\bar{\Omega}_j \cap \bar{\Omega}_i \ne \phi} a_h^j(w+w_L, w+w_L) \tag{3.39}$$

Therefore,

$$I_1 \le J_1 + J_2 \le C(1 + \log H/h) \sum_{\bar\Omega_j \cap \bar\Omega_i \ne \phi} a_h^j(w + w_L, w + w_L). \tag{3.40}$$

With the mean theorem and the inverse estimate we can prove that

$$I_2 \le C|I_H \bar w_L|^2_{2,\Omega_i} \le C(1 + \log H/h) \sum_{\bar\Omega_j \cap \bar\Omega_i \ne \phi} a_h^j(w + w_L, w + w_L). \tag{3.41}$$

From (3.34),(3.35), (3.40) and (3.41) we have

$$I_{11}(\bar w_L - \bar w_\perp) \le C(1 + \log^2 H/h) \sum_{\bar\Omega_j \cap \bar\Omega_i \ne \phi} a_h^j(w + w_L, w + w_L). \tag{3.42}$$

We have the similar estimates for the rest terms of $I(\bar w_L - \bar w_\perp)$ on the right side of (3.32). Hence (3.31) holds. This completes the proof. #

Proof of Theorem 1. As in section 2,we decompose $w \in S_0^h(\Omega)$ into $w = w_P + w_H + w_V$. With $w_H = w_E + w_V$, we have (as noted in section 2)

$$a_h(w, w) = a_h(w_P, w_P) + a_h(w_H, w_H)$$

and

$$B(w, w) = a_h(w_P, w_P) + B(w_H, w_H).$$

Hence, it suffices to compare $a_h(w_H, w_H)$ and $B(w_H, w_H)$. More specifically, the proof will be completed when we have shown that

$$a_h(w_H, w_H) \le CB(w_H, w_H), \tag{3.43}$$

and

$$B(w_H, w_H) \le C(1 + \log^2 H/h)a_h(w_H, w_H). \tag{3.44}$$

Consider a subdomain Ω_i. Using (3.6) and (3.13) yields that

$$\begin{aligned} a_h^i(w_H, w_H) \le & 2\{a_h^i(w_E, w_E) + a_h^i(w_V, w_V)\} \\ \le & C \sum_{jk \in \beta_i} \{\langle \partial_s \bar w_E, \partial_s \bar w_E \rangle_{H^{1/2}_{00}(\Gamma_{ij}} + \langle \partial_n \bar w_E, \partial_n \bar w_E \rangle_{H^{1/2}_{00}(\Gamma_{ij})}\} \\ & + C \sum_{jk \in \beta_i} \{(w_V(v_j) - w_V(v_k) - D\bar w_V(v_j)(v_j - v_k))^2 H^{-2} \\ & + (D\bar w_V(v_j) - D\bar w_V(v_k))^2\} + C \sum_m (\partial_n(w_V - (w_V)_I)(m))^2. \end{aligned} \tag{3.45}$$

Summing with respect to i gives (3.43). In view of (3.22) applied to w_V(replacing w by w_V in (3.22)) and (3.23) applied to w_E (replacing w and w_L in (3.23) by w_E and w_V,respectively) and using the fact that $I_H \bar w_E = 0$ we have on each Ω_i,

$$\sum_{jk \in \beta_i} \{\langle \partial_s \bar w_E, \partial_s \bar w_E \rangle_{H^{1/2}_{00}(\Gamma_{ij})} + \langle \partial_n \bar w_E, \partial_n \bar w_E \rangle_{H^{1/2}_{00}(\Gamma_{ij})}\}$$

$$+\sum_{jk\in\beta_i}\{(w_V(v_i)-w_V(v_j)-D\bar{w}_V(v_i)(v_i-v_j))^2H^{-2}$$
$$+(D\bar{w}_V(v_i)-D\bar{w}_V(v_j))^2\}+\sum_j\partial_n(w_V-(w_V)_I)(m_j))^2$$
$$\leq C(1+\log^2 H/h)\sum_{\bar{\Omega}_j\cap\bar{\Omega}_i\neq\phi}a_h^j(w_H,w_H), \tag{3.46}$$

and summing with respect to i and noting that the number of $\bar{\Omega}_j\cap\bar{\Omega}_i\neq\phi\leq C$ gives (3.44) which completes the proof of the theorem in the case where interior vertices are present.

In the case without internal cross points, i.e.where all the vertices and edge midpoints of Ω_i belong to $\partial\Omega$,the result is trivial. This completes the proof of Theorem 1.

Proof of Proposition 3. To prove Proposition 3 we set $\tilde{\Pi}_h$ be an interpolation operator such that

$$\begin{cases}\tilde{\Pi}_h v(p)=v(p), \quad \forall p \text{ vertex of } T,\ T\in J_h, T\subset\Omega_i;\\ \partial_n\tilde{\Pi}_h v(m)=\dfrac{1}{|e|}\displaystyle\int_e v(s)ds, \quad \text{midpoint } m \text{ of } J_h.\end{cases}$$

We can easily show that

$$|\tilde{\Pi}_h|_{2,T}\leq C|v|_{2,T}. \tag{3.47}$$

First, we show that

$$\Big|v(m)-\frac{1}{|e|}\int_e v(s)ds\Big|\leq C|v|_{H^1(\tau)}, \tag{3.48}$$

for $v\in H^1(\tau), v|_e$ is a polynomial. Here C depends on degree of the polynomial.

Set $\tilde{v}=v(x)-\frac{1}{|\tau|}\int_\tau vdx$. The key point is

$$\int_e\int_e\Big(\frac{v(x)-v(y)}{x-y}\Big)^2ds(x)ds(y)=0\Longrightarrow v(x)=\frac{1}{|e|}\int_e v(x)ds$$
$$\Longrightarrow\max_{x\in e}\Big|v(x)-\frac{1}{|e|}\int_e v(x)ds\Big|=0. \tag{3.49}$$

It follows from (3.49) and a homogeneity argument using reference triangles that

$$\Big|v(x)-\frac{1}{|e|}\int_e v(x)\Big|=\Big|\tilde{v}(x)-\frac{1}{|e|}\int_e\tilde{v}(x)ds\Big|\leq\max_{x\in e}\Big|\tilde{v}(x)-\frac{1}{|e|}\int_e\tilde{v}(x)dx\Big|$$
$$\leq\int_e\int_e\Big(\frac{\tilde{v}(x)-\tilde{v}(y)}{x-y}\Big)^2ds(x)ds(y)\leq C|\tilde{v}|^2_{1/2,\partial\tau}. \tag{3.50}$$

Therefore, by the trace theorem and Poincaré inequality we obtain

$$\Big|v(x)-\frac{1}{|e|}\int_e v(x)\Big|\leq C\|\tilde{v}\|_{1,\tau}\leq C|v|_{1,\tau}. \tag{3.51}$$

Second, it follows (3.51) that

$$\Big|\partial_m v(m) - \frac{1}{|e|}\int_e \frac{\partial v}{\partial n} ds\Big| \le C|v|_{H^2(\tau)}. \tag{3.52}$$

Finally, from (3.52) we have

$$|\Pi'_h v - \tilde{\Pi}_h v|_{H^2(\tau)} \le C\Big|\frac{\partial v}{\partial n}(m) - \frac{1}{|e|}\int_e \frac{\partial v}{\partial n} v ds\Big| \le C|v|_{H^2(\tau)}. \tag{3.53}$$

(3.3) follows from (3.53) and (3.47). #

Remark. We can easily get similar results for many other nonconforming plate elements [6] as well as for various conforming elements.

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