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COMPUTATION OF HOPF BRANCHES BIFURCATING FROM A HOPF/PITCHFORK POINT FOR PROBLEMS WITH Z_2–SYMMETRY*[1)]

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Abstract

This paper is concerned with the computation of Hopf branches emanating from a Hopf/Pitchfork point in a two-parameter nonlinear problem satisfying a Z_2-symmetry condition. Our aim is to present a new approach to the theoretical and computational analysis of the bifurcating Hopf branches at this singular point by using the system designed to calculate Hopf points and exploring its symmetry. It is shown that a Hopf/Pitchfork point is a pitchfork bifurcation point in the system. Hence standard continuation and branch-switching can be used to compute these Hopf branches. In addition, an effect method based on the extended system of the singular points is developed for the computation of branch of secondary (non-symmetric) Hopf points. The implementation of Newton's method for solving the extended system is also discussed. A numerical example is given.

Key words: Hopf/pitchfork point, Z_2-symmetry, Hopf point, bifurcation, Extended system

1. Introduction

This paper is devoted to the calculation of branches of Hopf points which emanate from a certain singular point of a two parameter nonlinear system

$$g(x, \lambda, \alpha) = 0 \quad g : X \times R \times R \to X \tag{1.1}$$

where X is a real Hilbert space, λ a bifurcation parameter, α an additional control parameter, and g is a smooth mapping. We assume

(H1) g is Z_2-symmetric: there exists a linear operator $s : X \to X$ satisfying (I: identical operator in X)

$$s \neq I, s^2 = I, sg(x, \lambda, \alpha) = g(sx, \lambda, \alpha) \text{ for } (x, \lambda, \alpha) \in X \times R^2. \tag{1.2}$$

It is well known that (1.2) induces the splitting

$$X = X_s \oplus X_a, \tag{1.3a}$$

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$$X_s := \{x \in X, sx = x\}, \quad X_a := \{x \in X, sx = -x\}. \tag{1.3b}$$

We say that x is symmetric if $x \in X_s$, and antisymmetric if $x \in X_a$.

Equation (1.1) is often studied as a first step towards the understanding of the evolution equation

$$\frac{dx}{dt} = g(x, \lambda, \alpha). \tag{1.4}$$

In particular, the transition from steady-state solutions to periodic solutions in (1.4) typically occurs at a Hopf point. Such a point is usually recognized by the occurrence of a pair of purely imaginary eigenvalues of g_x, the linearization of g with respect to x of the steady-state equation (1.1).

A Hopf/Pitchfork point (HP-point) $(x_0, \lambda_0, \alpha_0)$ is defined as a solution with $x_0 \in X_s$ of (1.1) where $g_x^0 := g_x(x_0, \lambda_0, \alpha_0)$ has a pair of simple pure imaginary eigenvalues, and a simple zero eigenvalue with an antisymmetric eigenvector. In the case of $X = R^n$ and $g(0, \lambda, \alpha) = 0$ for all λ and α, the coalescence of pitchfork and Hopf bifurcation points for a two-parameter system with Z_2-symmetry has been investigated by Langford and Iooss[8], Guckenheimer and Holmes[5] using Birkhoff normal form, respectively. They found interesting secondary Hopf bifurcation (Hopf bifurcation on the bifurcating non-symmetric steady-state solution branch) and, in addition, aperiodic (chaotic) motion from a periodic orbit in a neighborhood of the degenerate point.

In this paper, we will contribute to the analysis and the computation of Hopf points near a HP-point, which forms a foundation for understanding the complex dynamics of (1.4). The main analytical and numerical tool will be the following extended system (1.5a, b) for Hopf points which was given by Jepson[6], Griewank and Reddien[3]; here α will be used as a bifurcation parameter:

$$G(x, \phi_1, \phi_2, \lambda, \beta, \alpha) := \left\{ \begin{array}{c} g(x, \lambda, \alpha) \\ g_x(x, \lambda, \alpha)\phi_1 - \beta\phi_2 \\ g_x(x, \lambda, \alpha)\phi_2 + \beta\phi_1 \\ \langle l, \phi_1 \rangle - 1 \\ \langle l, \phi_2 \rangle \end{array} \right\} = 0, \tag{1.5a}$$

$$G : X \times X \times X \times R \times R \times R \to X \times X \times X \times R \times R := Y. \tag{1.5b}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in X and $l \in X$ is a normalizing vector. The system has already been used to study bifurcation phenomena near a Takens-Bogdanov point in the cases where symmetry is not broken (Spence, Cliffe and Jepson [10]) and where symmetry is broken (Wu, Spence and Cliffe[15]), respectively. Following the preliminaries of Section 2, we devote ourself to a straightforward analysis of the fact that, under some non-degenerate condition, a HP-point is a pitchfork bifurcation point in the system $G = 0$ with respect to α, see Section 3. The analysis relies on the result on symmetry breaking pitchfork bifurcation in Werner and Spence[12] by defining a symmetry of (1.5) similar to (1.2). One theoretical by-product of the analysis is that it gives a self-contained proof for the existence of the branch of secondary (non-symmetric) Hopf points emanating from a HP-point. This is also useful for numerical computations -for Hopf points on the symmetric and on the non-symmetric steady-state branches as well. Our treatment is different from those of the papers[5,8] and allows $x_0 \neq 0$ in a HP-point.

We will give in Section 4 an extended system determining HP-points, show how to implement Newton's method to solve it efficiently and point out how to jump onto the branch of secondary Hopf points by the system. In Section 5 we will illustrate our method with an example where we compute symmetric and non-symmetric Hopf branches intersecting at a HP-point.

2. Preliminaries

Throughout the paper, we assume, in addition to hypothesis (H1), that the following hypotheses hold.

(H2) There exists a solution $(x_0, \lambda_0, \alpha_0)$ with symmetric x_0 of (1.1) such that the operator $g_x^0(:= g_x(x_0, \lambda_0, \alpha_0))$ has algebraically simple eigenvalues 0 and $\pm i\beta_0$ ($i = \sqrt{-1}$, $\beta_0 > 0$), and no other eigenvalue on the imaginary axis. In addition, the eigenvector ϕ^0 corresponding to the eigenvalue 0 is antisymmetric, i.e. $\phi^0 \in X_a$.

It follows from H1-H2 that there exist $\delta_0 > 0$ and a continuously differentiable function $\hat{x}$ from $D_{\delta_0} = \{(\lambda, \mu) : (\lambda, \mu) \in R^2,\ |\lambda - \lambda_0| < \delta_0,\ |\alpha - \alpha_0| < \delta_0\}$ into X_s such that for fixed $\alpha \in (\alpha_0 - \delta_0,\ \alpha_0 + \delta_0)$, the curve

$$c^s(\alpha) := \{(\hat{x}(\lambda, \alpha), \lambda) : \lambda_0 - \delta_0 < \lambda < \lambda_0 + \delta_0\} \tag{2.1}$$

is a unique branch of symmetric solutions of equation (1.1) near the HP-point. Let $A(\lambda, \alpha) = g_x(\hat{x}(\lambda, \alpha), \lambda, \alpha)$. Then $A(\lambda, \alpha)$ has continuously differentiable eigenvalues $\gamma(\lambda, \alpha) \pm i\beta(\lambda, \alpha)$ and $\sigma(\lambda, \alpha)$ satisfying $\gamma(\lambda_0, \alpha_0) = 0 = \sigma(\lambda_0, \alpha_0)$ and $\beta(\lambda_0, \alpha_0) = \beta_0$. These eigenvalues are unique for (λ, α) in some neighborhood of (λ_0, α_0). We also assume that the steady-state solution $\hat{x}(\lambda, \alpha_0)$ loses stability with respect to steady-state and Hopf bifurcation simultaneously when λ goes beyond λ_0:

(H3) $\sigma_\lambda := \dfrac{\partial \sigma(\lambda, \alpha)}{\partial \lambda}\Big|_{(\lambda_0, \alpha_0)} > 0, \qquad \gamma_\lambda := \dfrac{\partial \gamma(\lambda, \alpha)}{\partial \lambda}\Big|_{(\lambda_0, \alpha_0)} > 0,$

and employ natural generalization of the Hopf condition[8]:

(H4) $\Big|\dfrac{\partial(\gamma, \sigma)}{\partial(\lambda, \alpha)}\Big|_{(\lambda_0, \alpha_0)} \neq 0.$

For the bifurcation analysis of system (1.5a,b) near a HP-point, we give the representations of $\gamma_\lambda, \sigma_\lambda, \gamma_\alpha$, σ_α, β_λ and β_α ($\gamma_\alpha, \sigma_\alpha$, β_λ and β_α are defined similar to γ_λ and σ_λ) in terms of eigenvectors and conjugate eigenvectors of g_x^0 with respect to the eigenvalues 0 and $\pm i\beta_0$.

Let ϕ^0, φ^0 span the null spaces of g_x^0, $[g_x^0]^*$ ($[g_x^0]^*$ denote the dual of operator g_x^0, etc.), respectively. Under hypothesis (H2), the two vectors may be chosen such that $< \varphi^0, \phi^0 >= 1$. Then we obtain

$$\sigma_\lambda = \langle \varphi^0, (g_{xx}^0 e^0 + g_{x\lambda}^0)\phi^0 \rangle, \sigma_\alpha = \langle \varphi^0, (g_{xx}^0 f^0 + g_{x\alpha}^0)\phi^0 \rangle, \tag{2.2}$$

where $e^0(f^0) \in X_s$ is the unique solution of the following equation

$$g_x^0 z + g_\lambda^0(g_\alpha^0) = 0. \tag{2.3}$$

Furthermore, we have[1,12]:

Proposition 2.1. *Let hypotheses* $(H1)$-$(H3)$ *be satisfied at a point* $x_0 \in X_s$, *and* λ_0, α_0 *in* R. *Then there exist* $\delta_1 > 0$ *such that for fixed* $\alpha \in (\alpha_0 - \delta_1, \alpha_0 + \delta_1)$ *a unique branch* $c^a(\alpha)$ *of non-symmetric solutions of* (1.1) *bifurcates from the branch* $c^s(\alpha)$ *of symmetric solutions of* (1.1) *near the HP-point.*

Let $\phi_0 = \phi_0^1 + i\phi_0^2$ be an eigenvector corresponding to the simple imaginary eigenvalue $-\beta_0 i$ of g_x^0. Note that both ϕ_0^1 and ϕ_0^2 lie either in X_s or in X_a simultaneously. We might assume

$$\phi_0^1 \text{ and } \phi_0^2 \in X_s; \tag{2.4}$$

the other case may be discussed in a similar way.

Let

$$\text{Null } ([g_x^0 + i\beta_0 I]^*) = \text{span } \{\varphi_0 = \varphi_0^1 + i\varphi_0^2\}. \tag{2.5}$$

The eigenvector φ_0 can be chosen such that[2,9]

$$\langle \varphi_0^1 + i\varphi_0^2, \phi_0 \rangle = 2, \qquad \langle \varphi_0^1 - i\varphi_0^2, \phi_0 \rangle = 0 \tag{2.6}$$

From (2.6) we obtain

$$\langle \varphi_0^1, \phi_0^1 \rangle = 1, \langle \varphi_0^2, \phi_0^2 \rangle = 1, \langle \varphi_0^2, \phi_0^1 \rangle = 0, \langle \varphi_0^1, \phi_0^2 \rangle = 0. \tag{2.7}$$

Consequently, by using a method similar to that of the Appendix of Roose and Hlavaček[9], we obtain

$$\gamma_\lambda = \frac{1}{2}(\langle \varphi_0^1, (g_{xx}^0 e^0 + g_{x\lambda}^0)\phi_0^1 \rangle + \langle \varphi_0^2, (g_{xx}^0 e^0 + g_{x\lambda}^0)\phi_0^2 \rangle), \tag{2.8a}$$

$$\gamma_\alpha = \frac{1}{2}(\langle \varphi_0^1, (g_{xx}^0 f^0 + g_{x\alpha}^0)\phi_0^1 \rangle + \langle \varphi_0^2, (g_{xx}^0 f^0 + g_{x\alpha}^0)\phi_0^2 \rangle), \tag{2.8b}$$

$$\beta_\lambda = \frac{1}{2}(\langle \varphi_0^2, (g_{xx}^0 e^0 + g_{x\lambda}^0)\phi_0^1 \rangle - \langle \varphi_0^1, (g_{xx}^0 e^0 + g_{x\lambda}^0)\phi_0^2 \rangle), \tag{2.8c}$$

$$\beta_\alpha = \frac{1}{2}(\langle \varphi_0^2, (g_{xx}^0 f^0 + g_{x\alpha}^0)\phi_0^1 \rangle - \langle \varphi_0^1, (g_{xx}^0 f^0 + g_{x\alpha}^0)\phi_0^2 \rangle). \tag{2.8d}$$

In addition, since $-i\beta$ is an algebraically simple eigenvalue of g_x^0, we have

$$\langle \varphi_0, (x_1 + ix_2) \rangle = 0, \ \forall \ x_j \in X_a (j = 1, 2). \tag{2.9}$$

3. Bifurcation Analysis

In this section, we will define a symmetry on $Y = X \times X \times X \times R \times R$ to ensure that G in (1.5a,b) satisfies a symmetry relation of the same type as (1.2) and recapitulate some standard theory[12]. Let us make precise the requirements on l, the normalizing function in (1.5a). We require that, for $\phi_0^j (j = 1, 2)$ where (2.4) holds, l satisfy:

$$\langle l, s\cdot \rangle = \langle l, \cdot \rangle, \qquad \langle l, \phi_0^1 \rangle - 1 = 0, \qquad \langle l, \phi_0^2 \rangle = 0. \tag{3.1}$$

This condition is not restrictive. In fact for any $\tilde{l}$ satisfying $\langle \tilde{l}, \phi_0^1\rangle - 1 = 0$, $\langle \tilde{l}, \phi_0^2\rangle = 0$, we can choose l according to the following formula (see papers[14−16] for similar discussions)

$$\langle l, \cdot\rangle = \frac{\langle \tilde{l}, (I+s)\cdot\rangle}{\langle 2\tilde{l}, \phi_0^1\rangle}$$

which clearly satisfies (3.1).

Our first result is that G inherits the symmetry of g.

Proposition 3.1. *For $y = (x, \phi_1, \phi_2, \lambda, \beta) \in Y$, define a linear operator $S : Y \to Y$ by*

$$Sy = (sx, s\phi_1, s\phi_2, \lambda, \beta). \tag{3.2a}$$

Assume $(H1)$ *and* (3.1). *Then*

$$S \neq I_1, \qquad S^2 = I_1, \qquad SG(y, \alpha) = G(Sy, \alpha) \ \ \textit{for all} \ \ (y, \alpha) \in Y \times R \tag{3.2b}$$

where I_1 is the identity on Y.

Accordingly, we may split Y into

$$Y = Y_S \oplus Y_A \tag{3.3}$$

where $Y_S = X_s \times X_s \times X_s \times R \times R$ and $Y_A = X_a \times X_a \times X_a \times \{0\} \times \{0\}$. Let $y_0 := (x_0, \phi_0^1, \phi_0^2, \lambda_0, \beta_0)$ and let l be chosen such that (3.1) holds. Then $G(y_0, \alpha_0) = 0$ with $y_0 \in Y_S$ and we have the following:

Proposition 3.2. *Let $(x_0, \lambda_0, \alpha_0)$ be a HP-point satisfying hypotheses $(H1)$-$(H3)$ and assume that* (3.1) *holds. Then (with $G_y^0 := G_y(y_0, \alpha_0)$)*

$$\textit{Null}\ (G_y^0) = \textit{span}\ \{\Phi_0\}, \qquad \Phi_0 = (\phi^0, u_0, v_0, 0, 0) \in Y_A$$

and

$$\textit{Null}\ ([G_y^0]^*) = \textit{span}\ \{\Psi_0\}, \qquad \Psi_0 = (\varphi^0, 0, 0, 0, 0)$$

where $u_0 + iv_0 \in X_a + iX_a$ is the unique solution of equation (3.9).

Proof. Let $y = (x, u, v, s, t) \in Y$. Then $y \in$ Null (G_y^0) iff

$$g_x^0 x + s g_\lambda^0 = 0, \tag{3.4a}$$
$$(g_x^0 + i\beta_0 I)(u + iv) = -(g_{xx}^0 x + s g_{x\lambda}^0 + itI)(\phi_0^1 + i\phi_0^2), \tag{3.4b}$$
$$\langle l, u\rangle = 0, \tag{3.4c}$$
$$\langle l, v\rangle = 0. \tag{3.4d}$$

Under hypotheses (H1)-(H2), we obtain from (2.3) and (3.4a)

$$x = k\phi^0 + se^0 \tag{3.5}$$

with some $k \in R$. Substituting (3.5) into (3.4b) leads to

$$(g_x^0 + i\beta_0 I)(u + iv) = -[kg_{xx}^0\phi^0 + s(g_{xx}^0 e^0 + g_{x\lambda}^0) + itI](\phi_0^1 + i\phi_0^2). \tag{3.6}$$

Let us consider first the set of following homogeneous equations in unknown $\tilde{u}, \tilde{v}, \tilde{s}, \tilde{t}$:

$$(g_x^0 + i\beta_0 I)(\tilde{u} + i\tilde{v}) = -[\tilde{s}(g_{xx}^0 e^0 + g_{x\lambda}^0) + i\tilde{t}I](\phi_0^1 + i\phi_0^2), \tag{3.7a}$$

$$\langle l, \tilde{u}\rangle = 0, \tag{3.7b}$$

$$\langle l, \tilde{v}\rangle = 0. \tag{3.7c}$$

Taking inner product with φ_0 in (3.7a) and using (2.8), one obtains $2\gamma_\lambda\tilde{s} = 0$, $2\tilde{t} - 2\beta_\lambda\tilde{s} = 0$. Under hypothesis (H3), one obtains $\tilde{s} = \tilde{t} = 0$. Substituting these $\tilde{s}$ and $\tilde{t}$ into (3.7a) and combining with (3.7b,c), by the choice of l, we have $\tilde{u} = \tilde{v} = 0$. Hence the set of homogeneous equations (3.7a-c) has only the trivial solution.

Let (u_0, v_0, s_0, t_0) be the solution of the nonhomogenous equations (3.6) with $k = 1$ and (3.4c,d). By the previous discussion, we see that (u_0, v_0, s_0, t_0) is uniquely determined. Under hypotheses (H1) and (H2), from (2.4) one deduces

$$g_{xx}^0\phi^0\phi_0^j \in X_a, \quad j = 1, 2. \tag{3.8}$$

Therefore, we have $s_0 = t_0 = 0$ by using (2.9), and equation (3.6) ($k = 1$) becomes

$$(g_x^0 + i\beta_0 I)(u_0 + iv_0) = -(g_{xx}^0\phi^0\phi_0^1 + ig_{xx}^0\phi^0\phi_0^2). \tag{3.9}$$

Since the restriction $g_x^0 + i\beta_0 I|_{X_a+iX_a}$ is nonsingular, from (3.9) we can uniquely obtain the special solution $u_{01} + iv_{01} \in X_a + iX_a$ to (3.9) and the general solutions of (3.9) may be written as $u_0 + iv_0 = (a + ib)(\phi_0^1 + i\phi_0^2) + u_{01} + iv_{01}$ where $a, b \in R$. Substituting the expression into (3.4c) and (3.4d) and using (3.1) lead to $a = b = 0$. Therefore $u_0 + iv_0 = u_{01} + iv_{01}$. Finally, the only nontrivial solution to equations (3.4a-d) for $k = 1$ is essentially given by $\Phi_0 = (\phi^0, u_0, v_0, 0, 0) \in Y_A$.

The proof of Null $([G_y^0]^*) =$ span $\{(\varphi^0, 0, 0, 0, 0)\}$ is similar. $\diamond$

A immediate consequence of this proposition is:

Theorem 3.3. *Under the same assumptions as for Proposition* 3.2, *there exists locally one solution branch $C_H^S(\alpha)$ in $Y_S \times R$ of* (1.5*a*, *b*) *passing through* $(y_0, \alpha_0) = (x_0, \phi_0^1, \phi_0^2, \lambda_0, \beta_0, \alpha_0)$. *It can be parameterized by* $(\tilde{y}(\alpha), \alpha) = (\tilde{x}(\alpha), \tilde{\phi}_1(\alpha), \tilde{\phi}_2(\alpha), \tilde{\lambda}(\alpha), \tilde{\beta}(\alpha), \alpha)$ *with tangent vector* $(T_0, 1)$ *at* (y_0, α_0) *where* $T_0 \in Y_S$ *is defined by*

$$G_y^0 T_0 + G_\alpha^0 = 0. \tag{3.10}$$

Proof. Note that $y_0 \in Y_S$ and that Y_S is an invariant subspace in the sense that $G_y^0 u \in Y_S, \forall u \in Y_S$. With Proposition 3.2, the restriction $G_y^0|_{Y_S}$ is nonsingular, and so the implicit function theorem implies the existence and uniqueness of $\tilde{y}(\alpha) \in Y_S$ for each α near α_0. The last statement of the theorem follows from evaluating the following equation at $\alpha = \alpha_0$

$$\frac{d}{d\alpha}G(\tilde{y}(\alpha), \alpha) = G_y^0\dot{\tilde{y}} + G_\alpha = 0, \dot{\tilde{y}} \in Y_S. \tag{3.11}$$

This ends the proof. $\diamond$

From the branch $C_H^S(\alpha)$ of solutions in $Y_S \times R$ of (1.5a,b), one can get the branch $c_h^s(\alpha) = (\tilde{x}(\alpha), \tilde{\lambda}(\alpha), \alpha)$ of symmetric Hopf points of (1.1).

The question whether -by variation of α-there also arise Hopf points on the bifurcating branch $c^a(\alpha)$ of non-symmetric solutions of (1.1), can be attacked by means of the following theorem based on Proposition 3.2.

Theorem 3.4. *Let* $(x_0, \lambda_0, \alpha_0)$ *be a HP-point satisfying hypotheses* $(H1)$-$(H4)$ *and let* l *in* $(1.5a, b)$ *be chosen such that* (3.1) *holds. Then* (y_0, α_0) *is a* S*-breaking pitchfork bifurcation point in system* $(1.5a, b)$.

Proof. By Proposition 3.2, we see firstly that (y_0, α_0) is a simple S-breaking singular point in $G(y, \alpha) = 0$. The pitchfork non-degeneracy[10−12] is given by

$$\frac{d}{d\alpha} \ll \Psi_0, G_y(\tilde{x}(\alpha), \tilde{\phi}_1(\alpha), \tilde{\phi}_2(\alpha), \tilde{\lambda}(\alpha), \tilde{\beta}(\alpha), \alpha)\Phi_0 \gg |_{\alpha=\alpha_0} \neq 0. \tag{3.12}$$

where $\ll \cdot, \cdot \gg$ denotes the corresponding inner product in Y. Since

$$\ll \Psi_0, G_y(.)\Phi_0 \gg = \langle \varphi_0, g_x(.)\phi_0 \rangle$$

inequality (3.12) becomes

$$\Delta := \langle \varphi_0, (g^0_{xx}\dot{\tilde{x}}(\alpha_0) + g^0_{x\lambda}\dot{\tilde{\lambda}}(\alpha_0) + g^0_{x\alpha})\phi_0 \rangle \neq 0. \tag{3.13}$$

By the above definition of T_0 (see (3.10) and (3.11)), $T_0 = (\dot{\tilde{x}}(\alpha_0), \dot{\tilde{\phi}}_1(\alpha_0), \dot{\tilde{\phi}}_2(\alpha_0), \dot{\tilde{\lambda}}(\alpha_0), \dot{\tilde{\beta}}(\alpha_0)) := (x_1, u_1, v_1, \lambda_1, \omega_1) \in Y_S$ satisfies the following equations

$$g^0_x x_1 + \lambda_1 g^0_\lambda = -g^0_\alpha, \tag{3.14a}$$
$$(g^0_x + i\beta_0 I)(u_1 + iv_1) + (g^0_{xx}x_1 + \lambda_1 g^0_{x\lambda} + i\omega_1 I)(\phi^1_0 + i\phi^2_0) = -g^0_{x\alpha}(\phi^1_0 + i\phi^2_0), \tag{3.14b}$$
$$\langle l, u_1 \rangle = 0 \tag{3.14c}$$
$$\langle l, v_1 \rangle = 0. \tag{3.14d}$$

From (3.14a) and (2.3) we obtain

$$x_1 = \lambda_1 e^0 + f^0. \tag{3.15}$$

Substituting (3.15) into (3.14b) one obtains

$$(g^0_x + i\beta_0 I)(u_1 + iv_1) = -[\lambda_1(g^0_{xx}e^0 + g^0_{x\lambda}) + (g^0_{xx}f^0 + g^0_{x\alpha}) + i\omega_1 I](\phi^1_0 + i\phi^2_0). \tag{3.16}$$

Taking inner product with φ_0 in (3.16) and using (2.8a,b) one obtains

$$\lambda_1 = -\frac{\gamma_\alpha}{\gamma_\lambda}. \tag{3.17}$$

Substituting (3.15) and (3.17) into (3.13) and using (2.2) and (2.8) we have

$$\Delta = \langle \varphi^0, (g^0_{xx}f^0 + g^0_{x\alpha})\phi^0 + \lambda_1(g^0_{xx}e^0 + g^0_{x\lambda})\phi^0 \rangle = \frac{\gamma_\lambda\sigma_\alpha - \gamma_\alpha\sigma_\lambda}{\gamma_\lambda}. \tag{3.18}$$

Under hypotheses H3 and H4, from (3.18) one deduces that inequility (3.13) holds. This completes the proof. ⋄

Remark 3.5. Consider the eigenvalue problem along the branch $c_h^s(\alpha)$ of symmetric Hopf points of (1.1) with $\eta(0) = 0$ and $\phi(0) = \phi^0$: $g_x(\tilde{x}(\alpha), \tilde{\lambda}(\alpha), \alpha)\phi(\alpha) = \eta(\alpha)\phi(\alpha)$. By differentiating with respect to α, evaluating at $\alpha = \alpha_0$, and taking inner product with φ^0, one can show that $\frac{d}{d\alpha}\eta(\alpha)|_{\alpha=\alpha_0} = \frac{\gamma_\lambda \sigma_\alpha - \gamma_\alpha \sigma_\lambda}{\gamma_\lambda}$. Therefore the nondegenerate condition of $S-$breaking bifurcation in $G = 0$ is equivalent to $\frac{d}{d\alpha}\eta(\alpha)|_{\alpha=\alpha_0} \neq 0$. Consequently, det $g_x(\tilde{x}(\alpha), \tilde{\lambda}(\alpha), \alpha)$ changes sign at $\alpha = \alpha_0$ (HP-point).

Remark 3.6. For fixed α, each zero of $G(y, \alpha) = 0$ leads to a Hopf point of $g(x, \lambda, \alpha) = 0$ with respect to λ. The branch $C_H^A(\alpha)$ not in $Y_S \times R$ of $G = 0$ leads to "secondary" (non-symmetric) Hopf points of $g(x, \lambda, \alpha) = 0$ for fixed α near α_0. By using Proposition 2.1, we know that the"secondary" Hopf point lies on the branch $c^a(\alpha)$ of non-symmetric solutions of (1.1) bifurcating from a s-breaking pitchfork bifurcation point on the branch $c^s(\alpha)$ of symmetric solutions of (1.1) for α near α_0.

We give a schematic illustration of the solution surface of $g(x, \lambda, \alpha) = 0$ near a HP-point in Fig. 1.

Fig. 1. Schematic diagram of the solution surface of $g(x, \lambda, \alpha) = 0$ near a HP-point.
Q: surface of symmetric solutions $c^s(\alpha_1)$: branch of symmetric solutions for $\alpha = \alpha_1$
$c^a(\alpha_1)$: branch of non-symmetric solutions for $\alpha = \alpha_1$

4. A Regular System Determining HP-Points and Computation of Branches of Hopf Points

Theorem 3.4 does not yet provide a regular system which permits to find systematically a HP-point on some branch of symmetric solutions of (1.1). But the theorem indicates that a HP-point corresponds to a non-degenerate S-breaking pitchfork bifurcation point in system (1.5a,b). Hence it is easy to see that, under the hypotheses of Theorem 3.4, a regular system for determining HP-points is ($y = (x, \phi_1, \phi_2, \lambda, \beta) \in Y_S$, $\theta = (\phi, u_1, v_1, 0, 0) \in Y_A$, the normalizing vector L in Y is chosen such as $\ll L, \Phi_0 \gg = \langle l_1, \phi^0 \rangle = 1$, $l_1 \in X$)

$$\hat{E}(y,\theta,\alpha) = \hat{E}(x,\phi_1,\phi_2,\lambda,\beta,\phi,u_1,v_1,\alpha) := \left\{ \begin{array}{c} G(y,\alpha) \\ G_y(y,\alpha)\theta \\ \ll L,\theta \gg -1 = 0 \end{array} \right\}$$

$$= \left\{ \begin{array}{c} g(x,\lambda,\alpha) \\ g_x(x,\lambda,\alpha)\phi_1 - \beta\phi_2 \\ g_x(x,\lambda,\alpha)\phi_2 + \beta\phi_1 \\ \langle l,\phi_1 \rangle - 1 \\ \langle l,\phi_2 \rangle \\ g_x(x,\lambda,\alpha)\phi \\ g_x(x,\lambda,\alpha)u_1 - \beta v_1 + g_{xx}(x,\lambda,\alpha)\phi_1\phi \\ g_x(x,\lambda,\alpha)v_1 + \beta u_1 + g_{xx}(x,\lambda,\alpha)\phi_2\phi \\ \langle l_1,\phi \rangle - 1 \end{array} \right\} = 0, \tag{4.1a}$$

$$\hat{E} : \tilde{Y} \to \tilde{Y} \text{ with } \tilde{Y} := Y_s \times Y_a \times R = X_s \times X_s \times X_s \times R \times R \times X_a \times X_a \times X_a \times R \tag{4.1b}$$

where $l \in X$ is suitable normalizing vector satisfying (3.1).

Using Theorem 3.1 of paper[12], we have

Theorem 4.1. *Let* $(x_0, \lambda_0, \alpha_0)$ *be a HP -point of* $g(x,\lambda,\alpha)$ *satisfying hypotheses* $H1$-$H4$*, and let* l *and* l_1 *be chosen such that* (3.1) *holds. Then the system* (4.1) *has a solution* $x_0, \phi_0^1, \phi_0^2, \lambda_0, \beta_0$*,* ϕ^0, u_0, v_0 *and* α_0*, and its linearization at this solution is non-singular.*

The regularity immediately suggests that HP-points can be computed by solving (4.1a-b), at least for the case $X = R^n$. The numerical details are discussed in the latter part.

We now move on to the computation of branches of Hopf points bifurcating from a HP-point. Based on Proposition 3.3, the calculation of branch $c_h^s(\alpha)$ of symmetric Hopf points of (1.1) can easily be carried out by considering the restriction $G : Y_S \times R \to Y_S$ and continuing α. Concerning the computation of branch of secondary (non-symmetric) Hopf points of (1.1), we may use the pseudo-arc length branch switching method[7] to (1.5a,b) due to Theorem 3.4. The tangent vector $(\dot{z}_0, \dot{\alpha}_0) = (\Phi_0, 0) = (\phi^0, u_0, v_0, 0, 0, 0)$ needed in the approach in our pitchfork case is immediately found after the location of a HP-point (see (4.1a-b)). The usual "predictor-corrector" procedure may now be applied to jump on to the path of secondary (non-symmetric) Hopf points. But we should note a well known fact that Keller's approach[7] converges only in a cone with vertex at the HP-point. In order to overcome this drawback, we give another method based on Theorem 4.1 as in Wu[13].

we will firstly deduce an important conclusion from Theorem 4.1. The conclusion is based on the implicit function theorem applied to the system ($\varepsilon < \varepsilon_0$)

$$\tilde{E}(y,\theta,\alpha,\varepsilon) := \left\{ \begin{array}{c} G_1(y,\theta,\alpha,\varepsilon) \\ G_2(y,\theta,\alpha,\varepsilon) \\ \langle l_1,\phi \rangle - 1 \end{array} \right\} = 0, \tag{4.2a}$$

$$\tilde{E} : Y_s \times Y_a \times R \times R \to Y_s \times Y_a \times R. \tag{4.2b}$$

where

$$G_1(y,\theta,\alpha,\varepsilon) := \frac{1}{2}(G(y+\varepsilon\theta,\alpha)+G(y-\varepsilon\theta,\alpha)), \tag{4.2c}$$

$$G_2(y,\theta,\alpha,\varepsilon) := \begin{cases} G_y(y,\alpha)\theta, & \varepsilon = 0, \\ \dfrac{1}{2\varepsilon}(G(y+\varepsilon\theta,\alpha)-G(y-\varepsilon\theta,\alpha)), & \varepsilon \neq 0. \end{cases} \tag{4.2d}$$

Corollary 4.2. *Let the hypotheses of Theorem* 4.1 *hold. Then there exists a smooth branch* $(y(\varepsilon),\theta(\varepsilon),\ \alpha(\varepsilon)) \in Y_S \times Y_A \times R$ *(i.e.* $(x(\varepsilon),\ \phi_1(\varepsilon),\phi_2(\varepsilon),\lambda(\varepsilon),\beta(\varepsilon),$ $\phi(\varepsilon),u_1(\varepsilon),v_1(\varepsilon),\alpha(\varepsilon)) \in X_s \times X_s \times X_s \times R \times R \times X_a \times X_a \times X_a \times R$*) of solutions of* $\tilde{E} = 0$ *such that* $x(0) = x_0, \phi_1(0) = \phi_0^1,\ \phi_2(0) = \phi_0^2, \lambda(0) = \lambda_0, \beta(0) = \beta_0,$ $\phi(0) = \phi_0, u_1(0) = u_0, v_1(0) = v_0$ *and* $\alpha(0) = \alpha_0$*, and for each* $\varepsilon \neq 0$*,* $(x(\varepsilon) \pm \varepsilon\phi(\varepsilon),$ $\lambda(\varepsilon))$ *is secondary (non-symmetric) Hopf point of* (1.1) *with respect to* λ *for* $\alpha = \alpha(\varepsilon)$.

We suggest a computational procedure for branch $c_h^s(\alpha)$ of symmetric Hopf points, a HP-point and branch $c_h^a(\alpha)$ of secondary (non-symmetric) Hopf points of (1.1):

(i) fix α, calculate a Hopf point on the branch $c^s(\alpha)$ of symmetric solutions of (1.1);

(ii) vary α and follow the path $c_h^s(\alpha)$ of symmetric Hopf points of (1.1) by using the restriction $G|_{Y_S}$ and monitor the determinant of g_x along the branch;

(iii) detect the existence of a HP-point by a sign change of the determinant of g_x(see Remark 3.5).

(iv) Once an approximation to the HP-point is known it may be determined precisely by solving the regular system $\hat{E} = 0$.

(v) The extended system (4.2) (i.e. $\tilde{E} = 0$) can then be used to compute the branch c_h^a of secondary (non-symmetric) Hopf points of (1.1) by starting for $\varepsilon = 0$ with $(x,\phi_1,\phi_2,\lambda,\beta,\phi,u_1,v_1,\alpha) = (x_0,\phi_0^1,\phi_0^2,\lambda_0,\beta_0,\phi^0,u_0,v_0,\alpha_0)$ and continuing ε from $\varepsilon =$ 0.

Remark 4.3. In step(i), for given $\alpha = \alpha_1$, a starting point on the branch $C_S^H(\alpha)$ of solution of $G = 0$ can be found by the continuation of branch $c^s(\alpha_1)$ of symmetric solutions of (1.1) with respect to λ and monitoring the eigenvalue of the Jacobian matrix. Then an approximation $\phi_{11}(\alpha_1) + i\phi_{12}(\alpha_1)$ to $\tilde{\phi}_1(\alpha_1) + i\tilde{\phi}_2(\alpha_1)$ can be calculated by an inverse iteration procedure, and we can choose the normalizing vector $l = c_1\phi_{11}(\alpha_1) + c_2\phi_{12}(\alpha_1)$ where c_1 and c_2 satisfy $c_1\langle\phi_{11}(\alpha_1),\phi_{11}(\alpha_1)\rangle + c_2\langle\phi_{11}(\alpha_1),$ $\phi_{12}(\alpha_1)\rangle = 1, c_1\langle\phi_{12}(\alpha_1),\phi_{11}(\alpha_1)\rangle + c_2\langle\phi_{12}(\alpha_1),\phi_{12}(\alpha_1)\rangle = 0$. Then l can be chosen according to (3.1). For steps (iii) and (iv), the normalizing vector l_1 can be chosen as in paper[12].

Remark 4.4. The regularity of the HP-point as the solution in Theorem 4.1 implies our method computing the branch $C_H^A(\alpha)$ of solutions not in $Y_S \times R$ of $G = 0$ converges in a full neighborhood of the HP-point. We can directly calculate the branch $c_h^a(\alpha)$ of non-symmetric Hopf points of (1.1) without tracing the bifurcating branch $c^a(\alpha)$ of non-symmetric solutions of (1.1) for each α near α_0.

We now discuss Newton's method applied to (4.1) to calculate HP-points for the finite dimensional case $X = R^n$. Noting that the 7th and 8th equations in (4.1) are uncoupled from the others, we need solve only the system consisting of the 1st-6th and 9th equations in (4.1). Since $R^n = R_s^n \oplus R_a^n$ $(R_s^n := X_s, R_a^n := X_a)$ we have $n = n_s + n_a$ where $n_s = \dim\ R_s^n$, $n_a = \dim R_a^n$. In general it will be easy to identify $R_s^n(R_a^n)$ with

$R^{n_s}(R^{n_a})$ using the isomorphism $I_s : R^n_s \to R^{n_s}$, $I_s(x) = x^s(I_a : R^n_a \to R^{n_a}, I_a(x) = x^a, etc.)$. Then HP-points are determined by

$$\bar{E}(x^s, \phi_1^s, \phi_2^s, \phi^a, \lambda, \beta, \alpha) := \begin{Bmatrix} g^s(x^s, \lambda, \alpha) \\ g_x^s(x^s, \lambda, \alpha)\phi_1^s - \beta\phi_2^s \\ g_x^s(x^s, \lambda, \alpha)\phi_2^s + \beta\phi_1^s \\ g_x^a(x^s, \lambda, \alpha)\phi \\ (l^s)^t\phi_1^s - 1 \\ (l^s)^t\phi_2^s \\ (l_1^a)^t\phi^a - 1 \end{Bmatrix}, \tag{4.3a}$$

$$\bar{E} : W \to W := R^{n_s} \times R^{n_s} \times R^{n_s} \times R^{n_a} \times R \times R \times R \tag{4.3b}$$

where

$$\begin{aligned} & x^s = I_s x, \phi_1^s = I_s\phi_1, \phi_2^s = I_s\phi_2, \\ & \phi^a = I_a\phi, \\ & g^s(x^s, \lambda, \alpha) = I_s g(x, \lambda, \alpha), \\ & g_x^a(x^s, \lambda, \alpha) = I_a g_x(x, \lambda, \alpha)\phi, \\ & g_x^s(x^s, \lambda, \alpha)\phi_j^s + (-1)^j\beta\phi_{3-j}^s = I_s(g_x(x, \lambda, \alpha)\phi_j + (-1)^j\beta\phi_{3-j}),\ j = 1, 2, \\ & (l^s)^t\phi_j^s = l^t\phi_j,\ j = 1, 2, \\ & (l_1^a)^t\phi^a = {l_1}^t\phi. \end{aligned}$$

System (4.3a,b) can be solved efficiently by combing the methods described in Werner and Spence[12] and Griewank and Reddien[3]. The details are omitted.

Once a HP-point has been computed, we can use the following equations (cf. (3.9)) to solve (u_0, v_0) in Φ_0 (see Proposition 3.2):

$$I_a(g_x^0 + i\beta_0 I){I_a}^{-1}(u_{01} + iv_{01}) = -I_a(g_{xx}^0\phi^0\phi_0^1 + ig_{xx}^0\phi^0\phi_0^2), \tag{4.4a}$$

$$u_0 = {I_a}^{-1}u_{01}, v_0 = {I_a}^{-1}v_{01}. \tag{4.4b}$$

In the implementation, we can also use the difference approximation for second order derivatives of g[7].

In a similar way one can use the implementation of Newton's method to trace the initial bifurcating branch $C_A^H(\alpha)$ of $G = 0$ by using the extended system (4.2a-d).

5. A Numerical Example

For numerical demonstrations, we consider a specific reaction-diffusion model, often called the Brusselator[2,4]

$$\frac{\partial u}{\partial t} = D_1\frac{\partial^2 u}{\partial \xi^2} + u^2 v - (B+1)u + A, \tag{5.1a}$$

$$\frac{\partial v}{\partial t} = D_2\frac{\partial^2 v}{\partial \xi^2} - u^2 v + Bu. \tag{5.1b}$$

with boundary conditions

$$u(0,t) = u(\pi,t) = A, \tag{5.2a}$$
$$v(0,t) = v(\pi,t) = B/A. \tag{5.2b}$$

In (5.1,2) u, v, A and B represent chemical concentrations; u and v are unknown, while A and B are independent of ξ and t. As customary, we shall treat $\lambda := A$ as the bifurcation parameter and $\alpha := B$ as the auxiliary or control parameter, D_1 and D_2 are diffusion constants. The bifurcation behavior of this system with respect to A and B has been extensively studied[2,4]. In the computations we used the values $D_1 = 0.04, D_2 = 0.2$.

The steady-state equation of (5.1,2) has the Z_2-symmetry given by

$$s_1 \begin{bmatrix} u \\ v \end{bmatrix} (\xi) = \begin{bmatrix} u(\pi-\xi) \\ v(\pi-\xi) \end{bmatrix}, 0 \le \xi \le \pi. \tag{5.3}$$

We applied the usual $O(h^2)$- discretization on an equidistant grid with mesh size h and discretization points $x_i = ih(i = 1, \cdots, 2m;\ m = (\pi/h - 1)/2)$. The approximating system can be written as

$$\frac{du_i}{dt} = D_1 \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} - (B+1)u_i + {u_i}^2 v_i + A, \tag{5.4a}$$
$$\frac{dv_i}{dt} = D_2 \frac{v_{i-1} - 2v_i + v_{i+1}}{h^2} - {u_i}^2 v_i + Bu_i, \quad i = 1, 2, \cdots, 2m. \tag{5.4b}$$

Defining $x = (x_1, x_2, \cdots, x_n)^t = (u_1, v_1, u_2, v_2, \cdots, u_{2m}, v_{2m})^t$ with $n = 4m$, and taking into account the boundary conditions, one can rewrite (5.4) into the form of (1.4).

Let

$$E_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in R^{2\times 2}, \quad F_p = \begin{bmatrix} & & & & E_p \\ & & & E_p & \\ & & \cdot & & \\ & \cdot & & & \\ \cdot & & & & \\ E_p & & & & \end{bmatrix} \in R^{2m\times 2m} \tag{5.5a}$$

and

$$s = \begin{bmatrix} 0 & F_p \\ F_p & 0 \end{bmatrix} \in R^{4m\times 4m}. \tag{5.5b}$$

Then (1.2) holds for s in (5.5) and for g in the right-hand side of (5.4).

We are mainly interested in the computations of branches $c_h^s(\alpha)$ and $c_h^a(\alpha)$ of Hopf points bifurcating from an HP-point. Using the computational procedure of Section 4, we have calculated $c_h^s(\alpha)$, located the HP-point, and computed $c_h^a(\alpha)$ of (5.4) for $n = 80$. In Fig.2 we have drawn the projections of branches of Hopf points bifurcating from the HP-point with $x_0 = (0.421, 3.366, 0.421, 3.366, \cdots)^t$, $\lambda_0 = 0.421$, and $\alpha_0 = 1.417$ on the parameter spaces.

Fig. 2(a). Branches of Hopf points for Eqs.(5.1) and (5.2) with (D_1, D_2)=(0.04,0.2). Projections on the (B, A)-plane.
—: branch of symmetric Hopf points;
$\cdots\square\cdots$: branch of non-symmetric Hopf points

Fig. 2 (b). Branches of Hopf points for Eqs.(5.1) and (5.2) with (D_1, D_2)=(0.04,0.2). Projections on the (B, β)-plane.
—: branch of symmetric Hopf points;
$\cdots\square\cdots$: branch of non-symmetric Hopf points

After a secondary Hopf point far from the HP-point is obtained by the continuation, one may use other techniques[3] for solving equations (1.5) to trace the branch of secondary Hopf points.

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