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QUADRILATERAL FINITE ELEMENTS FOR PLANAR LINEAR ELASTICITY PROBLEM WITH LARGE LAMÉ CONSTANT[*1)]

Xiao-liang Cheng
(*Department of Mathematics, Hangzhou University, P.R. China.*)
Hong-ci Huang
(*Department of Mathematics, Hong Kong Baptist University, Hong Kong.*)
Jun Zou
(*Department of Mathematics, The Chinese University of Hong Kong, Hong Kong.*)

Abstract

In this paper, we discuss the quadrilateral finite element approximation to the two-dimensional linear elasticity problem associated with a homogeneous isotropic elastic material. The optimal convergence of the finite element method is proved for both the L^2-norm and energy-norm, and in particular, the convergence is uniform with respect to the Lamé constant λ. Also the performance of the scheme does not deteriorate as the material becomes nearly incompressible. Numerical experiments are given which are consistent with our theory.

Key words: Planar linear elasticity, optimal error estimates, large Lamé constant, locking phenomenon

1. Planar linear elasticity problem

The two-dimensional linear elasticity problem associated with a homogeneous isotropic elastic material with pure displacements can be modelled by the following elliptic boundary value problem:

$$-\mu\Delta\vec{u} - (\mu+\lambda)\nabla(\text{div }\vec{u}) = \vec{f}, \qquad \text{in } \Omega, \tag{1.1}$$

$$\vec{u} = \vec{0}, \quad \text{on } \partial\Omega, \tag{1.2}$$

where $\Omega \subset \Re^2$ is an open and bounded domain, $\vec{u} = (u_1, u_2)$ the displacement, $\vec{f}(x)$ the body force, and λ, μ the Lamé constants. Different equivalent formulations of (1.1)–(1.2) can be found in [3, 4, 11].

It is well known that the convergence rate for the standard displacement method using continuous linear finite elements deteriorates as the Lamé constant λ becomes large, i.e., the elastic material is nearly incompressible. Many finite element methods of higher order have been proposed which work uniformly well for all λ, see [2, 3, 1, 14]. However, all these elements are required to satisfy the Babuska-Brezzi-Ladyzenskaja condition for saddle point problems.

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In this paper, we use the simplest finite elements, i.e. the bilinear elements, which do not satisfy the Babuska-Brezzi-Ladyzenskaja condition, for the above elasticity problem. The key technique here is to use the reduced integration to deal with the second term in the equation (1.1). We apply the finite element method studied in [5] and [8] and use a variant of the stability condition for some subspaces of the global finite element space. We are able to prove the optimal error estimates in both the energy-norm and L^2-norm uniformly with respect to the Lamé constant λ, thus the convergence rate does not deteriorate even for nearly incompressible material.

Before ending this section, we introduce some notation used in the paper. For any positive integer m, $H^m(\Omega)$ denotes the usual Sobolev space of all square integrable functions over Ω with square integrable derivatives of order up to m, and its norm and semi-norm are denoted by $\|\cdot\|_m$ and $|\cdot|_m$. $H_0^1(\Omega)$ is the subspace of $H^1(\Omega)$ with its functions vanishing on the boundary $\partial\Omega$ (in the sense of trace). $L_0^2(\Omega)$ is the space of all square integrable functions over Ω with their mean values in Ω vanishing.

2. The Bilinear Element Method

We first consider a very simple and regular domain in this section, i.e. the domain Ω is a rectangle. But we shall show in Section 4 that the method addressed in this section can be naturally extended to more general domains which may be triangulated using quadrilateral elements. As usual, we assume the Lamé constants μ, λ are in the following ranges $0 < \mu_0 \le \mu \le \mu_1$, $0 < \lambda < \infty$.

By Green's formula, it is easy to derive the weak formulation of the system (1.1)–(1.2):

Problem (P). *Find* $\vec{u} \in [H_0^1(\Omega)]^2$ *such that*

$$\mu(\nabla\vec{u}, \nabla\vec{v}) + (\mu+\lambda)(\mathrm{div}\ \vec{u}, \mathrm{div}\ \vec{v}) = (\vec{f}, \vec{v}), \qquad \forall \vec{v} \in [H_0^1(\Omega)]^2, \tag{2.1}$$

where $(\cdot,\cdot)$ *denotes the inner product in* $L^2(\Omega)$ *or* $[L^2(\Omega)]^2$.

Let $\mathcal{T}^h$ be a triangulation of the domain Ω into rectangular elements of mesh size h, which is obtained by refining a coarse rectangular mesh by dividing each coarse element into four subelements by linking the mid-points of the opposite edges of the coarse element. We then define the bilinear finite element space V_h by

$$V_h = \{\vec{v}_h \in [H_0^1(\Omega)]^2 : \ \vec{v}_{h|_K} \in [Q_1(K)]^2, \ \forall K \in \mathcal{T}^h \ \}, \tag{2.2}$$

where $Q_l(K)$ (l positive integer) is the space of polynomials of degree less than or equal to l in each variable on K.

Then the finite element problem to Problem (P) is formulated as follows:

Problem (P$_h$). *Find* $\vec{u}_h \in V_h$ *such that*

$$\mu(\nabla\vec{u}_h, \nabla\vec{v}_h) + (\mu+\lambda)I_1(\mathrm{div}\ \vec{u}_h, \mathrm{div}\ \vec{v}_h) = (\vec{f}, \vec{v}_h), \qquad \forall \vec{v}_h \in V_h, \tag{2.3}$$

where $I_1(\cdot,\cdot)$ *denotes the one-point Gaussian quadrature in each element, i.e.,*

$$I_1(\mathrm{div}\vec{u}_h, \mathrm{div}\vec{v}_h) = \sum_{K\in\mathcal{T}^h} |K|\,\mathrm{div}\ \vec{u}_h(q_K)\ \mathrm{div}\ \vec{v}_h(q_K)$$

$$= \sum_{K\in\mathcal{T}^h} \frac{1}{|K|} \int_K \text{div } \vec{u}_h\, dx \int_K \text{div } \vec{v}_h\, dx, \tag{2.4}$$

where q_K is the central point of the element K and $|K|$ the area of K.

Remark 1. Recently, [3] analyzed the conforming and nonconforming linear triangular finite element methods for the system (1.1)-(1.2). In the nonconforming case, they used the mid-point values on each edge of elements as degrees of freedom, thus compared with the finite elements here, the degrees of freedom needed in [3] are about three times of ours. In the conforming case, they used a projection mapping finite element functions into functions defined on larger coarse elements to deal with the second term in the equation (2.1), thus compared with the finite elements here, their stiffness matrix has a double bandwidth as ours.

We remark that if one uses the exact integration for the second term in (2.1), we have the following scheme:

Problem ($\mathbf{P}_h^e$). *Find $\vec{u}_h^e \in V_h$ such that*

$$\mu(\nabla\vec{u}_h^e, \nabla\vec{v}_h) + (\mu+\lambda)(\text{div } \vec{u}_h^e,\ \text{div } \vec{v}_h) = (\vec{f}, \vec{v}_h), \qquad \forall \vec{v}_h \in V_h. \tag{2.5}$$

By the standard finite element theory[9], we immediately come to the following error estimates

$$\begin{aligned}&\mu\|\nabla(\vec{u}-\vec{u}_h^e)\|_0 + (\mu+\lambda)\|\text{div } (\vec{u}-\vec{u}_h^e)\|_0\\ \le& \inf_{\vec{v}_h\in V_h}\Big(\mu\|\nabla(\vec{u}-\vec{v}_h)\|_0 + (\mu+\lambda)\|\text{div } (\vec{u}-\vec{v}_h)\|_0\Big) \le C_{\mu,\lambda}\, h\, |\vec{u}|_2,\end{aligned}$$

and $\|\vec{u}-\vec{u}_h^e\|_0 \le C_{\mu,\lambda} h^2 |\vec{u}|_2$, where $C_{\mu,\lambda}$ will be very large when λ tends to infinity. We will show numerically in Section 5 that the locking phenomenon occurs with this scheme when the Lamé constant λ becomes large.

3. Error Estimates of the Finite Element Schemes

To analyzed the error estimates of the finite element problem (P_h), we first introduce an equivalent formulation of (2.1). Let

$$p = (\mu+\lambda)\ \text{div } \vec{u}, \tag{3.1}$$

then (2.1) can be written in the following mixed formulation:

Problem ($\mathbf{P}^*$). *Find $(\vec{u},\ p) \in [H_0^1(\Omega)]^2 \times L_0^2(\Omega)$ such that*

$$\mu(\nabla\vec{u}, \nabla\vec{v}) + (p,\ \text{div } \vec{v}) = (\vec{f},\ \vec{v}), \qquad \forall \vec{v} \in [H_0^1(\Omega)]^2, \tag{3.2}$$

$$(\mu+\lambda)(\text{div } \vec{u}, q) - (p,\ q) = 0, \quad \forall q \in L_0^2(\Omega). \tag{3.3}$$

Now we introduce a piecewise constant finite element space:

$$Q_h = \{q_h \in L_0^2(\Omega) : q_{h|_K} \in Q_0(K), \quad \forall K \in \mathcal{T}^h\ \}, \tag{3.4}$$

and a local averaged function p_h:

$$p_h|_K = \frac{\mu+\lambda}{|K|} \int_K \text{div } \vec{u}_h\, dx \quad \forall\, K \in \mathcal{T}^h. \tag{3.5}$$

Using the above notation, one can prove immediately that the finite element problem $(\mathbf{P}_h)$ is equivalent to the following

Problem $(\mathbf{P}_h^*)$. *Find* $(\vec{u}_h, p_h) \in V_h \times Q_h$ *such that*

$$\mu(\nabla\vec{u}_h, \nabla\vec{v}_h) + (p_h,\ \text{div}\ \vec{v}_h) = (\vec{f},\ \vec{v}_h), \quad \forall\vec{v}_h \in V_h, \tag{3.6}$$

$$(\mu+\lambda)(\text{div}\ \vec{u}_h, q_h) - (p_h,\ q_h) = 0, \quad \forall q_h \in Q_h. \tag{3.7}$$

It is well known that the pair of finite element spaces (V_h, Q_h) does not satisfy the *inf-sup* condition[13,12,1,5]. Thus we can not apply the classical saddle point approximation theory here for the error estimates. Instead we are going to use the "local checkerboard pattern" technique to derive the error estimates.

Using notations in [8], we can construct a subspace $\hat{V}_h$ of V_h and a subspace $\hat{Q}_h$ of Q_h such that $Q_h = \hat{Q}_h \oplus \tilde{Q}_h$ and

$$\beta_0\|\hat{q}_h\|_0 \le \sup_{\hat{v}_h\in\hat{V}_h} \frac{(\text{div}\hat{v}_h, \hat{q}_h)}{|\hat{v}_h|_1}, \quad \forall\hat{q}_h \in \hat{Q}_h, \tag{3.8}$$

$$(\text{div}\ \hat{v}_h, \tilde{q}_h) = 0, \quad \forall\tilde{q}_h \in \tilde{Q}_h, \quad \hat{v}_h \in \hat{V}_h \tag{3.9}$$

where β_0 is a constant independent of h. The subspace $\tilde{Q}_h$ is called the "local checkerboard pattern". Let $\mathcal{T}^{2h}$ be the coarse triangulation with rectangular elements of size $2h$, from which $\mathcal{T}^h$ is obtained by dividing each element in $\mathcal{T}^{2h}$ into four elements of size h. Then $\tilde{Q}_h = \text{span}\,\{\phi_M^1 :\ M \in \mathcal{T}^{2h}\}$ with ϕ_M^1 defined by

−1	+1
+1	−1

ϕ_M^1

and the subspace $\hat{Q}_h = \text{span}\,\{\phi_M^2, \phi_M^3, \phi_M^4 :\ M \in \mathcal{T}^{2h}\}$ with $\phi_M^2, \phi_M^3, \phi_M^4$ defined by

+1	+1
+1	+1

ϕ_M^2

+1	+1
−1	−1

ϕ_M^3

−1	+1
−1	+1

ϕ_M^4

It is easy to see that $\hat{Q}_h$ and $\tilde{Q}_h$ are orthogonal in $L^2(\Omega)$. The choice of $\hat{V}_h$ is discussed in detail in [8]. We now only state some approximation properties of the two subspaces $\hat{Q}_h$ and $\hat{V}_h$. For any $\vec{u} \in [H^2(\Omega) \cap H_0^1(\Omega)]^2$ and $p \in H^1(\Omega) \cap L_0^2(\Omega)$, we define their projections $\hat{u}^I \in \hat{V}_h$ and $\hat{p}^I \in \hat{Q}_h$ by $(\nabla\hat{u}^I,\ \nabla v_h) = (\nabla\vec{u},\ \nabla v_h), \forall\, v_h \in \hat{V}_h$ and $(\hat{p}^I,\ q_h) = (p,\ q_h), \forall\, q_h \in \hat{Q}_h$. Then we have (cf. [8])

$$|\vec{u} - \hat{u}^I|_1 \le C\,h|\vec{u}|_2, \quad \|p - \hat{p}^I\|_0 \le C\,h|p|_1. \tag{3.10}$$

Our main results are stated in the following theorem.

Theorem 1. *Let $\vec{u}$ and $\vec{u}_h$ be the solutions of Problem (P) and Problem (P_h), respectively, then $|\vec{u} - \vec{u}_h|_1 \le Ch\|\vec{f}\|_0$, where C is independent of h and the Lamé constant λ.*

Proof. We analyze the approximation (P_h) to (P) from their equivalent forms, i.e. (P_h^*) and (P^*). From these equivalent formulations, we have

$$\begin{aligned}&\mu(\nabla(\vec{u}-\vec{u}_h),\nabla\vec{v}_h)+(p-p_h,\ \mathrm{div}\ \vec{v}_h)=0,\quad \forall\vec{v}_h\in V_h,\\&(\mu+\lambda)(\mathrm{div}\ (\vec{u}-\vec{u}_h),q_h)-(p-p_h,q_h)=0,\quad \forall q_h\in Q_h.\end{aligned}\tag{3.11}$$

Then for the projections $\hat{u}^I\in\hat{V}_h$ and $\hat{p}^I\in\hat{Q}_h$ of $\vec{u}$ and p, we have

$$\begin{aligned}\mu\|\nabla(\vec{u}-\vec{u}_h)\|_0^2&=\mu(\nabla(\vec{u}-\vec{u}_h),\nabla(\vec{u}-\hat{u}^I))+\mu(\nabla(\vec{u}-\vec{u}_h),\nabla(\hat{u}^I-\vec{u}_h)\\&\le C\,h\,|\vec{u}|_2\,\|\nabla(\vec{u}-\vec{u}_h)\|_0-(p-p_h,\ \mathrm{div}(\hat{u}^I-\vec{u}_h)),\end{aligned}$$

and

$$-(p-p_h,\ \mathrm{div}(\hat{u}^I-\vec{u}_h))=-(p-\hat{p}^I,\ \mathrm{div}(\hat{u}^I-\vec{u}_h))-(\hat{p}^I-p_h,\ \mathrm{div}\ (\hat{u}^I-\vec{u}_h)),$$

whose first term can be estimated directly from (3.10), namely

$$\begin{aligned}-(p-\hat{p}^I,\ \mathrm{div}\ (\hat{u}^I-\vec{u}_h))&\le 2\|p-\hat{p}^I\|_0\|\nabla(\hat{u}^I-\vec{u}_h)\|_0\\&\le Ch|p|_1\,(\|\nabla(\hat{u}^I-\vec{u})\|_0+\|\nabla(\vec{u}-\vec{u}_h)\|_0)\\&\le Ch\,|p|_1\,(h\,|\vec{u}|_2+\|\nabla(\vec{u}-\vec{u}_h)\|_0),\end{aligned}$$

and whose second term can be analysed as follows: using (3.8), (3.3) and (3.7), we denote

$$p_h=\hat{p}_h\oplus\tilde{p}_h,\quad \hat{p}_h\in\hat{Q}_h,\quad \tilde{p}_h\in\tilde{Q}_h,$$

then we get

$$\begin{aligned}-(\hat{p}^I-p_h,\mathrm{div}(\hat{u}^I-\vec{u}_h))&=-\,(\hat{p}^I-\hat{p}_h,\mathrm{div}\hat{u}^I)+(\hat{p}^I-p_h,\mathrm{div}\vec{u}_h)\\&=-\,(\hat{p}^I-\hat{p}_h,\mathrm{div}(\hat{u}^I-\vec{u}))-(\hat{p}^I-\hat{p}_h,\mathrm{div}\vec{u})+(\hat{p}^I-p_h,\mathrm{div}\vec{u}_h)\\&\le\|\hat{p}^I-\hat{p}_h\|_0\,\|\mathrm{div}(\hat{u}^I-\vec{u})\|_0-\frac{1}{\mu+\lambda}((p,\hat{p}^I-\hat{p}_h)-(p_h,\hat{p}^I-p_h))\\&\le C\,h\,\|\hat{p}^I-\hat{p}_h\|_0\,|\vec{u}|_2-\frac{1}{\mu+\lambda}(\hat{p}^I-p_h,\hat{p}^I-p_h).\end{aligned}$$

Using the last two estimates, we derive that

$$\begin{aligned}\mu\|\nabla(\vec{u}-\vec{u}_h)\|_0^2+\frac{1}{\mu+\lambda}\|\hat{p}^I-p_h\|_0^2\le&\,Ch(\|\nabla(\vec{u}-\vec{u}_h)\|_0+\|\hat{p}^I-\hat{p}_h\|_0)(|\vec{u}|_2+|p|_1)\\&+Ch^2\,|p|_1\,|\vec{u}|_2.\end{aligned}$$

But from (3.8) and (3.9), we know that

$$\begin{aligned}\|\hat{p}^I-\hat{p}_h\|_0&\le\frac{1}{\beta_0}\sup_{\hat{v}\in\hat{V}_h}\frac{(\mathrm{div}\ \hat{v},\hat{p}^I-\hat{p}_h)}{|\hat{v}|_1}\le C\,h\,|p|_1+\frac{1}{\beta_0}\sup_{\hat{v}\in\hat{V}_h}\frac{\mu(\nabla(\vec{u}-\vec{u}_h),\nabla\hat{v})}{|\hat{v}|_1}\\&\le C\,h\,|p|_1+C\,\|\nabla(\vec{u}-\vec{u}_h)\|_0.\end{aligned}\tag{3.12}$$

Thus we have

$$\mu\|\nabla(\vec{u}-\vec{u}_h)\|_0^2 + \frac{1}{\mu+\lambda}\|\hat{p}^I - p_h\|_0^2 \le C\,h^2\,(|\vec{u}|_2 + |p|_1)^2. \tag{3.13}$$

Using the estimate of the solution to the system (1.1)-(1.2) (cf. [3], p.325):

$$\|\vec{u}\|_2 + \|p\|_1 \le C\|\vec{f}\|_0 \tag{3.14}$$

where C is a constant independent of the Lamé constant λ, Theorem 1 follows from (3.13). □

Theorem 2. *Let $\vec{u}$ and $\vec{u}_h$ be the solutions of Problem (P) and Problem (P_h), respectively, then*

$$\|\vec{u}-\vec{u}_h\|_0 \le Ch^2\|\vec{f}\|_0, \tag{3.15}$$

where C is independent of h and the Lamé constant λ.

Proof. Introduce the following dual problem: find $\vec{\phi} \in [H_0^1(\Omega)]^2$ such that

$$\mu(\nabla\vec{\phi}, \nabla\vec{\psi}) + (\mu+\lambda)(\text{div}\ \vec{\phi}, \text{div}\ \vec{\psi}) = (\vec{g}, \vec{\psi}), \quad \forall\vec{\psi} \in [H_0^1(\Omega)]^2,$$

where $\vec{g} \in [L^2(\Omega)]^2$ is given. It is equivalent to the mixed formulation:

Find $(\vec{\phi},\ m) \in [H_0^1(\Omega)]^2 \times L_0^2(\Omega)$ such that

$$\mu(\nabla\vec{\phi}, \nabla\vec{\psi}) + (m,\ \text{div}\vec{\psi}) = (\vec{g},\ \vec{\psi}), \quad \forall\vec{\psi} \in [H_0^1(\Omega)]^2, \tag{3.16}$$

$$(\mu+\lambda)(\text{div}\ \vec{\phi}, n) - (m,\ n) = 0, \quad \forall n \in L_0^2(\Omega). \tag{3.17}$$

The following a priori estimates are known (see [3], p.325): $\|\vec{\phi}\|_2 + \|m\|_1 \le C\,\|\vec{g}\|_0$, where C is a constant independent of the Lamé constant λ.

Note that

$$\|\vec{u}-\vec{u}_h\|_0 = \sup_{\vec{g}\in[L^2(\Omega)]^2} \frac{(\vec{g}, \vec{u}-\vec{u}_h)}{\|\vec{g}\|_0}, \tag{3.18}$$

and

$$(\vec{g}, \vec{u}-\vec{u}_h) = \mu(\nabla\vec{\phi}, \nabla(\vec{u}-\vec{u}_h)) + (m,\ \text{div}(\vec{u}-\vec{u}_h)). \tag{3.19}$$

Let $\hat{\phi}^I$ and $\hat{m}^I$ be the projections of $\vec{\phi}$ and m similarly defined as the previous $\hat{u}^I$ and $\hat{p}^I$, then

$$\begin{aligned}
\mu(\nabla\vec{\phi}, \nabla(\vec{u}-\vec{u}_h)) =& \mu(\nabla(\vec{\phi}-\hat{\phi}^I), \nabla(\vec{u}-\vec{u}_h)) + \mu(\nabla\hat{\phi}^I, \nabla(\vec{u}-\vec{u}_h)) \\
=& \mu(\nabla(\vec{\phi}-\hat{\phi}^I), \nabla(\vec{u}-\vec{u}_h)) - (p-p_h,\ \text{div}\ \hat{\phi}^I) \\
=& \mu(\nabla(\vec{\phi}-\hat{\phi}^I), \nabla(\vec{u}-\vec{u}_h)) - (\hat{p}^I-\hat{p}_h,\ \text{div}\ \hat{\phi}^I) \\
=& \mu(\nabla(\vec{\phi}-\hat{\phi}^I), \nabla(\vec{u}-\vec{u}_h)) + (\hat{p}^I-\hat{p}_h,\ \text{div}\ (\vec{\phi}-\hat{\phi}^I)) \\
& - (\hat{p}^I-\hat{p}_h,\ \text{div}\ \vec{\phi}),
\end{aligned} \tag{3.20}$$

and from (3.11),

$$\begin{aligned}
(m,\ \text{div}(\vec{u}-\vec{u}_h)) &= (m-\hat{m}^I,\ \text{div}(\vec{u}-\vec{u}_h)) + (\hat{m}^I,\ \text{div}(\vec{u}-\vec{u}_h)) \\
&= (m-\hat{m}^I,\ \text{div}(\vec{u}-\vec{u}_h)) + \frac{1}{\mu+\lambda}(p-p_h, \hat{m}^I)
\end{aligned}$$

$$= (m - \hat{m}^I,\ \operatorname{div}(\vec{u} - \vec{u}_h)) + \frac{1}{\mu+\lambda}(\hat{p}^I - \hat{p}_h, \hat{m}^I), \tag{3.21}$$

$$(\hat{p}^I - \hat{p}_h, \operatorname{div}\vec{\phi}) = \frac{1}{\mu+\lambda}(\hat{p}^I - \hat{p}_h, m). \tag{3.22}$$

Then from (3.19)–(3.22) and (3.12), we have

$$\begin{aligned}(\vec{g}, \vec{u} - \vec{u}_h) =& \mu(\nabla(\vec{\phi} - \hat{\phi}^I), \nabla(\vec{u} - \vec{u}_h)) + (\hat{p}^I - \hat{p}_h,\ \operatorname{div}(\vec{\phi} - \hat{\phi}^I)) + (m - \hat{m}^I,\ \operatorname{div}(\vec{u} - \vec{u}_h)) \\ \le& C\, h^2\, \|\vec{f}\|_0(\|\vec{\phi}\|_2 + \|m\|_1) + C\, h\, \|\vec{\phi}\|_2\, \|\hat{p}^I - \hat{p}_h\|_0 \le C\, h^2\, \|\vec{f}\|_0\, \|\vec{g}\|_0,\end{aligned}$$

where the term $\|\hat{p}^I - \hat{p}_h\|_0$ is bounded by using (3.12) and Theorem 1. Now Theorem 2 follows from this and (3.18). □

4. Some Extensions

We have proved in the last section that the finite element scheme (P_h) is effective and able to overcome the locking phenomenon when the Lamé constant λ is large. But for small Lamé constant λ, the reduced integration used in (P_h) may affect the accuracy of the finite element solution. To find a finite element scheme which is effective both for large and small Lamé constant λ, we propose the following combined scheme of (P_h) and (P_h^e):

Problem (P_h^c). *Find $\vec{u}_h^c \in V_h$ such that*

$$\mu(\nabla\vec{u}_h^c, \nabla\vec{v}_h) + \alpha(\operatorname{div}\, \vec{u}_h^c, \operatorname{div}\, \vec{v}_h) + (\mu+\lambda-\alpha)I_1(\operatorname{div}\, \vec{u}_h^c, \operatorname{div}\, \vec{v}_h) = (\vec{f}, \vec{v}_h), \quad \forall \vec{v}_h \in V_h, \tag{4.1}$$

where $0 \le \alpha \le \mu + \lambda$ is a constant.

The choice of α is interesting. The practical rules may be set up on the basis of numerical experiments. In general, we choose small α if λ is large to avoid the locking. It seems that a suitable choice of α may increase the accuracy slightly in the numerical tests. The scheme (P_h^c) is more suitable for all $\lambda \in (0, \infty)$ than the schemes (P_h) or (P_h^e). By the method of [6, 10, 15], the $h-$version shows the locking of order h^{-2}. So we should let

$$\frac{\mu+\lambda-\alpha}{\alpha} \ge O(h^{-2}). \tag{4.2}$$

Using the bilinear form $\mu(\nabla\vec{u}_h^c,\ \nabla\vec{v}_h) + \alpha(\operatorname{div}\vec{u}_h^c,\ \operatorname{div}\, \vec{v}_h)$ instead of the bilinear form $\mu(\nabla\vec{u}_h,\ \nabla\vec{v}_h)$ and $p^c = (\mu+\lambda-\alpha)\operatorname{div}\vec{u}$ instead of $p = (\mu+\lambda)\operatorname{div}\vec{u}$ and

$$p_h^c|_K = \frac{\mu+\lambda-\alpha}{|K|}\int_K \operatorname{div}\, \vec{u}_h^c\, dx$$

instead of p_h defined in (3.5), we can establish the following optimal error estimates in the same way as for Theorems 1 & 2:

Theorem 3. *Let $\vec{u}$ and $\vec{u}_h^c$ be the solutions of Problem (P) and Problem (P_h^c), respectively, then*

$$|\vec{u} - \vec{u}_h^c|_1 \le Ch\|\vec{f}\|_0, \quad \|\vec{u} - \vec{u}_h^c\|_0 \le Ch^2\|\vec{f}\|_0, \tag{4.3}$$

where C is independent of h and λ but only depending on α.

If the domain Ω is a general two dimensional convex polygon instead of the rectangular domain discussed so far, we can extend the considered methods, e.g. the scheme (P_h), as follows. We assume the quadrilateral triangulation $\mathcal{T}^h$ is achieved from a quadrilateral coarse mesh by dividing each coarse element into four quadrilaterals by linking the midpoints of the opposite edges of the coarse element. Each element of $\mathcal{T}^h$ is assumed to be a convex quadrilateral of mesh size h.

Let $K \in \mathcal{T}^h$ be a quadrilateral with four nodes $A_j(x_1^j, x_2^j)$, $j = 1, 2, 3, 4$, then there exists a bilinear mapping $F_K : \hat{K} \to K$, with $\hat{K} = [-1, 1] \times [-1, 1]$ the reference element in $\xi\eta$-coordinate, defined by

$$\begin{aligned} x_1^K(\xi,\eta) =& \frac{1}{4}(1+\xi)(1+\eta)x_1^1 + \frac{1}{4}(1-\xi)(1+\eta)x_1^2 \\ &+ \frac{1}{4}(1-\xi)(1-\eta)x_1^3 + \frac{1}{4}(1+\xi)(1-\eta)x_1^4, \\ x_2^K(\xi,\eta) =& \frac{1}{4}(1+\xi)(1+\eta)x_2^1 + \frac{1}{4}(1-\xi)(1+\eta)x_2^2 \\ &+ \frac{1}{4}(1-\xi)(1-\eta)x_2^3 + \frac{1}{4}(1+\xi)(1-\eta)x_2^4. \end{aligned}$$

Then we can define the quadrilateral finite element space

$$V_h = \Big\{(v_1, v_2) \in [H_0^1(\Omega)]^2 : \ v_j|_K = \hat{v}_j \circ F_K^{-1}, \ \hat{v}_j \in Q_1(\hat{K}), \ j = 1, 2, \ \forall K \in \mathcal{T}^h \Big\}. \tag{4.4}$$

Then the finite element problem to the variational problem (P) is formulated as:

Problem ($\mathbf{I}_h$). *Find $\vec{u}_h \in V_h$ such that $\mu(\nabla\vec{u}_h, \nabla\vec{v}_h) + (\mu+\lambda)I_1(\text{div}\ \vec{u}_h, \text{div}\ \vec{v}_h) = (\vec{f}, \vec{v}_h)$, $\forall \vec{v}_h \in V_h$. Here $I_1(\cdot,\cdot)$ denotes the one-point numerical quadrature in each element, i.e.,*

$$\begin{aligned} I_1(\text{div}\ \vec{u}_h, \text{div}\ \vec{v}_h) &= \sum_{K\in\mathcal{T}^h} |K|\, \text{div}\ \vec{u}_h(q_K)\, \text{div}\ \vec{v}_h(q_K) \\ &= \sum_{K\in\mathcal{T}^h} \frac{1}{|K|} \int_K \text{div}\ \vec{u}_h\, dx \int_K \text{div}\ \vec{v}_h\, dx, \end{aligned} \tag{4.5}$$

where q_K is the centroid point of the element K. The second equality is valid for general quadrilaterals. In fact, by the definition of the space V_h in (4.4), for any $\vec{w} = (w_1, w_2) \in V_h|_K$, we have

$$\int_K \text{div}\ \ \vec{w} dx_1 dx_2 = \int_{\hat{K}} \Big(\frac{\partial\hat{w}_1}{\partial\xi}\frac{\partial\xi}{\partial x_1^K} + \frac{\partial\hat{w}_1}{\partial\eta}\frac{\partial\eta}{\partial x_1^K} + \frac{\partial\hat{w}_2}{\partial\xi}\frac{\partial\xi}{\partial x_2^K} + \frac{\partial\hat{w}_2}{\partial\eta}\frac{\partial\eta}{\partial x_2^K} \Big) |J_K| d\xi d\eta,$$

where $\hat{w}_j = w_j \circ F_K, j = 1, 2$ and J_K is the Jacobian matrix

$$J_K = \begin{pmatrix} \partial x_1^K/\partial\xi & \partial x_1^K/\partial\eta \\ \partial x_2^K/\partial\xi & \partial x_2^K/\partial\eta \end{pmatrix} \tag{4.6}$$

We know that (for one term as an example)

$$\frac{\partial\hat{w}_1}{\partial\xi}\frac{\partial\xi}{\partial x_1^K}|J_K| = \frac{\partial\hat{w}_1}{\partial\xi}\frac{\partial x_2^K}{\partial\eta} \in Q_1(\hat{K}). \tag{4.7}$$

Since the one–point numerical quadrature is exact for all functions in $Q_1(\hat{K})$ and $F_K(0,0) = q_K$, the second equation in (4.5) is true.

From [5], there still exist spaces $\hat{V}_h$, $\hat{Q}_h$ and $\tilde{Q}_h$ such that (3.8) and (3.9) are satisfied. So we can establish the same optimal error estimates as those stated in §3.

5. A Numerical Experiment

In this section, we show some numerical experiments for the finite element schemes considered in Section 2. We borrow the example used in [2]: the domain Ω is a unit square, and $\mu = 1$, $\vec{f} = (f_1, f_2)$ is taken to be

$$f_1 = \pi^2\Big[4\sin 2\pi y(-1 + 2\cos 2\pi x) - \cos\pi(x+y) + \frac{2}{1+\lambda}\sin\pi x\sin\pi y\Big],$$
$$f_2 = \pi^2\Big[4\sin 2\pi x(-1 + 2\cos 2\pi y) - \cos\pi(x+y) + \frac{2}{1+\lambda}\sin\pi x\sin\pi y\Big].$$

The exact solution $\vec{u} = (u_1, u_2)$ is

$$u_1 = \sin 2\pi y(-1 + \cos 2\pi x) + \frac{1}{1+\lambda}\sin\pi x\sin\pi y,$$
$$u_2 = \sin 2\pi x(-1 + \cos 2\pi y) + \frac{1}{1+\lambda}\sin\pi x\sin\pi y.$$

We divide Ω into $(N+1)^2$ squares and $h = \dfrac{1}{N+1}$, then (2.3) can be written as

$$[\mu A + (\lambda+\mu)B]U_h = F_h,$$

and (2.5) as

$$[\mu A + (\lambda+\mu)C]U_h^e = F_h.$$

Table 1 shows the numerical results for the finite element scheme (P_h), we can see the optimal convergence in L^2-norm clearly. Table 2 shows the numerical results of the scheme (P_h^e) with exact integration. We see the locking phenomenon occurs when the Lamé constant λ becomes large. In fact, by calculation, for $h = 1/10$ we have rank$(B) = 98$ and rank$(C) = 162$. The matrix C is full rank and thus nonsingular, so the locking occurs for the scheme (P_h^e) when λ is large. Finally for the combined scheme (P_h^c), we can expect a slightly more accurate solution with an appropriately chosen α. For example, when $h = 1/20$ and $\lambda = 999$, $\|\vec{u} - \vec{u}_h\|_0 = 0.0285$ and $\|\vec{u} - \vec{u}_h^c\|_0 = 0.0068$ for $\alpha = 10$, so the combined scheme gives a better result.

Table 1 The error $\|\vec{u} - \vec{u}_h\|_0$ and the relative error $\dfrac{\|\vec{u} - \vec{u}_h\|_0}{\|\vec{u}\|_0}$

h	$\lambda = 9$	$\lambda = 99$	$\lambda = 999$	$\lambda = 9999$
$\frac{1}{8}$	0.1839 (10.62%)	0.1848 (10.67%)	0.1849 (10.68%)	0.1849 (10.68%)
$\frac{1}{16}$	0.0445 (2.57%)	0.0447 (2.58%)	0.0448 (2.59%)	0.0448 (2.59%)
$\frac{1}{32}$	0.0111 (0.64%)	0.0111 (0.64%)	0.0111 (0.64%)	0.0111 (0.64%)
$\frac{1}{64}$	0.0027 (0.16%)	0.0027 (0.16%)	0.0027 (0.16%)	0.0027 (0.16%)

Table 2 The error $\|\vec{u}-\vec{u}_h^e\|_0$ and the relative error $\frac{\|\vec{u}-\vec{u}_h^e\|_0}{\|\vec{u}\|_0}$

h	$\lambda=9$	$\lambda=99$	$\lambda=999$	$\lambda=9999$
$\frac{1}{8}$	0.0319 (1.84%)	0.8829 (50.98%)	1.5914 (91.88%)	1.7169 (99.13%)
$\frac{1}{16}$	0.0101 (0.58%)	0.3825 (22.08%)	1.3037 (75.27%)	1.6772 (96.84%)
$\frac{1}{32}$	0.0027 (0.16%)	0.1171 (0.67%)	0.7586 (43.80%)	1.5354 (88.65%)
$\frac{1}{64}$	0.0007 (0.01%)	0.0301 (1.74%)	0.2773 (16.01%)	1.1362 (65.60%)

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