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POTENTIAL INVERSION PROBLEMS FOR COUPLED SYSTEM OF DOWNGOING AND UPCOMING ONE-WAY WAVE EQUATIONS*1)

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Abstract

By using wave splitting method the formulation of the two-dimensional potential inversion problem is set up in terms of the coupled system for downgoing and upcoming wavefields. The boundary counditions on the characteristic surface needed for solving the problem are derived by singularity analysis. Two stability theorems are given for the direct problems of the system treated as Cauchy problems in the direction of depth.

Key words: 2-D potential inversion, Wave splitting, Singularity analysis.

1. Introduction

In this paper the following potential inversion problem is considered

$$\Big[\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial z^2}+v(x,z)\Big]p(x,z,t)=0,\quad x\in R, z>0, t>0, \tag{1.1}$$

$$p(x,z,0)=\frac{\partial p}{\partial t}(x,z,0)=0, \tag{1.2}$$

$$p(x,0,t)=\delta(t), \tag{1.3}$$

$$\frac{\partial}{\partial z}p(x,0,t)=h(x,t). \tag{1.4}$$

That is, giving an impulse at the surface $z=0$, to determine the wavefield p and potential v from the impulse response h.

In one-dimensional case, by factorizing the wave operator, the wavefield can be split into upcoming and downgoing waves, so the wave equation can be easily reduced to a coupled first-order system. The direct problem and the coefficient-inversion problem can be treated as Cauchy problems in time and in the direction of depth regarded as the time-like variable[1]. These problems are well-posed because the time and space

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variables are exchangeable in one-dimensional case. The numerical solutions for wavefields and unknown coefficients in inverse problems can be obtained layer-by-layer by finite difference methods. There are two difficulties to extent this technique into multidimensional case. The first consists in the ill-posedness of the problems, both direct and inverse, treated as Cauchy problems in the direction of depth, which is non-time-like in the multi-dimensional case. Another lies in the factorization of the wave operator, because the so-called square-root operator is not a differential one.

There are some approaches to get rid of these difficulties. A.E. Yagle and P. Raadhakrishan (1992) [2] used the clipped filter for some cutoff wave number of the lateral variable in order to avoid instability. V.H. Weston (1987) [3] used the relationship between the Dirichlet and Neumann data of the wave to construct the square-root operator. The square-root operator was considered also in L. Fishman (1991) [4] as Weyl pseudo-differential operator.

In this paper we treat only the radiation part of wavefield by neglecting the evanescent wave, which vanished rapidly with increasing of depth. In this case the square-root operator can be represented as an integral of a parameter-dependent operator. Using this representation, the two-dimensional wave equation can be reduced also to a coupled system, as in one-dimensional case, in which the two main equations describe the propagation of downgoing and upcoming wavefields and their coupling. The principal parts of these equations are exactly the one-way wave equations, familiar in geophysics for migration problems. Other equations in the system, needed for determining the auxiliary functions involved in the main equations, are one-dimensional wave equations only in the lateral variable. All these equations can easily be approximated and discretized by finite difference methods

The boundary conditions on the characteristic surface needed for solving the system of equations are derived by analyzing the propagation of the singularity.

We also give two theorems on the stability of the direct problems of the coupled system treated as Cauchy problems in the direction of depth.

2. Approximation of Square-root Operator and Wavefield Splitting

The two dimensional wave operator can be factorized as follows [5]:

$$\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial z^2} = \Big(\Lambda + \frac{\partial}{\partial z}\Big)\Big(\Lambda - \frac{\partial}{\partial z}\Big),$$

where Λ is a pseudo-differential operator, so-call square-root operator, with the "symbol"

$$\lambda(k_x, \omega) = i\sqrt{\omega^2 - {k_x}^2}.$$

We treat only the radiation part of wavefield and neglect the evanescent wave, that is, we consider only the case of $\omega^2 \geq {k_x}^2$. In this case, the following formula is true

$$\frac{1}{\pi}\int_{-1}^{1} \sqrt{1-s^2}\frac{k_x^2 ds}{\omega^2 - s^2 k_x^2} = \frac{1}{\omega}(\omega - \sqrt{\omega^2 - k_x^2}).$$

So the square-root operator Λ can be represented in the integral form

$$\Lambda = \frac{\partial}{\partial t}(I - R),$$

where R is a pseudo-differential operator which is defined as

$$R = \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-s^2}Q(s)ds, \tag{2.1}$$

and $q(s,x,z,t) = Q(s)p(x,z,t)$ satisfies

$$\Big(\frac{\partial^2}{\partial t^2} - s^2\frac{\partial^2}{\partial x^2}\Big)q = \frac{\partial^2}{\partial x^2}p. \tag{2.2}$$

We define the upcoming wave U_1 and downgoing wave D_1 by

$$U_1 = \frac{\partial U}{\partial t} = \frac{1}{2}\Big(\Lambda + \frac{\partial}{\partial z}\Big)p, \tag{2.3}$$

$$D_1 = \frac{\partial D}{\partial t} = \frac{1}{2}\Big(\Lambda - \frac{\partial}{\partial z}\Big)p. \tag{2.4}$$

Then the two dimensional wave equation (1.1) can be transformed into the coupled system of one-way wave equations

$$\frac{\partial}{\partial t}\Big(\Lambda - \frac{\partial}{\partial z}\Big)U + \frac{1}{2}vp = 0, \tag{2.5}$$

$$\frac{\partial}{\partial t}\Big(\Lambda + \frac{\partial}{\partial z}\Big)D + \frac{1}{2}vp = 0, \tag{2.6}$$

$$\frac{\partial p}{\partial z} = \frac{\partial U}{\partial t} - \frac{\partial D}{\partial t}, \tag{2.7}$$

and the corresponding auxiliary equations

$$\Big(\frac{\partial^2}{\partial t^2} - s^2\frac{\partial^2}{\partial x^2}\Big)q_U = \frac{\partial^2 U}{\partial x^2}, \tag{2.8}$$

$$\Big(\frac{\partial^2}{\partial t^2} - s^2\frac{\partial^2}{\partial x^2}\Big)q_D = \frac{\partial^2 D}{\partial x^2}. \tag{2.9}$$

Now the potential inversion problem can be reformulated to determine the upcoming wave U_1 and downgoing wave D_1, wavefield p and the unknown potential $v(x,z)$, satisfying the system (2.5)–(2.9) in the domain $\Omega = \{(x,z,t),\ x \in R,\ z > 0, t > z\}$, with the necessary initial and boundary conditions.

By using the Gaussian quadrature formula the square-root operator can be approximated by [5] [6]

$$\Lambda_n = \frac{\partial}{\partial t} - \frac{\partial}{\partial t}\sum_{k=1}^{n} a_k Q(s_k), \tag{2.10}$$

where

$$s_k = \cos\Big(\frac{k\pi}{n+1}\Big), \quad a_k = \frac{1}{n+1}\sin^2\Big(\frac{k\pi}{n+1}\Big).$$

3. Singularity Analysis

In order to solve the inverse problem, the boundary conditions on the characteristic surface $t = z + 0$ should be derived. The impulse condition is $p(x, 0, t) = \delta(t)$, the singularity of wavefield will propagate along the characteristic surface $t = z$. By singularity analysis, the impulse condition can be transformed into the conditions on the characteristic surface.

The general solution of our problem can be written as [7]

$$p(x, z, t) = \delta(t - z) + a(x, z, t)H(t - z), \tag{3.1}$$

where H is the Heaviside function and $a(x, z, t)$ is a regular function. Substituting (3.1) in (1.1) we get

$$\begin{aligned} H(t - z)\Big(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial z^2}\Big)a(x, z, t) + 2(a_t + a_z)\delta(t - z) \\ + v(x, z)[\delta(t - z) + a(x, z, t)H(t - z)] = 0. \end{aligned}$$

Analyzing the singularity in the above expression we have

$$\begin{aligned} & a_t + a_z = -\frac{v(x, z)}{2}, \ \forall t = z, \\ & \Big(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial z^2}\Big)a(x, z, t) + v(x, z)a(x, z, t) = 0, \ \forall t > z. \end{aligned} \tag{3.2}$$

So

$$\begin{aligned} & p(x, z, t = z + 0) = a(x, z, t = z + 0) = -\frac{1}{2}\int_0^z v(x, y)dy \doteq g(x, z), \\ & \Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial z}\Big)\Big(\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\Big)a(x, z, t) = \Big[\frac{\partial^2}{\partial x^2} - v(x, z)\Big]a(x, z, t). \end{aligned} \tag{3.3}$$

Integrating the above expression along the line $t = z + 0$, we have

$$\begin{aligned} \Big(\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\Big)p(x, z, t = z + 0) &= \int_0^z \Big[\frac{\partial^2}{\partial x^2} - v(x, y)\Big]g(x, y)dy - \frac{\partial}{\partial z}p(x, 0, t = +0) \\ &\doteq g_D(x, z) - h(x, z, t = +0). \end{aligned} \tag{3.4}$$

From (3.2) and (3.3) we get

$$\Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial z}\Big)p(x, z, t = z + 0) = -\frac{v(x, z)}{2}. \tag{3.5}$$

Suppose the function $q(x, z, t)$ in (2.1) can be expanded as

$$q(x, z, t) = A_q(x, z)\delta(t - z) + A_q^{(1)}(x, z)H(t - z) + A_q^{(2)}(x, z, t)H_1(t - z), \tag{3.6}$$

where $H_1(t) = \dfrac{t + \mid t \mid}{2}$, and $H_1' = H$. Substituting (3.1) and (3.6) into equation (2.2) we get

$$A_q\delta'' + A_q^{(1)}\delta' + A_q^{(2)}\delta + 2\frac{\partial A_q^{(2)}}{\partial t}H + \frac{\partial^2 A_q^{(2)}}{\partial t^2}H_1$$

$$-s^2\frac{\partial^2 A_q}{\partial x^2}\delta - s^2\frac{\partial^2 A_q^{(1)}}{\partial x^2}H - s^2\frac{\partial^2 A_q^{(2)}}{\partial x^2}H_1 = \frac{\partial^2 a}{\partial x^2}H.$$

Comparing the coefficients of $\delta'', \delta', \delta$ on the both sides of this equation we know

$$A_q(x,z) = 0, \quad A_q^{(1)}(x,z) = 0, \quad A_q^{(2)}(x,z,t=z+0) = 0.$$

So (3.6) becomes

$$q(x,z,t) = A_q^{(2)}(x,z,t)H_1(t-z).$$

This yields

$$q(x,z,t=z+0) = \frac{\partial q}{\partial t}(x,z,t=z+0) = 0. \tag{3.7}$$

Substituting (3.5),(3.7) into the definition (2.3) of upcoming wave we obtain

$$\frac{\partial U}{\partial t}(x,z,t=z+0) = -\frac{v(x,z)}{4}. \tag{3.8}$$

Substituting (3.4),(3.7) into the definition (2.4) of downgoing wave we obtain

$$\frac{\partial D}{\partial t}(x,z,t=z+0) = \frac{1}{2}g_D(x,z) - \frac{1}{2}h(x,t=+0). \tag{3.9}$$

Suppose $D(x,z,t)$ can be expressed as

$$D(x,z,t) = A_D(x,z)\delta(t-z) + A_D^{(1)}(x,z)H(t-z) + A_D^{(2)}(x,z,t)H_1(t-z).$$

Substituting this expression and (3.1),(3.7) into the definition (2.4) of downgoing wave we have

$$D(x,z,t=z+0) = a(x,z,t=z+0) = p(x,z,t=z+0) = g(x,z). \tag{3.10}$$

Suppose the function q_D in equation (2.9) can be expanded as

$$q_D(x,z,t) = A_{q_D}(x,z)\delta(t-z) + A_{q_D}^{(1)}(x,z)H(t-z) + A_{q_D}^{(2)}(x,z,t)H_1(t-z).$$

Substituting this expansion and (3.10) into (2.9) yields

$$q_D(x,z,t=z+0) = \frac{\partial q_D}{\partial t}(x,z,t=z+0) = 0. \tag{3.11}$$

4. New Formulation of Potential Inversion Problem

By summarizing the above discussion, we reduce the potential inversion problem as follows

$$\frac{\partial}{\partial t}\Big(\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\Big)U - \frac{\partial^2}{\partial t^2}\Big[\sum_{k=1}^{n} a_k q_U(s_k)\Big] + \frac{vp}{2} = 0, \tag{4.1}$$

$$\frac{\partial}{\partial t}\Big(\frac{\partial}{\partial t}+\frac{\partial}{\partial z}\Big)D-\frac{\partial^2}{\partial t^2}\Big[\sum_{k=1}^{n}a_k q_D(s_k)\Big]+\frac{vp}{2}=0, \tag{4.2}$$

$$\frac{\partial p}{\partial z}=\frac{\partial U}{\partial t}-\frac{\partial D}{\partial t}, \tag{4.3}$$

$$\Big(\frac{\partial^2}{\partial t^2}-s_k^2\frac{\partial^2}{\partial x^2}\Big)q_U(s_k,x,z,t)=\frac{\partial^2}{\partial x^2}U(x,z,t), \tag{4.4}$$

$$\Big(\frac{\partial^2}{\partial t^2}-s_k^2\frac{\partial^2}{\partial x^2}\Big)q_D(s_k,x,z,t)=\frac{\partial^2}{\partial x^2}D(x,z,t). \tag{4.5}$$

The boundary conditions at $z=0$ are

$$\frac{\partial U}{\partial t}(x,0,t)=-\frac{\partial D}{\partial t}(x,0,t)=\frac{1}{2}h(x,t),\quad p(x,0,t)=0,\ \forall t>0 \tag{4.6}$$

which are obtained from conditions (3) (4) and the definitions of $\frac{\partial U}{\partial t}$ and $\frac{\partial D}{\partial t}$. As the results of singularity analysis we get the conditions on the characteristic surface $t=z+0$

$$D(x,z,t=z+0)=-\int_0^z\frac{v(x,y)}{2}dy=g(x,z), \tag{4.7}$$

$$\frac{\partial D}{\partial t}(x,z,t=z+0)=\frac{1}{2}\int_0^z\Big[\frac{\partial^2}{\partial x^2}-v(x,y)\Big]g(x,y)dy-\frac{1}{2}h(x,0), \tag{4.8}$$

$$q_D(x,z,t=z+0)=\frac{\partial q_D}{\partial t}(x,z,t=z+0)=0, \tag{4.9}$$

$$v(x,z)=-4\frac{\partial U}{\partial t}(x,z,t=z+0). \tag{4.10}$$

In order to limit the behavior of the upgoing wave U_1 (and U) which is related to the behavior of $v(x,z)$, we should impose some conditions on $t\to\infty$. For example all upcoming waves $U(x,z,t)$, $U_1(x,z,T)$ and $q_U(x,z,t)$, $\frac{\partial q_U}{\partial t}(x,z,t)$ will tend to 0, which consistent with sommerfeld radiation condition. For convenience in numerical solution we transfer these conditions to sufficient large value T.

$$U(x,z,T)=\frac{\partial U}{\partial t}(x,z,T)=q_U(x,z,T)=\frac{\partial q_U}{\partial t}(x,z,T)=0. \tag{4.11}$$

The problem of (4.1)–(4.5) with conditions (4.6)–(4.11) is closed by giving appropriate boundary conditions on the direction x.

The system of (4.1)–(4.5) is different from that proposed by Zhang and Song (1993) [8]. The coupling term is treated more simple and more convenient to numerical implementation.

If taking different values of n in (2.10),we can get different orders of approximate one-way wave equations.For example,if taking $n=1$ we get the so-called 15° approximate equations.And if taking $n=2$ we get the so-called 45° approximate equations.

It is important to point out that the order of all above equations is no more than two for all values of n. So it is very simple to discrete the equations and to do some theoretical analysis for the corresponding difference equations.On the other hand,the forms of all equations are the same for all values of n,so it is possible for us to handle them in a uniform manner.All of these features are valuable for practical computations.

5. Theorems of Stability

The potential inversion problem is an initial value problem in the direction z. In multi-dimensional case it is ill-posed in the original formulation with the wave equation. In order to get rid of this difficulty we factorized the wave operator approximately and split the wave equation into a coupled system of one-way wave equations. In this section we give two theorems of energy estimates which show that the direct problems of the above coupled system in the direction z are stable. Although it is very difficult to prove the stability of inverse problem, our proof demonstrates that there are essential differences between the system of one-way wave equations and the original wave equation.

For convenience we take $n = 1$ in (2.10). The system of one-way wave equations (4.1)–(4.5) becomes

$$\Big(\frac{\partial^2}{\partial t\partial z} - \frac{\partial^2}{\partial t^2} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\Big)U - \frac{vp}{2} = 0, \tag{5.1}$$

$$\Big(\frac{\partial^2}{\partial t\partial z} + \frac{\partial^2}{\partial t^2} - \frac{1}{2}\frac{\partial^2}{\partial x^2}\Big)D + \frac{vp}{2} = 0, \tag{5.2}$$

$$\frac{\partial p}{\partial z} = \frac{\partial}{\partial t}(U - D). \tag{5.3}$$

Denote

$$\Omega_1 = \{(x,z,t), x \in R,\ z > 0,\ 0 < t < T\},\ \text{and}\ \Omega = \{(x,z,t), x \in R, t > z > 0\}.$$

We consider the initial-boundary value problem of the above system in Ω_1 with the specified values of $p, \dfrac{\partial U}{\partial t}, \dfrac{\partial D}{\partial t}$ at $z = 0$, the specified values of $D, \dfrac{\partial D}{\partial t}$ at $t = 0$, and the specified values of $U, \dfrac{\partial U}{\partial t}$ at $t = T$. Then we have the following stability theorem:

Theorem 1. *Suppose p, U, D are solutions of the above problem, and $p, \dfrac{\partial U}{\partial t}, \dfrac{\partial U}{\partial x}, \dfrac{\partial D}{\partial t}, \dfrac{\partial D}{\partial x}$ are square integrable in $x \in R$, then the following energy estimate is valid.*

$$\begin{aligned}&\int_0^T dt \int dx\Big\{p^2 + \frac{1}{2}\Big[\Big(\frac{\partial U}{\partial t}\Big)^2 + \Big(\frac{\partial D}{\partial t}\Big)^2\Big]\Big\}(x,Z,t)\\ \le &\Big\{\int_0^T dt \int dx\Big\{p^2 + \frac{1}{2}\Big[\Big(\frac{\partial U}{\partial t}\Big)^2 + \Big(\frac{\partial D}{\partial t}\Big)^2\Big]\Big\}(x,0,t) + \frac{A}{2}\Big\} \cdot \exp\int_0^Z \Big[2 + \frac{1}{2}V(z)\Big]dz.\end{aligned}$$

where

$$\begin{aligned}V(z) &= \max\{|v(x,z)|, x \in R\},\\ A &= \int_0^Z dz \int dx\Big[\Big(\frac{\partial U}{\partial t}\Big)^2 + \frac{1}{2}\Big(\frac{\partial U}{\partial x}\Big)^2\Big](x,z,T)\end{aligned}$$

$$+\int_0^Z dz\int dx\Big[\Big(\frac{\partial D}{\partial t}\Big)^2+\frac{1}{2}\Big(\frac{\partial D}{\partial x}\Big)^2\Big](x,z,0).$$

Proof. Multiplying equation (5.1) by $2\dfrac{\partial U}{\partial t}$, integrating it with respect to x, taking part integration for the third term, and noticing that $\dfrac{\partial U}{\partial t}, \dfrac{\partial U}{\partial x}$ are square integrable in $x\in R$, we get

$$\int dx\Big[\frac{\partial}{\partial z}\Big(\frac{\partial U}{\partial t}\Big)^2-\frac{\partial}{\partial t}\Big(\frac{\partial U}{\partial t}\Big)^2-\frac{1}{2}\frac{\partial}{\partial t}\Big(\frac{\partial U}{\partial x}\Big)^2\Big]-\int vp\frac{\partial U}{\partial t}dx=0. \tag{5.4}$$

In the same manner we get the following equation from (5.2)

$$\int dx\Big[\Big(\frac{\partial}{\partial z}\Big(\frac{\partial D}{\partial t}\Big)^2+\frac{\partial}{\partial t}\Big(\frac{\partial D}{\partial t}\Big)^2+\frac{1}{2}\frac{\partial}{\partial t}\Big(\frac{\partial D}{\partial x}\Big)^2\Big]+\int vp\frac{\partial D}{\partial t}dx=0. \tag{5.5}$$

Integrating the above two equations respectively in the domain $z\in[0,Z]$, $t\in[0,T]$, we obtain

$$\begin{aligned}
&\int_0^T dt\int dx\Big(\frac{\partial U}{\partial t}\Big)^2(x,Z,t)+\int_0^Z dz\int dx\Big[\Big(\frac{\partial U}{\partial t}\Big)^2+\frac{1}{2}\Big(\frac{\partial U}{\partial x}\Big)^2\Big](x,z,0)\\
=&\int_0^T dt\int dx\Big(\frac{\partial U}{\partial t}\Big)^2(x,0,t)+\int_0^Z dz\int dx\Big[\Big(\frac{\partial U}{\partial t}\Big)^2+\frac{1}{2}\Big(\frac{\partial U}{\partial x}\Big)^2\Big](x,z,T)\\
&+\int_0^Z dz\int_0^T dt\int vp\frac{\partial U}{\partial t}dx, \qquad (5.6)\\
&\int_0^T dt\int dx\Big(\frac{\partial D}{\partial t}\Big)^2(x,Z,t)+\int_0^Z dz\int dx\Big[\Big(\frac{\partial D}{\partial t}\Big)^2+\frac{1}{2}\Big(\frac{\partial D}{\partial x}\Big)^2\Big](x,z,T)\\
=&\int_0^T dt\int dx\Big(\frac{\partial D}{\partial t}\Big)^2(x,0,t)+\int_0^Z dz\int dx\Big[\Big(\frac{\partial D}{\partial t}\Big)^2+\frac{1}{2}\Big(\frac{\partial D}{\partial x}\Big)^2\Big](x,z,0)\\
&-\int_0^Z dz\int_0^T dt\int vp\frac{\partial D}{\partial t}dx, \qquad (5.7)
\end{aligned}$$

Adding the above two equations we have

$$\begin{aligned}
&\int_0^T dt\int dx\Big[\Big(\frac{\partial U}{\partial t}\Big)^2+\Big(\frac{\partial D}{\partial t}\Big)^2\Big](x,Z,t)+\int_0^Z dz\int dx\Big[\Big(\frac{\partial U}{\partial t}\Big)^2+\frac{1}{2}\Big(\frac{\partial U}{\partial x}\Big)^2\Big](x,z,0)\\
&\quad+\int_0^Z dz\int dx\Big[\Big(\frac{\partial D}{\partial t}\Big)^2+\frac{1}{2}\Big(\frac{\partial D}{\partial x}\Big)^2\Big](x,z,T)\\
&=\int_0^T dt\int dx\Big[\Big(\frac{\partial U}{\partial t}\Big)^2+\Big(\frac{\partial D}{\partial t}\Big)^2\Big](x,0,t)+A+\int_0^Z dz\int_0^T dt\int vp\Big(\frac{\partial U}{\partial t}-\frac{\partial D}{\partial t}\Big)dx\\
&\le\int_0^T dt\int dx\Big[\Big(\frac{\partial U}{\partial t}\Big)^2+\Big(\frac{\partial D}{\partial t}\Big)^2\Big](x,0,t)+A\\
&\quad+\int_0^Z dz V(z)\int_0^T dt\int dx\Big\{p^2+\frac{1}{2}\Big[\Big(\frac{\partial U}{\partial t}\Big)^2+\Big(\frac{\partial D}{\partial t}\Big)^2\Big]\Big\}.
\end{aligned}$$

where

$$V(z)=\max\{|v(x,z)|,x\in R\}$$

$$A = \int_0^Z dz \int dx \Big[\Big(\frac{\partial U}{\partial T}\Big)^2 + \frac{1}{2}\Big(\frac{\partial U}{\partial x}\Big)^2 \Big](x,z,T) + \int_0^Z dz \int dx \Big[\Big(\frac{\partial D}{\partial t}\Big)^2 + \frac{1}{2}\Big(\frac{\partial D}{\partial x}\Big)^2 \Big](x,z,0).$$

So we have

$$\begin{aligned}\int_0^t dt \int dx \Big[\Big(\frac{\partial D}{\partial t}\Big)^2 + \Big(\frac{\partial D}{\partial t}\Big)^2 \Big](x,Z,t) &\le \int_0^T dt \int dx \Big[\Big(\frac{\partial U}{\partial t}\Big)^2 + \frac{\partial D}{\partial t}\Big)^2 \Big](x,0,t) + A \\ &+ \int_0^Z dz V(z) \int_0^T dt \int dx \Big\{ p^2 + \frac{1}{2}\Big[\Big(\frac{\partial U}{\partial t}\Big)^2 + \Big(\frac{\partial D}{\partial t}\Big)^2 \Big] \Big\},\end{aligned} \tag{5.8}$$

On the other hand, Multiplying (5.3) by $2p$ yields

$$\frac{\partial}{\partial z}(p^2) = 2p\Big(\frac{\partial U}{\partial t} - \frac{\partial D}{\partial t}\Big).$$

Integrating it in the domain $(x,z,t) \in \{x \in R, z \in [0,Z],\ t \in [0,T]\}$, we get

$$\begin{aligned}\int_0^T dt \int p^2(x,Z,t)dx &= \int_0^T dt \int p^2(x,0,t)dx + \int_0^Z dz \int_0^T dt \int 2p\Big(\frac{\partial U}{\partial t} - \frac{\partial D}{\partial t}\Big)dx \\ &\le \int_0^T dt \int p^2(x,0,t)dx \\ &\quad + 2\int_0^Z dz \int_0^T dt \int dx \Big\{ p^2 + \frac{1}{2}\Big[\Big(\frac{\partial U}{\partial t}\Big)^2 + \Big(\frac{\partial D}{\partial t}\Big)^2 \Big] \Big\}.\end{aligned} \tag{5.9}$$

Denote that

$$F(Z) = \int_0^t dt \int dx \Big\{ p^2 + \frac{1}{2}\Big[\Big(\frac{\partial U}{\partial t}\Big)^2 + \frac{\partial D}{\partial T}\Big)^2 \Big] \Big\}.$$

Adding $\frac{1}{2}$(5.8) to (5.9) we get the following inequality

$$F(Z) \le F(0) + \frac{A}{2} + \int_0^Z \Big[2 + \frac{1}{2}V(z)\Big] F(z)dz.$$

So we get the energy estimate

$$F(Z) \le \Big[F(0) + \frac{A}{2}\Big] \cdot \exp \int_0^Z \Big[2 + \frac{1}{2}V(z)\Big] dz.$$

The theorem is proved.

Corresponding to the inverse problem (1.4)–(2.4), we consider the direct initial boundary value problem of system (2.5)–(2.7) in Ω with the specified values of p, $\frac{\partial U}{\partial t}$, $\frac{\partial D}{\partial t}$ at $z = 0$, the specified values of U, $\frac{\partial U}{\partial t}$ at $t = T$, and the specified value of D at $t = z+0$. In a similiar way we can prove the following theorem:

Theorem 2. *Suppose p, U, D are solutions of the above problem, and $p, \frac{\partial U}{\partial t}, \frac{\partial U}{\partial x}, \frac{\partial D}{\partial t}, \frac{\partial D}{\partial x}$ are square integral in $x \in R$, then the following energy inequality is true.*

$$\begin{aligned}
&\int_z^T dt \int dx \Big\{ p^2 + \frac{1}{2}\Big[\Big(\frac{\partial U}{\partial t}\Big)^2 + \Big(\frac{\partial D}{\partial t}\Big)^2\Big]\Big\}(x, Z, t) \\
&\quad \leq \Big\{ \int_0^T dt \int dx \Big\{ p^2 + \frac{1}{2}\Big[\Big(\frac{\partial U}{\partial t}\Big)^2 + \Big(\frac{\partial D}{\partial t}\Big)^2\Big]\Big\}(x, 0, t) + \frac{B}{2}\Big\} \cdot \exp \int_0^Z \Big[2 + \frac{1}{2}V(z)\Big] dz.
\end{aligned}$$

where

$$\begin{aligned}
V(z) &= \max\{|v(x,z)|, x \in R\}, \\
B &= \int_0^Z dz \int dx \Big[\Big(\frac{\partial U}{\partial t}\Big)^2 + \frac{1}{2}\Big(\frac{\partial U}{\partial x}\Big)^2\Big](x, z, T) \\
&\quad + \int_0^Z dz \int dx \Big(\frac{\partial D}{\partial x}\Big)^2 (x, z, z).
\end{aligned}$$

In contrast with the ill-posed direct initial value problem in the direction z for the original two-dimensional wave equation, the above theorems proved that the corresponding direct initial value problem in the direction z for the system of one-way wave equations with sufficient regular $v(x,z)$ is well-posed. But the well-posedness of the potential inversion problem for the system remains to be proven.

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