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ON NUMEROV SCHEME FOR NONLINEAR TWO-POINTS BOUNDARY VALUE PROBLEM*

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Abstract

Nonlinear Jacobi iteration and nonlinear Gauss-Seidel iteration are proposed to solve the famous Numerov finite difference scheme for nonlinear two-points boundary value problem. The concept of supersolutions and subsolutions for nonlinear algebraic systems are introduced. By taking such solutions as initial values, the above two iterations provide monotone sequences, which tend to the solutions of Numerov scheme at geometric convergence rates. The global existence and uniqueness of solution of Numerov scheme are discussed also. The numerical results show the advantages of these two iterations.

Key words: Nonlinear two-points boundary value problem, New iterations for Nomerov scheme, Monotone approximations.

1. Introduction

In studying some problems arising in electromagnetism, biology, astronomy, boundary layer and other topics, we often meet nonlinear two-points boundary problem, i.e., finding $y \in C^0[0,1] \cap C^2(0,1)$ such that

$$\begin{cases} -y'' - f(x, y(x)) = 0, \quad 0 < x < 1, \\ y(0) = \alpha, \ y(1) = \beta \end{cases} \tag{1.1}$$

where α, β are certain constants, and $f(x,z) \in C^0(0,1) \times C^1(-\infty,\infty)$. Under some conditions on $f(x,z)$, we can use the framework of [1] to investigate the existence and uniqueness of its solutions. Also there are a lot of literature concerning its numerical solutions[2–4]. In particular, Numerov[5] proposed a famous finite difference scheme with the accuracy of fourth order, which has been used widely in many practical problems. Let N be any positive integer and $h = \dfrac{1}{N}$, $x_n = nh$, $0 \le n \le N$. Also, let $y_n = y(x_n)$, $f_n = f(x_n, y_n)$, and

$$Y = (y_1, \cdots, y_{N-1})^T, \quad F(Y) = (f_1, \cdots, f_{N-1})^T,$$
$$C = (\alpha, 0, \cdots, 0, \beta)^T, \quad D = \Big(\frac{1}{12} f(0,\alpha), 0, \cdots, 0, \frac{1}{12} f(1,\beta)\Big)^T.$$

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Moreover we introduce the symmetric tridiagonal matrices $J = (J_{i,j})$ and $B = (B_{i,j})$ with the following elements $J_{i,i} = 2$, $J_{i,i-1} = J_{i,i+1} = -1$, $1 \le i \le N-1$, $B_{i,i} = 5/6$, $B_{i,i-1} = B_{i,i+1} = 1/12$, $1 \le i \le N-1$. Then the Numerov scheme can be described as follows[5]

$$L_h(Y) \equiv JY - h^2(BF(Y) + D) - C = 0. \tag{1.2}$$

If $f(x,y)$ is nonlinear in y, then we need some iterations to solve (1.2). Henrici[6] and Less[7] considered the Newton iteration. Chawla[8] improved the results of [6,7]. He proposed a suitable initial approximation of the Newton procedure and obtained the sufficient conditions for the convergence when $-\infty < \frac{\partial f}{\partial z}(x,z) < \pi^2$. But such conditions involve an implicit equation for the mesh size h and it is difficult to solve it usually. In addition, we have to adopt an interior iteration for solving a linear system for each step of the exterior iteration, which costs a lot of computational time. The purpose of this paper is to develop two new iterations. In next section, we introduce nonlinear Jacobi iteration and nonlinear Gauss-Seidel iteration. Both of them avoid the interior iterations in [8], and so save a lot of work. Also, we introduce the concept of supersolutions and subsolutions, and prove that if we take such solutions as initial values, then the above iterations may provide two monotone sequences. They not only give us the up-bound and low-bound of the exact solution of (1.2), but also tend to it with geometric convergence rates. In Section 3, we consider global existence and uniqueness of solution of (1.2) as well as the global convergences of the new iterations. In the final section, we present the numerical results which agree the theoretical analysis and show the advantages of the two new approaches.

2. New Nonlinear Iterations

We now present nonlinear Jacobi iteration and nonlinear Gauss-Seidel iteration for (1.2). Let ω be a parameter. We decompose the matrices J and B as $J = \mathcal{D} - \mathcal{L} - \mathcal{U}$, $B = \mathcal{D}^* + \mathcal{L}^* + \mathcal{U}^*$, where $\mathcal{D}$ and $\mathcal{D}^*$ are diagonal matrices, $\mathcal{L}$ and $\mathcal{L}^*$ are lower-off diagonal matrices, $\mathcal{U}$ and $\mathcal{U}^*$ are upper-off diagonal matrices. Let $Y^{(m)}$ be the m'th iterated vector $(y_1^{(m)}, \cdots, y_{N-1}^{(m)})^T$ and $y_i^{(m)} = y^{(m)}(x_i)$. Then the nonlinear Jacobi iteration is defined as

$$(\mathcal{D} - \omega h^2 \mathcal{D}^*)Y^{(m)} = (\mathcal{L} + \mathcal{U})Y^{(m-1)} - \omega h^2 \mathcal{D}^* Y^{(m-1)} + h^2 BF(Y^{(m-1)}) + h^2 D + C, \tag{2.1}$$

while the Gauss-Seidel iteration is given by

$$(\mathcal{D} - \mathcal{L} - \omega h^2(\mathcal{D}^* + \mathcal{L}^*))Y^{(m)} = \mathcal{U}Y^{(m-1)} - \omega h^2(\mathcal{D}^* + \mathcal{L}^*)Y^{(m-1)} + h^2 BF(Y^{(m-1)}) + h^2 D + C. \tag{2.2}$$

Clearly both (2.1) and (2.2) do not need the interior iterations to solve $Y^{(m)}$ as long as $Y^{(m-1)}$ is known.

For theoretical analysis, we first introduce some notations and analyze the monotonicity of the matrix $J - \omega h^2 B$. Let $U = (u_1, \cdots, u_{N-1})^T$ and $V = (v_1, \cdots, v_{N-1})^T$. If $u_i \le v_i$ for all i, then we say that $U \le V$. If $U \le W \le V$, then it is denoted by $W \in \mathbf{K}(U, V)$. If all elements of a vector U or a matrix $A = (A_{i,j})$ are non-negative, then we say that $U \ge 0$ or $A \ge 0$, etc.. Furthermore if $AU \ge 0$ implies $U \ge 0$ for any vector U, then we say that A is a monotone matrix. In this case, $A_{i,i} > 0$ if $A_{i,j} \leq 0$ for $i \neq j$ (see page 131 of [3]). A necessary and sufficient condition for the monotonicity of A is the existence of the inverse $A^{-1} \geq 0$ (e.g. see Theorem 4.16 of [3]). In particular, a matrix A is of positive-type, if it fulfils the following conditions

(i) $A_{i,i} > 0$ and for $i \neq j$, $A_{i,j} \leq 0$;

(ii) $d_i = \dfrac{-\sum_{j\neq i} A_{i,j}}{A_{i,i}} \leq 1$ and the set $\mathcal{N}(A) = \{i | d_i < 1\}$ is not empty;

(iii) for any $i_1 \notin \mathcal{N}(A)$, there exists $i_2 \in \mathcal{N}(A)$ such that

$$A_{i_1,j_1}, A_{j_1,j_2}, \cdots, A_{j_q,i_2} \neq 0.$$

Any positive-type matrix is monotone[1,9].

Lemma 2.1. *If* $-\infty < \omega < 0$ *and* $h \leq \sqrt{-\dfrac{12}{\omega}}$*, then* $J - \omega h^2 B$ *is monotone.*

Proof. By the conditions of this lemma, $(J - \omega h^2 B)_{m,m} > 0$, $(J - \omega h^2 B)_{m,m-1} \leq 0$, $(J - \omega h^2 B)_{m,m+1} \leq 0$. The matrix $J - \omega h^2 B$ also satisfies the other conditions of positive-type matrix and so the conclusion follows.

Lemma 2.2. *If* $0 \leq \omega < 8$*, then* $J - \omega h^2 B$ *is monotone.*

Proof. Obviously J is a positive-type matrix and so the inverse $J^{-1} \geq 0$ exists. Let I be the identity matrix and $J - \omega h^2 B = J(I - \omega h^2 J^{-1} B)$. We know that if a matrix $A = I - S$, $S \geq 0$ and for certain norm $\| \cdot \|$, $\|S\| < 1$, then A is monotone (Theorem 3, Page 298 of [9]). Also the product of two monotone matrices is still monotone. Hence $J - \omega h^2 B$ is monotone provided that for certain norm of matrix $\| \cdot \|$, $\|\omega h^2 J^{-1} B\| < 1$. Let $J^{-1} = (J^{-1}_{i,j})$. It can be checked that

$$J^{-1}_{i,j} = \begin{cases} \dfrac{(N-j)i}{N}, & \text{for } i \leq j, \\ \dfrac{(N-i)j}{N}, & \text{for } i > j. \end{cases}$$

Clearly $\|B\|_\infty = 1$, and

$$\|J^{-1}\|_\infty = \max_{1 \leq i \leq N-1} \sum_{j=1}^{N-1} | J^{-1}_{i,j} | \leq \max_{1 \leq i \leq N-1} \frac{i}{2}(N-i) = \frac{N^2}{8}.$$

Thus $\|\omega h^2 J^{-1} B\|_\infty \leq \dfrac{1}{8}\omega h^2 N^2 < 1$ and the conclusion follows.

Lemma 2.3. *If* $8 \leq \omega < \pi^2$ *and* $h < \sqrt{\dfrac{12}{\pi^2}\Big(1 - \dfrac{\omega}{\pi^2}\Big)}$*, then* $J - \omega h^2 B$ *is monotone.*

Proof. The proof follows along the same line as in Lemma 2.2. We consider the auxiliary problem

$$\begin{cases} JU = \lambda U, \\ u_0 = u_N = 0. \end{cases}$$

The minimal eigenvalue $\lambda_1 = 4 \sin^2 \Big(\dfrac{\pi h}{2}\Big)$. Furthermore $\|J^{-1}\|_2$ equals μ_{N-1}, the maximal eigenvalue of J^{-1}. Since $\mu_{N-1} = \dfrac{1}{\lambda_1}$, we have $\|J^{-1}\|_2 = \dfrac{1}{4\sin^2(\frac{\pi h}{2})}$. Similarly

$\|B\|_2 = 1 - \frac{1}{3}\sin^2\left(\frac{\pi h}{2}\right) \leq 1$. Since $\sin^2 x \geq x^2\Big(1 - \frac{x^2}{3}\Big)$ for $x \geq 0$, we find that

$$\|\omega h^2 J^{-1} B\|_2 \leq \frac{\omega}{\pi^2\Big(1 - \frac{\pi^2 h^2}{12}\Big)} < 1$$

and so the conclusion follows.

Hereafter we define

$$h(\omega) = \begin{cases} \sqrt{-\frac{12}{\omega}}, & \omega < 0, \\ \text{arbitrary positive constant}, & 0 \leq \omega < 8, \\ \sqrt{\frac{12}{\pi^2}\Big(1 - \frac{\omega}{\pi^2}\Big)}, & 8 \leq \omega < \pi^2. \end{cases} \tag{2.3}$$

Next we introduce the concept of supersolutions and subsolutions. A vector $\tilde{Y}$ is called a supersolution of (1.2) if $L_h(\tilde{Y}) \geq 0$. Similarly a vector $\hat{Y}$ is called a subsolution of (1.2) if $L_h(\hat{Y}) \leq 0$. Clearly every solution of (1.2) is a supersolution as well as a subsolution. If $\hat{Y} \leq \tilde{Y}$, then we say that $(\hat{Y}, \tilde{Y})$ is an ordered pair for (1.2). There is no definitive result for the existence of such pairs. But they really exist under some conditions.

Lemma 2.4. *Assume that*

(i) $\frac{\partial f}{\partial z}(x, z) \leq \overline{M} < \pi^2$ *for* $x \in [0, 1]$ *and* $z \in (-\infty, \infty)$*;*

(ii) $h \leq h(\overline{M})$ *for* $\overline{M} < 8$ *and* $h < h(\overline{M})$ *for* $8 \leq \overline{M} < \pi^2$, $h(\overline{M})$ *being defined by* (2.3).

Then (1.2) *has at least one ordered pair of supersolution and subsolution.*

Proof. By condition (ii) and Lemmas 2.1–2.3, $J - \overline{M}h^2 B$ is monotone and so $(J - \overline{M}h^2 B)^{-1} \geq 0$. For any vector V, set $W = (J - \overline{M}h^2 B)^{-1} \mid L_h(V) \mid \geq 0$. Let $\tilde{Y} = V + W$, $\hat{Y} = V - W$. Then $\hat{Y} \leq V \leq \tilde{Y}$, and

$$\begin{aligned} L_h(\tilde{Y}) &= JW + JV - h^2(BF(\tilde{Y}) + D) - C \\ &= \mid L_h(V) \mid + \overline{M}h^2 BW + L_h(V) + h^2 BF(V) - h^2 BF(\tilde{Y}) \\ &\geq h^2 B(\overline{M}W + F(V) - F(\tilde{Y})) = h^2 B(\overline{M}I - F'(\Theta))W \end{aligned}$$

where $\Theta \in \mathbf{K}(V, \tilde{Y})$. Thus $L_h(\tilde{Y}) \geq 0$ and $\tilde{Y}$ is a supersolution of (1.2). Similarly, it can be verified that $\hat{Y}$ is a subsolution of (1.2). This completes the proof.

Now we begin to analyze the nonlinear iteration (2.1) and (2.2). If we take $Y^{(0)} = \tilde{Y}$, then we denote the corresponding sequence by $\{\overline{Y}^{(m)}\}$ for both iterations. If we take $Y^{(0)} = \hat{Y}$, then we denote the corresponding sequence by $\{\underline{Y}^{(m)}\}$. We first deal with the nonlinear Jacobi iteration (2.1).

Theorem 2.1. *Assume that*

(i) $(\hat{Y}, \tilde{Y})$ *is an ordered pair of supersolution and subsolution for* (1.2)*;*

(ii) there exists a constant M_0 *such that* $\frac{\partial f}{\partial z}(x, z) \geq M_0$, *for all* x_i *and* $z \in \mathbf{K}(\hat{Y}, \tilde{Y})$*;*

(iii) $\omega = \underline{M} \leq \min(M_0, \pi^2)$ *and* $\omega \neq \pi^2$ *in* (2.1)*;*

(iv) $h \leq h(\underline{M})$ *for* $\underline{M} < 8$ *and* $h < h(\underline{M})$ *for* $\underline{M} \geq 8$.

Then the sequence $\{\overline{Y}^{(m)}\}$ *is a nonincreasing sequence of supersolutions and converges to the maximal solution of* (1.2) *in* $\mathbf{K}(\hat{Y},\tilde{Y})$*, denoted by* $\overline{Y}$*, while the sequence* $\{\underline{Y}^{(m)}\}$ *is a nondecreasing sequence of subsolutions and converges to the minimal solution of* (1.2) *in* $\mathbf{K}(\hat{Y},\tilde{Y})$*, denoted by* $\underline{Y}$*. Moreover for all* $m \geq 1$*,*

$$\hat{Y} = \underline{Y}^{(0)} \leq \underline{Y}^{(1)} \leq \cdots \leq \underline{Y}^{(m)} \leq \underline{Y} \leq \overline{Y} \leq \overline{Y}^{(m)} \leq \cdots \leq \overline{Y}^{(1)} \leq \overline{Y}^{(0)} = \tilde{Y}. \tag{2.4}$$

If in addition, the following condition is satisfied

(*v*) $\dfrac{\partial f}{\partial z}(x,z) \leq \overline{M} < \pi^2$ *for all* x_i *and* $z \in \mathbf{K}(\hat{Y},\tilde{Y})$*,*

then $\overline{Y} = \underline{Y}$ *is the unique solution of* (1.2) *in* $\mathbf{K}(\hat{Y},\tilde{Y})$ *provided* $h < \min(h(\underline{M}),\, h(\overline{M}))$.

Proof. Firstly, we note that Jacobi iterative scheme (2.1) with $\omega = \underline{M}$ is equivalent to

$$\begin{aligned}(\mathcal{D} - \underline{M}h^2\mathcal{D}^*)Y^{(m)} =& (\mathcal{L} + \mathcal{U} + \underline{M}h^2(\mathcal{L}^* + \mathcal{U}^*))Y^{(m-1)} \\ &- \underline{M}h^2BY^{(m-1)} + h^2BF(Y^{(m-1)}) + h^2D + C.\end{aligned} \tag{2.5}$$

Now we use induction. Suppose that $\overline{Y}^{(m)} \in \mathbf{K}(\hat{Y},\tilde{Y})$ is a supersolution and let $Z^{(m)} = \overline{Y}^{(m+1)} - \overline{Y}^{(m)}$. Then

$$(\mathcal{D} - \underline{M}h^2\mathcal{D}^*)Z^{(m)} = -L_h(\overline{Y}^{(m)}) \leq 0. \tag{2.6}$$

By condition (iv) and Lemmas 2.1–2.3, we know that $J - \underline{M}h^2B$ is monotone and so $\mathcal{D} - \underline{M}h^2\mathcal{D}^* > 0$. Therefore (2.6) implies

$$\overline{Y}^{(m+1)} \leq \overline{Y}^{(m)} \leq \tilde{Y}. \tag{2.7}$$

Now let

$$F'(Y) = \operatorname{diag}\Big(\frac{\partial f}{\partial y}(x_1,y_1), \cdots, \frac{\partial f}{\partial y}(x_{N-1},y_{N-1})\Big).$$

Then from (2.5) and $L_h(\hat{Y}) \leq 0$, we have

$$\begin{aligned}&(\mathcal{D} - \underline{M}h^2\mathcal{D}^*)(\overline{Y}^{(m+1)} - \hat{Y}) \geq (\mathcal{L} + \mathcal{U} + \underline{M}h^2(\mathcal{L}^* + \mathcal{U}^*))(\overline{Y}^{(m)} - \hat{Y}) - \underline{M}h^2B(\overline{Y}^{(m)} \\ &\quad - \hat{Y}) + h^2B(F(\overline{Y}^{(m)}) - F(\hat{Y})) \\ &= (\mathcal{L} + \mathcal{U} + \underline{M}h^2(\mathcal{L}^* + \mathcal{U}^*))(\overline{Y}^{(m)} - \hat{Y}) - h^2B(\underline{M}I - F'(\Theta^{(m)}))(\overline{Y}^{(m)} - \hat{Y})\end{aligned} \tag{2.8}$$

where $\Theta^{(m)} \in \mathbf{K}(\hat{Y},\overline{Y}^{(m)}) \subseteq \mathbf{K}(\hat{Y},\tilde{Y})$ and thus $\underline{M}I - F'(\Theta^{(m)}) \leq 0$. If $\underline{M} < 0$ and $h \leq h(\underline{M})$, then we know from the proof of Lemma 2.1 that $(\mathcal{L}+\mathcal{U}+\underline{M}h^2(\mathcal{L}^*+\mathcal{U}^*)) \geq 0$. If $\underline{M} > 0$, the same conclusion is valid. Therefore the right side of (2.8) is non-negative. Thus $\overline{Y}^{(m+1)} \geq \hat{Y}$ and so $\overline{Y}^{(m+1)} \in \mathbf{K}(\hat{Y},\tilde{Y})$. Furthermore from (2.5),

$$\begin{aligned}L_h(\overline{Y}^{(m+1)}) &= -(\mathcal{L} + \mathcal{U} + \underline{M}h^2(\mathcal{L}^* + \mathcal{U}^*))Z^{(m)} + \underline{M}h^2BZ^{(m)} \\ &\quad - h^2B(F(\overline{Y}^{(m+1)}) - F(\overline{Y}^{(m)})) \\ &= -(\mathcal{L} + \mathcal{U} + \underline{M}h^2(\mathcal{L}^* + \mathcal{U}^*))Z^{(m)} + h^2B(\underline{M}I - F'(\Theta))Z^{(m)}\end{aligned}$$

where $\Theta = (\theta_1, \cdots, \theta_{N-1})^T \in \mathbf{K}(\overline{Y}^{(m+1)}, \overline{Y}^{(m)}) \subseteq \mathbf{K}(\hat{Y}, \tilde{Y})$. By $Z^{(m)} \le 0$ and the same reason as above we conclude that $L_h(\overline{Y}^{(m+1)}) \ge 0$. Thus $\overline{Y}^{(m+1)}$ is also a supersolution and so the induction is completed.

Similarly we can prove that $\underline{Y}^{(m)}$ is a subsolution and

$$\hat{Y} \le \underline{Y}^{(m)} \le \underline{Y}^{(m+1)}. \tag{2.9}$$

Next, we use induction to prove that

$$\underline{Y}^{(m)} \le \overline{Y}^{(m)}. \tag{2.10}$$

Assume that (2.10) holds for m and let $Z^{(m)} = \overline{Y}^{(m+1)} - \underline{Y}^{(m+1)}$. Then by (2.5),

$$\begin{aligned}(\mathcal{D} - \underline{M}h^2\mathcal{D}^*)Z^{(m)} =&(\mathcal{L} + \mathcal{U} + \underline{M}h^2(\mathcal{L}^* + \mathcal{U}^*))(\overline{Y}^{(m)} - \underline{Y}^{(m)}) \\ &- h^2B(\underline{M}I - F'(\Phi))(\overline{Y}^{(m)} - \underline{Y}^{(m)})\end{aligned}$$

where $\Phi \in \mathbf{K}(\underline{Y}^{(m)}, \overline{Y}^{(m)}) \subseteq \mathbf{K}(\hat{Y}, \tilde{Y})$. By an argument similar to that in the previous paragraph, we obtain $Z^{(m)} \ge 0$, and thus (2.10) holds also for $m+1$. The combination of (2.7), (2.9) and (2.10) leads to

$$\hat{Y} = \underline{Y}^{(0)} \le \underline{Y}^{(1)} \le \cdots \le \underline{Y}^{(m)} \le \overline{Y}^{(m)} \le \cdots \le \overline{Y}^{(1)} \le \overline{Y}^{(0)} = \tilde{Y}. \tag{2.11}$$

Hence there exist the limits

$$\lim_{m\to\infty} \underline{Y}^{(m)} = \underline{Y}, \quad \lim_{m\to\infty} \overline{Y}^{(m)} = \overline{Y}. \tag{2.12}$$

By letting $m \to \infty$ in (2.1), we see that both $\overline{Y}$ and $\underline{Y}$ are the solutions of (1.2) in $\mathbf{K}(\hat{Y}, \tilde{Y})$. The combination of (2.11) and (2.12) leads to (2.4).

If Y^* is any other solution of (1.2) in $\mathbf{K}(\hat{Y}, \tilde{Y})$, then it is also a subsolution of (1.2). By taking Y^* as initial value of (2.1) and an argument as in the previous paragraph, we find that $Y^* \le \overline{Y}$. Similarly we can verify that $Y^* \ge \underline{Y}$. Therefore $\overline{Y}$ and $\underline{Y}$ are the maximal solution and the minimal solution of (1.2) in $\mathbf{K}(\hat{Y}, \tilde{Y})$, respectively.

We now turn to the local uniqueness of solution. Let condition (v) hold. It suffices to show $\overline{Y} = \underline{Y}$. Let $Z = \overline{Y} - \underline{Y}$. Then $Z \ge 0$ and $JZ = h^2BF'(\Theta)Z$ where $\Theta \in \mathbf{K}(\underline{Y}, \overline{Y}) \subseteq \mathbf{K}(\hat{Y}, \tilde{Y})$. Therefore, $JZ \le \overline{M}h^2BZ$ or $(J - \overline{M}h^2B)Z \le 0$. If $h < \min(h(\underline{M}), h(\overline{M}))$, then we have from Lemmas 2.1–2.3 that $Z \le 0$ and thus $Z = 0$. This completes the proof.

We now estimate the errors of iteration. Let Y be the solution of (1.2) in $\mathbf{K}(\hat{Y}, \tilde{Y})$ and $Y^{(m)}$ the iterated solution $\overline{Y}^{(m)}$ or $\underline{Y}^{(m)}$ given in Theorem 2.1. Let $\rho(\varphi)$ be the spectral radius of φ and $\varphi = (\mathcal{D} - \underline{M}h^2\mathcal{D}^*)^{-1}(\mathcal{D} - J - \underline{M}h^2\mathcal{D}^* + \overline{M}h^2B)$.

Theorem 2.2. *Let conditions (i)–(v) of Theorem 2.1 hold and $h < \min(h(\underline{M}), h(\overline{M}))$. Then there exists a positive constant δ such that $\rho(\varphi) + \delta < 1$ and for certain vector norm $\|\cdot\|$, $\|Y^{(m)} - Y\| \le (\rho(\varphi) + \delta)^m \|Y^{(0)} - Y\|$.*

Proof. Let $Z^{(m)} = Y^{(m)} - Y$. Then

$$(\mathcal{D} - \underline{M}h^2\mathcal{D}^*)Z^{(m)} = (\mathcal{L} + \mathcal{U} + \underline{M}h^2(\mathcal{L}^* + \mathcal{U}^*))Z^{(m-1)}$$

$$\begin{aligned}&-\underline{M}h^2BZ^{(m-1)}+h^2B(F(Y^{(m-1)})-F(Y))\\&=(\mathcal{L}+\mathcal{U}+\underline{M}h^2(\mathcal{L}^*+\mathcal{U}^*))Z^{(m-1)}+h^2B(-\underline{M}I+F'(\Theta))Z^{(m-1)}\end{aligned}$$

where Θ lies between $Y^{(m-1)}$ and Y, and thus $\Theta\in\mathbf{K}(\hat{Y},\tilde{Y})$. Therefore $0\le F'(\Theta)-\underline{M}I\le(\overline{M}-\underline{M})I$, from which and $h<\min(h(\overline{M}),h(\underline{M}))$, we have

$$\begin{aligned}|\,Z^{(m)}\,|&\le(\mathcal{D}-\underline{M}h^2\mathcal{D}^*)^{-1}(\mathcal{L}+\mathcal{U}+\underline{M}h^2(\mathcal{L}^*+\mathcal{U}^*)+(\overline{M}-\underline{M})h^2B)\,|\,Z^{(m-1)}\,|\\&=(\mathcal{D}-\underline{M}h^2\mathcal{D}^*)^{-1}(\mathcal{D}-J-\underline{M}h^2\mathcal{D}^*+\overline{M}h^2B)\,|\,Z^{(m-1)}\,|=\varphi\,|\,Z^{(m-1)}\,|\end{aligned}$$

where $|\,U\,|$ denotes the vector $(|\,u_1\,|,\cdots,|\,u_{N-1}\,|)^T$. By Lemmas 2.1–2.3, $J-\overline{M}h^2B$ is a monotone matrix. Indeed it is also a M-matrix (see page 131 of [3]). As we know, if a matrix $A=Q-S$, $Q^{-1}\ge0$ and $S\ge0$, then we say that the above expression is a regular splitting of A. On the other hand, any regular splitting of monotone matrix A is convergent, i.e., $\rho(Q^{-1}S)<1$ (see §2.4.17 of [10]). Clearly $J-\overline{M}h^2B=(\mathcal{D}-\underline{M}h^2\mathcal{D}^*)-(\mathcal{D}-J-\underline{M}h^2\mathcal{D}^*+\overline{M}h^2B)$ is a regular splitting and so $\rho(\varphi)<1$. Therefore there exists a positive constant δ such that $\rho(\varphi)+\delta<1$, and for certain norm $\|\cdot\|$, $\|Z^{(m)}\|\le\|\varphi\|\cdot\|Z^{(m-1)}\|\le(\rho(\varphi)+\delta)\|Z^{(m-1)}\|$ which completes the proof.

We now turn to the Gauss-Seidel iteration (2.2).

Theorem 2.3. *Assume that*

(i) conditions (i),(ii) of Theorem 2.1 *hold;*

(ii) $\omega=\underline{\underline{M}}\le\min(M_0,0)$ *and* $h\le h(\underline{\underline{M}})$ *in* (2.2).

Then the corresponding conclusions of Theorem 2.1 *are also valid for the iteration* (2.2). *If in addition,*

(iii) condition (v) of Theorem 2.2 *holds and* $h<\min(h(\underline{\underline{M}}),\,h(\overline{M}))$,

then $\overline{Y}=\underline{Y}$ *is the unique solution of* (1.2) *in* $\mathbf{K}(\hat{Y},\tilde{Y})$.

Proof. The iteration (2.2) with $\omega=\underline{\underline{M}}$ is equivalent to

$$\begin{aligned}(\mathcal{D}-\mathcal{L}-\underline{\underline{M}}h^2(\mathcal{D}^*+\mathcal{L}^*))Y^{(m)}=&(\mathcal{U}+\underline{\underline{M}}h^2\mathcal{U}^*)Y^{(m-1)}-\underline{\underline{M}}h^2BY^{(m-1)}\\&+h^2BF(Y^{(m-1)})+h^2D+C.\end{aligned}\tag{2.15}$$

By an argument similar to that in the proof of Lemma 2.1, we can show that $\mathcal{D}-\mathcal{L}-\underline{\underline{M}}h^2(\mathcal{D}^*+\mathcal{L}^*)$ is monotone and $\mathcal{U}+\underline{\underline{M}}h^2\mathcal{U}^*\ge0$. The rest of proof is same as in Theorem 2.1.

Now let $\varphi=(\mathcal{D}-\mathcal{L}-\underline{\underline{M}}h^2(\mathcal{D}^*+\mathcal{L}^*))^{-1}(\mathcal{D}-J-\mathcal{L}-\underline{\underline{M}}h^2(\mathcal{D}^*+\mathcal{L}^*)+\overline{M}h^2B)$.

Theorem 2.4. *Let conditions (i)–(iii) of Theorem* 2.3 *hold. Then the conclusion of Theorem* 2.2 *is also valid for the iteration* (2.2).

From Theorems 2.1–2.4, we see that the iterations (2.1) and (2.2) not only avoid the interior iterations but also provide the monotone sequences of supersolutions and subsolutions. They give the up-bounds and low-bounds of the exact solutions of (1.2) and also tend to it monotonically with geometric convergence rates. Indeed we can construct the corresponding sequences for the continuous version (1.1), which have the same properties. Thus Numerov scheme preserves the feature of the original problem, and (2.1) and (2.2) also preserve this feature. So they could provide better numerical results and save computational time.

3. Global Solution and Global Convergence

In this section, we consider the global solution of (1.2) and the global convergences of (2.1) and (2.2). We need the following lemma.

Lemma 3.1. *Assume that*

(i) for all $x \in [0,1]$ and $z \in (-\infty, \infty)$,

$$-\infty < \underline{M} \le \frac{\partial f}{\partial z}(x,z) \le \overline{M} < \pi^2;$$

(ii) $h < h(\underline{M}, \overline{M})$ and

$$h(\underline{M},\overline{M}) = \begin{cases} \sqrt{-\dfrac{12}{\underline{M}}}, & \overline{M} < 8 \text{ and } \underline{M} < 0, \\ \text{arbitrary positive constant}, & 0 \le \overline{M} < 8 \text{ and } \underline{M} \ge 0, \\ \sqrt{\dfrac{12}{\pi^2}\Big(1 - \dfrac{\overline{M}}{\pi^2}\Big)}, & \overline{M} \ge 8 \text{ and } \underline{M} \ge 0, \\ \min\left\{\sqrt{-\dfrac{12}{\underline{M}}}, \sqrt{\dfrac{12}{\pi^2}\Big(1 - \dfrac{\overline{M}}{\pi^2}\Big)}\right\}, & \overline{M} \ge 8 \text{ and } \underline{M} < 0. \end{cases} \tag{3.1}$$

Then for any vector Y, the matrix $J - h^2BF'(Y)$ is monotone.

Proof. There are four different cases.

(i) $\overline{M} < 8$ and $\underline{M} < 0$. If $\overline{M} \le 0$, then the conclusion follows from an argument as in Lemma 2.1. If $0 \le \overline{M} < 8$, then we write

$$F'(Y) = \mathcal{U}^+ + \mathcal{U}^-, \quad \mathcal{U}^+ > 0,\ \mathcal{U}^- \le 0. \tag{3.2}$$

Let $\overline{J} = J - h^2B\mathcal{U}^-$. By the same technique as in the proof of Lemma 2.1, we get the monotonicity of $\overline{J}$ as long as $h \le \sqrt{-\frac{12}{\underline{M}}}$. Next we write

$$J - h^2BF'(Y) = \overline{J}(I - h^2\overline{J}^{-1}B\mathcal{U}^+). \tag{3.3}$$

Clearly both J and $\overline{J}$ are monotone, and $\overline{J} \ge J$. Hence $\overline{J}^{-1}\overline{J}J^{-1} \ge \overline{J}^{-1}JJ^{-1}$ and so $J^{-1} \ge \overline{J}^{-1} \ge 0$. Thus $\|\overline{J}^{-1}\|_\infty \le \|J^{-1}\|_\infty \le \frac{N^2}{8}$. By the same reason as in the proof of Lemma 2.2, we know that $J - h^2BF'(Y)$ is monotone for all $h \le \sqrt{-\frac{12}{\underline{M}}}$. This leads to the conclusion.

(ii) $0 \le \overline{M} < 8$ and $\underline{M} \ge 0$. In this case, $0 \le F'(Y) < 8$ and so the conclusion follows along the same line as in Lemma 2.2.

(iii) $\overline{M} \ge 8$ and $\underline{M} \ge 0$. In this case, $F'(Y) \ge 0$ and so the conclusion comes along the same line as in Lemma 2.3.

(iv) $\overline{M} \ge 8$ and $\underline{M} < 0$. In this case, we decompose $F'(Y)$ by (3.2) and let $\overline{J}$ be the same as before. By the same technique as in the proof of Lemma 2.1, we find that

$\overline{J}$ is monotone as long as $h \le \sqrt{-\dfrac{12}{\underline{M}}}$. Next we also consider (3.3) and the auxiliary problem

$$\begin{cases} \overline{J}^T\overline{J}Z = \mu Z, \\ z_0 = z_N = 0. \end{cases}$$

Clearly the minimal eigenvalue μ_1 is not less than λ_1^2 given in the proof of Lemma 2.3. Thus

$$\|\overline{J}^{-1}\|_2 \le \|J^{-1}\|_2 = \frac{1}{4\sin^2\left(\dfrac{\pi h}{2}\right)}.$$

Then the conclusion comes as in Lemma 2.3.

Theorem 3.1 *Let condition* (*i*) *of Lemma* 3.1 *hold. Then*

(*i*) *if* $h < \min(h(\underline{M}), h(\overline{M}))$, *then* (1.2) *has at least a solution;*

(*ii*) *if* $h < h(\underline{M}, \overline{M})$, *then* (1.2) *has a unique solution.*

Proof. The conclusion (i) comes from Lemma 2.4 and Theorem 2.1. We now prove the second one. Let Y_1 and Y_2 be two solutions, and $Z = Y_1 - Y_2$. Then

$$JZ = h^2BF(Y_1) - h^2BF(Y_2) = h^2BF'(\Theta)Z$$

or $(J - h^2BF'(\Theta))Z = 0$, where Θ lies between Y_1 and Y_2. From Lemma 3.1, $J - h^2BF'(\Theta)$ is monotone provided $h < h(\underline{M}, \overline{M})$. Hence, $Z = 0$ as required.

If the conditions of Theorem 3.1 are fulfilled, then the iterations (2.1) and (2.2) have the global convergences.

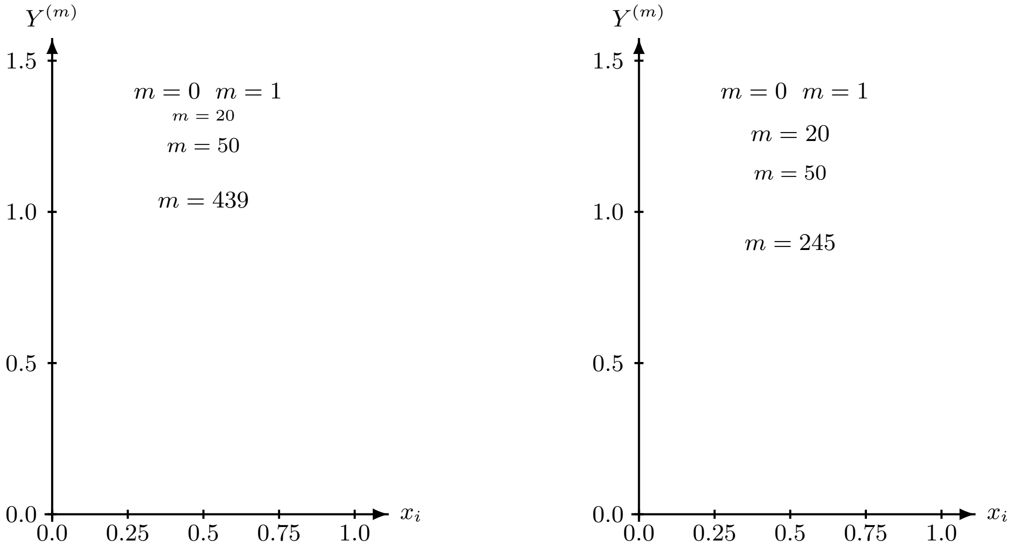
4. The Numerical Results

This section is devoted to numerical results. We consider the following problem

$$\begin{cases} -y'' - f(x, y(x)) = 0, \\ y(0) = y(1) = 0. \end{cases} \tag{4.1}$$

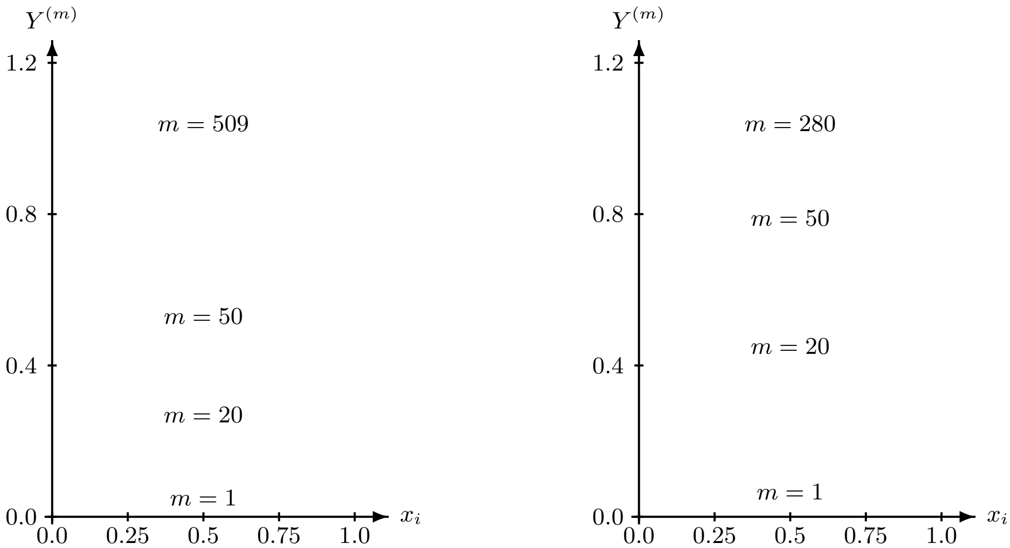
We first take $f(x, z) = -z^2 + \pi^2\sin(\pi x) + \sin^2(\pi x)$. In this case, it is easy to check that (4.1) has a supersolution $\tilde{y} = \dfrac{\pi^2+1}{2}x(1-x)$ and a subsolution $\hat{y} \equiv 0$ in the usual sense[3,4]. Furthermore $-\dfrac{\pi^2+1}{4} \le \dfrac{\partial f}{\partial z}(x, z) \le 0$ for all $0 < x < 1$ and $\hat{y} \le z \le \tilde{y}$. Thus we can prove that (4.1) has a unique solution $y = \sin \pi x$ such that $\hat{y} \le y \le \tilde{y}$. We use Numerov scheme (1.2) to solve (4.1). Then we use the nonlinear Jacobi iteration (2.1) and nonlinear Gauss-Seidel iteration (2.2) for the resulting nonlinear system (1.2). In this case, this system has a supersolution $\tilde{Y} = (\tilde{y}(x_1), \cdots, \tilde{y}(x_{N-1}))^T$ with $\tilde{y}(x_i) = \dfrac{\pi^2+1}{2}x_i(1-x_i)$ for $0 \le i \le N$, and a subsolution $\hat{Y} = (\hat{y}(x_1), \cdots, \hat{y}(x_{N-1}))^T$ with $\hat{y}(x_i) \equiv 0$ for $0 \le i \le N$. Also $-\frac{\pi^2+1}{4} \le \dfrac{\partial f}{\partial z}(x_i, z) \le 0 < \pi^2$ for $z \in \mathbf{K}(\hat{Y}, \tilde{Y})$ and $1 \le i \le N-1$. Thus (1.2) has a unique solution Y such that $\hat{Y} \le Y \le \tilde{Y}$. We take $N = 20$ in (1.2) and $\omega = -4$ in (2.1) and (2.2). Let $Y^{(m)}$ be the m'th iterated vector. If we take $Y^{(0)} = \tilde{Y}$, then we get decreasing sequences (see Fig. 4.1). If we take $Y^{(0)} = \hat{Y}$, then we get increasing sequences (see Fig. 4.2). Fig. 4.1 and Fig. 4.2 coincide with the theoretical analysis in Section 2. Next we take $N = 10$. In the case $Y^{(0)} = \tilde{Y}$, for the iteration (2.1) and $m \geq 134$, we have

$$\|Y^{(m+1)} - Y^{(m)}\|_\infty \leq 10^{-5}. \tag{4.2}$$



Jacobi iteration

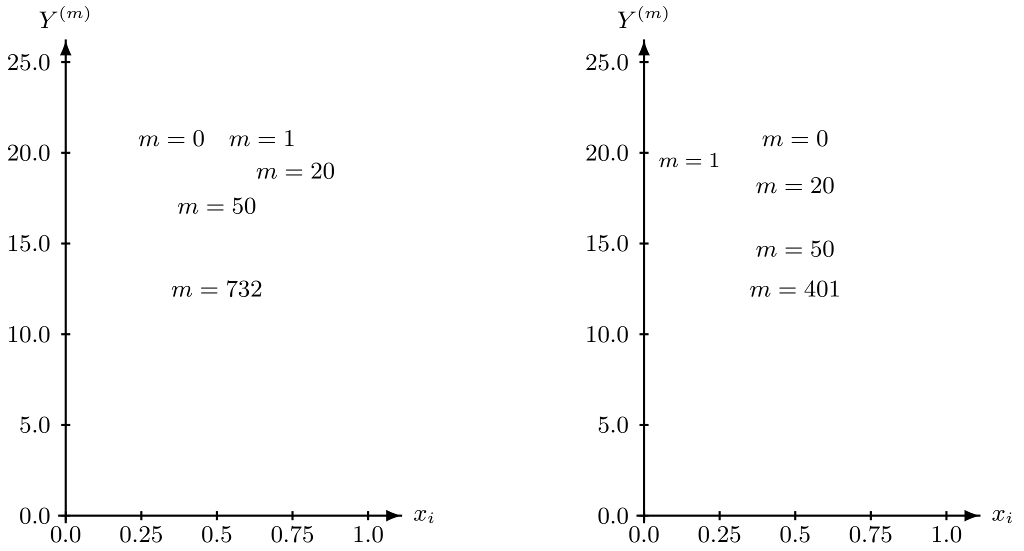
Fig. 4.1. Gauss-seidel iteration



Jacobi iteration

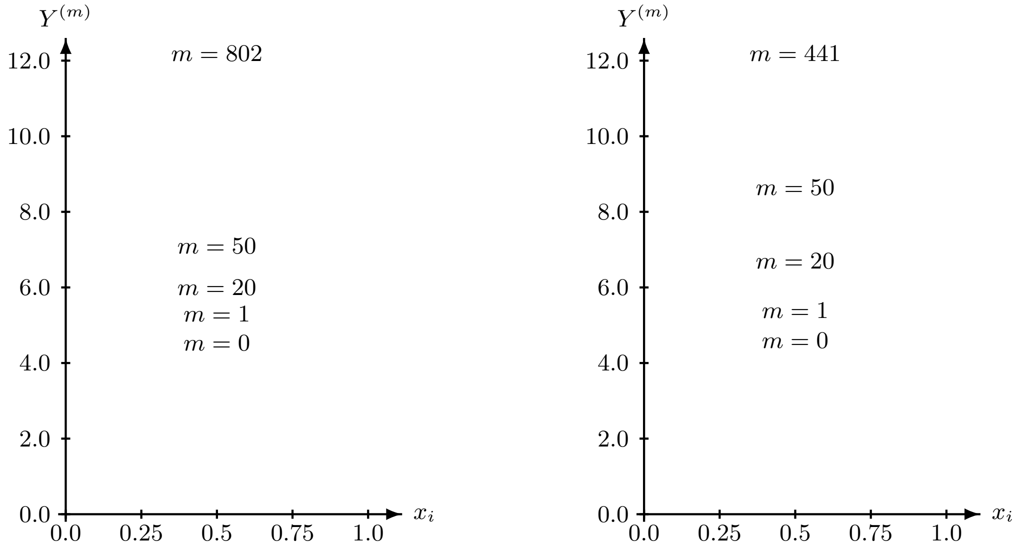
Fig. 4.2. Gauss-seidel iteration

We have the same results for the iteration (2.2) and $m \geq 75$. In the case $Y^{(0)} = \hat{Y}$, (4.2) holds for (2.1) with $m \geq 152$, and for (2.2) with $m \geq 84$.



Jacobi iteration

Fig. 4.3. Gauss-seidel iteration



Jacobi iteration

Fig. 4.4. Gauss-seidel iteration

We next take $f(x,z) = 20z - z^2$. In this case, it is easy to check that (4.1) has a supersolution $\tilde{y} \equiv 20$ and a subsolution $\hat{y} = 5\sin \pi x$ in the usual sense[3,4]. Furthermore $-20 \leq \dfrac{\partial f}{\partial z}(x,z) \leq 20$ for all $0 < x < 1$ and $\hat{y} \leq z \leq \tilde{y}$. Thus (4.1) has a positive solution y such that $\hat{y} \leq y \leq \tilde{y}$. We use Numerov scheme (1.2) to solve (4.1). Then we use the nonlinear Jacobi iteration (2.1) and nonlinear Gauss-Seidel iteration (2.2) for the resulting nonlinear system (1.2). In this case, this system has a supersolution $\tilde{Y} = (\tilde{y}(x_1), \cdots, \tilde{y}(x_{N-1}))^T$ with $\tilde{y}(x_i) \equiv 20$ for $1 \le i \le N-1$, and $\tilde{y}(x_0) = \tilde{y}(x_N) = 0$. It has a subsolution $\hat{Y} = (\hat{y}(x_1), \cdots, \hat{y}(x_{N-1}))^T$ with $\hat{y}(x_i) = 5 \sin i\pi h$ for $0 \le i \le N$. Also $-20 \le \dfrac{\partial f}{\partial z}(x_i, z) \le 20$, for $z \in \mathbf{K}(\hat{Y}, \tilde{Y})$ and $1 \le i \le N-1$. We take $N = 20$ in (1.2) and $\omega = -20$ in (2.1) and (2.2). Let $Y^{(m)}$ be the m'th iterated vector. If we take $Y^{(0)} = \tilde{Y}$, then we get decreasing sequences (see Fig. 4.3). If we take $Y^{(0)} = \hat{Y}$, then we get increasing sequences (see Fig. 4.4). Fig. 4.3 and Fig. 4.4 coincide with the theoretical analysis in Section 2. We find that the limits $\overline{Y} = \underline{Y}$ and so there is only one solution in $\mathbf{K}(\hat{Y}, \tilde{Y})$. Thus the condition for the uniqueness of solution in Theorem 2.1 and 2.3 are only a sufficient condition. Next we take $N = 10$. In the case $Y^{(0)} = \tilde{Y}$, for the iteration (2.1) and $m \ge 219$, (4.2) holds. We have the same result for the iteration (2.2) and $m \ge 128$. In the case $Y^{(0)} = \hat{Y}$, (4.2) holds for (2.1) with $m \ge 242$, and for (2.2) with $m \ge 141$. We find that $Y^{(m)}$ does not tend to the limit very fast, since $\max\limits_{z \in (\hat{Y}, \tilde{Y})} \dfrac{\partial f}{\partial z} = 20 > \pi^2$ which destroys the conditions of Theorems 2.2 and 2.4.

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