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BIVARIATE RATIONAL INTERPOLANTS WITH RECTANGLE-HOLE-STRUCTURE*1)

Jie-qing Tan
(*Institute of Applied Mathematics, Hefei University of Technology, Hefei 230009, China*)

Abstract

Bivariate vector valued rational interpolants are established by means of Thiele-type branched continued fractions and Samelson inverse over rectangular grids with holes, characterisation theorem with topologic structure is brought in light and uniqueness theorem in some sense is obtained.

Key words: Branched continued fraction, Interpolation, Vector-grid

1. Introduction

Given a set of distinct real points $\{x_i,\ i=0,1,2,\cdots,\ n : x_i \in \mathbf{R}\}$ and a set of complex vector data $\{\vec{v}^{(i)},\ i=0,1,2,\cdots,\ n : \vec{v}^{(i)} \in \mathbf{C^d}\}$, Graves-Morris showed[5] that the vector valued Thiele type continued fraction

$$\vec{S}(x)=\vec{b}^{(0)}+\frac{x-x_0}{\vec{b}^{(1)}}\,+\,\frac{x-x_1}{\vec{b}^{(2)}}\,+\cdots+\,\frac{x-x_{n-1}}{\vec{b}^{(n)}}$$

can serve to interpolate the given vectors. The construction process is closely ralated to the adoption of the Samelson inverse for vectors

$$\vec{v}^{\,-1}=\frac{\vec{v}^*}{|\vec{v}|^2}, \tag{1.1}$$

where $\vec{v}^*$ denotes the complex conjugate of vector $\vec{v}$. It was proved that $\vec{S}(x)$ is a vector valued rational function with numerator being a d-dimensional polynomial of degree n and denominator being a polynomial of degree $2[n/2]$, here and in the sequel of this paper, $[x]$ represents the integer function.

Let points $(x_i, y_j)\in\mathbf{R^2}$ $(i=0,1,\cdots,n; j=0,1,\cdots,m)$ be given and be arranged in the following table

$$\begin{array}{cccc}(x_0,y_0) & (x_1,y_0) & \cdots & (x_n,y_0)\\ (x_0,y_1) & (x_1,y_1) & \cdots & (x_n,y_1)\\ \vdots & \vdots & \ddots & \vdots\\ (x_0,y_m) & (x_1,y_m) & \cdots & (x_n,y_m)\end{array} \tag{1.2}$$

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which we call rectangular point-grid and denote by $\Pi^{n,m}$. Suppose d-dimensional vector $\vec{v}_{ij}$ is associated with the point (x_i, y_j) in $\Pi^{n,m}$ and let these $\vec{v}_{ij}$'s be arranged as follows

$$\begin{matrix} \vec{v}_{00} & \vec{v}_{10} & \cdots & \vec{v}_{n0} \\ \vec{v}_{01} & \vec{v}_{11} & \cdots & \vec{v}_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ \vec{v}_{0m} & \vec{v}_{1m} & \cdots & \vec{v}_{nm} \end{matrix} \tag{1.3}$$

which is called vector-grid and is denoted by $\vec{V}^{n,m}$.

Definition 1.1. *A $d-$dimensional vector valued polynomial*

$$\vec{N}(x,y) = (N_1(x,y), N_2(x,y), \cdots, N_d(x,y))$$

is said to be of degree n and denoted by $\partial\vec{N} = n$ if $\partial N_i(x,y) \le n$ for $i = 1, 2, \cdots, d$ and $\partial N_j(x,y) = n$ for some $j (1 \le j \le d)$.

Definition 1.2. *Denote by H_n the collection of all bivariate polynomials with total degree not exceeding n and by $\vec{H}_n$ the collection of d dimensional bivariate vector valued polynomials of degree n, then*

$$\vec{H}_{n,m} = \{\vec{N}(x,y)/M(x,y) | \vec{N}(x,y) \in \vec{H}_n, M(x,y) \in H_m\}$$

is called the collection of bivariate vector valued rational functions of type (n/m).

Making use of Samelson inverse and inverse differences, Zhu et al constructed the following Thiele-type branched continued fraction[9]

$$\vec{R}(x,y) = \vec{s}_0(y) + \frac{x - x_0}{\vec{s}_1(y)} + \cdots + \frac{x - x_{n-1}}{\vec{s}_n(y)}, \tag{1.4}$$

where

$$\vec{s}_l(y) = \vec{b}_{l,0}(x_0, \cdots, x_l; y_0) + \frac{y - y_0}{\vec{b}_{l,1}(x_0, \cdots, x_l; y_0, y_1)} + \cdots + \frac{y - y_{m-1}}{\vec{b}_{l,m}(x_0, \cdots, x_l; y_0, \cdots, y_m)}, \tag{1.5}$$

and $\vec{b}_{i,j}(x_0, \cdots, x_i; y_0, \cdots, y_j)$ are computed through the following recursive process

$$\vec{b}_{0,0}(x_i, y_j) = \vec{v}_{ij}, \quad i = 0, 1, \cdots, n; \ j = 0, 1, \cdots, m \tag{1.6}$$

$$\vec{b}_{0,j}(x_0; y_0, \cdots, y_j) = \frac{y_j - y_{j-1}}{\vec{b}_{0,j-1}(x_0; y_0, \cdots, y_{j-2}, y_j) - \vec{b}_{0,j-1}(x_0; y_0, \cdots, y_{j-2}, y_{j-1})} \tag{1.7}$$

$$\vec{b}_{i,0}(x_0, \cdots, x_i; y_0) = \frac{x_i - x_{i-1}}{\vec{b}_{i-1,0}(x_0, \cdots, x_{i-2}, x_i; y_0) - \vec{b}_{i-1,0}(x_0, \cdots, x_{i-2}, x_{i-1}; y_0)} \tag{1.8}$$

$$\begin{aligned} \vec{b}_{i,j}(x_0, \cdots, x_i; y_0, \cdots, y_j) = &(y_j - y_{j-1}) \Big/ [\vec{b}_{i,j-1}(x_0, \cdots, x_i; y_0, \cdots, y_{j-2}, y_j) \\ &- \vec{b}_{i,j-1}(x_0, \cdots, x_i; y_0, \cdots, y_{j-2}, y_{j-1})] \end{aligned} \tag{1.9}$$

It was shown in [9] that $\vec{R}(x,y) \in \vec{H}_{nm+n+m, 2[(nm+n+m)/2]}$ and

$$\vec{R}(x_i, y_j) = \vec{v}_{ij}, \quad i = 0, 1, \cdots, n; \ j = 0, 1, \cdots, m.$$

which surely extends the results obtained by Graves-Morris[5] in univariate vector case and by Siemaszko[6] in bivariate scalar case. But it does not include Cuyt and Verdonk's results[1,3], where a kind of symmetric branched continued fractions is considered.

The motivation for us to study irregular structures is based on such a recognization that for some scattered data set of points (see Fig. 1.1), whatever a new numbering of (x_i, y_j) is made, the obtained picture of the points always looks irregular, and therefore the method of general order multivariate rational Hermite interpolants cannot be used to deal with this situation since the data set of points in this case does not satisfy the inclusion property[2], a very important property dominating whether the rational Hermite interpolants exist or not. In this paper, we consider the bivariate vector valued rational interpolants over rectangular mesh with rectangular holes (this long terminology is cited as BVRIHs in the sequel of the text). For scalar case we refer to [4].

$$\begin{array}{cccccc}
 & x_0 & x_1 & x_2 & x_3 & x_4 \\
y_0 & \star & \star & \star & \star & \star \\
y_1 & \star & & & \star & \star \\
y_2 & \star & & \star & & \star \\
y_3 & \star & \star & & & \star \\
y_4 & \star & \star & \star & \star & \star
\end{array}$$

Fig.1.1

2. Characterisation of BVRIHs

Suppose the set of points is $\Pi = \Pi^{n,m} - \Pi^{c_1,c_2;r_1,r_2}$, where $\Pi^{c_1,c_2;r_1,r_2} = \{x_{c_1+1}, \cdots, x_{c_2-1}\} \times \{y_{r_1+1}, \cdots, y_{r_2-1}\}$, and the set of vector data is $\vec{V} = \{\vec{v}_{i,j} \in \vec{V}^{n,m} | (x_i, y_j) \in \Pi\}$. The structure of Π, as shown in Fig.2.1, looks as if a subrectangular block had been moved outside of $\Pi^{n,m}$, and it may also be regarded as a left-over resulting from covering $\Pi^{n,m}$ with a empty mesh of the same size as $\Pi^{c_1,c_2;r_1,r_2}$. The latter desciption about the structure of Π, as is to be seen, provides us with better and more understanding for the characterisation of interpolant over Π.

$$\begin{array}{cccccccc}
 & x_0 & \cdots & x_{c_1} & \cdots & x_{c_2} & \cdots & x_n \\
y_0 & * & \cdots & * & \cdots & * & \cdots & * \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
y_{r_1} & * & \cdots & * & \cdots & * & \cdots & * \\
\vdots & \vdots & \ddots & \vdots & B & \vdots & \ddots & \vdots \\
y_{r_2} & * & \cdots & * & \cdots & * & \cdots & * \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
y_m & * & \cdots & * & \cdots & * & \cdots & *
\end{array}$$

Fig.2.1

In Fig.2.1 B (for "block") is a sub-rectangle size of which is $(c_2-c_1-1)\times(r_2-r_1-1)$ (here by size we mean the number of contained points) and on which no interpolation is taken into consideration. Now we may construct a bivariate vector valued rational interpolant on Π by means of the following branched continued fraction

$$\vec{R}(x,y)=\vec{s}_0(y)+\frac{x-x_0}{\vec{s}_1(y)}+\cdots+\frac{x-x_{n-1}}{\vec{s}_n(y)}, \tag{2.1}$$

where for $i=0,1,\cdots,c_1,c_2,c_2+1,\cdots,n$

$$\vec{s}_i(y)=\vec{b}_{i,0}+\frac{y-y_0}{\vec{b}_{i,1}}+\cdots+\frac{y-y_{m-1}}{\vec{b}_{i,m}} \tag{2.2}$$

and for $i=c_1+1,\cdots,c_2-1$

$$\begin{aligned}\vec{s}_i(y)=&\vec{b}_{i,0}\\&+\frac{y-y_0}{\vec{b}_{i,1}}+\cdots+\frac{y-y_{r_1-1}}{\vec{b}_{i,r_1}}+\frac{y-y_{r_1}}{\vec{b}_{i,r_2}}+\frac{y-y_{r_2}}{\vec{b}_{i,r_2+1}}+\cdots+\frac{y-y_{m-1}}{\vec{b}_{i,m}}.\end{aligned} \tag{2.3}$$

Let

$$\vec{R}_1(x,y)=\frac{\vec{A}_1(x,y)}{B_1(x,y)}=\vec{s}_{c_2}(y)+\frac{x-x_{c_2}}{\vec{s}_{c_2+1}(y)}+\cdots+\frac{x-x_{n-1}}{\vec{s}_n(y)}, \tag{2.4}$$

from the characterisation theorem in [9] we know that

$$|\vec{A}_1(x,y)|^2=B_1(x,y)D_1(x,y)$$

and

$$\partial\vec{A}_1(x,y)=M-1,\ \partial B_1(x,y)=2\Big[\frac{M-1}{2}\Big],\ \partial D_1(x,y)=2\Big[\frac{M}{2}\Big],$$

where

$$M=(m+1)(n+1-c_2). \tag{2.5}$$

Let

$$\begin{aligned}\vec{R}_2(x,y)=&\frac{\vec{A}_2(x,y)}{B_2(x,y)}=\vec{s}_{c_2-1}(y)+\frac{x-x_{c_2-1}}{R_1(x,y)}\\=&\frac{\vec{a}_{c_2-1}(y)}{b_{c_2-1}(y)}+\frac{(x-x_{c_2-1})\vec{A}_1(x,y)}{D_1(x,y)},\end{aligned} \tag{2.6}$$

by Graves-Morris' characterisation theorem we have

$$|\vec{A}_2(x,y)|^2=B_2(x,y)D_2(x,y)$$

and

$$\begin{aligned}&\partial\vec{A}_2(x,y)=\begin{cases}M+N, & M\cdot N \text{ is even}\\ M+N-1, & M\cdot N \text{ is odd}\end{cases}\\&\partial B_2(x,y)=2[M/2]+2[N/2]\\&\partial D_2(x,y)=\begin{cases}2\Big[\frac{M+1}{2}\Big]+2\Big[\frac{N+1}{2}\Big], & M\cdot N \text{ is even}\\ 2\Big[\frac{M+1}{2}\Big]+2\Big[\frac{N+1}{2}\Big]-2, & M\cdot N \text{ is odd}\end{cases}\end{aligned}$$

where

$$N = m + 1 + r_1 - r_2. \tag{2.7}$$

Let

$$\begin{aligned}\vec{R}_3(x,y) =&\frac{\vec{A}_3(x,y)}{B_3(x,y)} = \vec{s}_{c_2-2}(y) + \frac{x - x_{c_2-2}}{\vec{R}_2(x,y)}\\ =&\frac{\vec{a}_{c_2-2}(y)}{b_{c_2-2}(y)} + \frac{(x - x_{c_2-2})\vec{A}_2(x,y)}{D_2(x,y)},\end{aligned} \tag{2.8}$$

then it follows

$$|\vec{A}_3(x,y)|^2 = B_3(x,y)D_3(x,y)$$

and

$$\begin{aligned}\partial\vec{A}_3(x,y) =&\begin{cases} M + 2N + 1, & M \cdot N \text{ is even} \\ M + 2N, & M \cdot N \text{ is odd}\end{cases}\\ \partial B_3(x,y) =&\begin{cases} 2[N/2] + M + N, & M + N \text{ is even} \\ 2[N/2] + M + N + 1, & M + N \text{ is odd}\end{cases}\\ \partial D_3(x,y) =&\begin{cases} M + 2N + 2, & M \text{ is even} \\ M + 2N + 1, & M \text{ is odd.}\end{cases}\end{aligned}$$

Similarly if we let

$$\vec{R}_4(x,y) = \frac{\vec{A}_4(x,y)}{B_4(x,y)} = \vec{s}_{c_2-3}(y) + \frac{x - x_{c_2-3}}{\vec{R}_3(x,y)} \tag{2.9}$$

and

$$\vec{R}_5(x,y) = \frac{\vec{A}_5(x,y)}{B_5(x,y)} = \vec{s}_{c_2-4}(y) + \frac{x - x_{c_2-4}}{\vec{R}_4(x,y)}, \tag{2.10}$$

then

$$\begin{aligned}\partial\vec{A}_4(x,y) &= \begin{cases} M + 3N + 2, & M \cdot N \text{ is even} \\ M + 3N + 1, & M \cdot N \text{ is odd}\end{cases}\\ \partial B_4(x,y) &= 2[M/2] + 2[3N/2] + 2\\ \partial\vec{A}_5(x,y) &= \begin{cases} M + 4N + 3, & M \cdot N \text{ is even} \\ M + 4N + 2, & M \cdot N \text{ is odd}\end{cases}\\ \partial B_5(x,y) &= \begin{cases} 2[N/2] + M + 3N + 2, & M + N \text{ is even} \\ 2[N/2] + M + 3N + 3, & M + N \text{ is odd.}\end{cases}\end{aligned}$$

Let

$$\vec{T}_0(x,y) = \vec{R}_{c_2-c_1}(x,y) = \frac{\vec{A}_{c_2-c_1}(x,y)}{B_{c_2-c_1}(x,y)} = \vec{s}_{c_1+1}(y) + \frac{x - x_{c_1+1}}{\vec{R}_{c_2-c_1-1}(x,y)} \tag{2.11}$$

and

$$M^* = M + (c_2 - c_1 - 1)(N + 1) - 1, \tag{2.12}$$

then it can be shown by induction that

$$|\vec{A}_{c_2-c_1}(x,y)|^2 = B_{c_2-c_1}(x,y)D_{c_2-c_1}(x,y)$$

where

$$\partial\vec{A}_{c_2-c_1}(x,y) = \begin{cases} M^*, & M\cdot N \text{ is even} \\ M^*-1, & M\cdot N \text{ is odd} \end{cases}$$

$$\partial B_{c_2-c_1}(x,y) = 2\Big[\frac{M}{2}\Big] + 2\Big[\frac{(c_2-c_1-1)N}{2}\Big] + c_2 - c_1 - 2$$

if $c_2 - c_1$ is even,

$$\partial B_{c_2-c_1}(x,y) = \begin{cases} 2[N/2] + M + (c_2-c_1-2)(N+1) - 1, & M+N \text{ is even} \\ 2[N/2] + M + (c_2-c_1-2)(N+1), & M+N \text{ is odd} \end{cases}$$

if $c_2 - c_1$ is odd, and

$$\partial D_{c_2-c_1}(x,y) = 2\Big(M^* - \Big[\frac{M^*}{2}\Big]\Big).$$

For any interger k, define

$$mod_2 k = \begin{cases} 0, & \text{if } k = 2[k/2] \\ 1, & \text{if } k = 2[k/2]+1. \end{cases} \tag{2.13}$$

Let

$$\bar{m} = mod_2 m,\ \ \bar{M}^* = mod_2 M^*,\ \ \bar{N} = mod_2 N \tag{2.14}$$

and let

$$\vec{T}_1(x,y) = \frac{\vec{U}_1(x,y)}{V_1(x,y)} = \vec{s}_{c_1}(y) + \frac{x - x_{c_1}}{\vec{T}_0(x,y)}, \tag{2.15}$$

then there exists a polynomial $W_1(x,y)$ such that

$$|\vec{U}_1(x,y)|^2 = V_1(x,y)W_1(x,y)$$

and

$$\partial\vec{U}_1(x,y) = \begin{cases} M^*+m+1, & \text{if } (\bar{m},\bar{M}^*,\bar{N}) \in \{(0,0,0),(*,1,*)\} \\ M^*+m, & \text{if } (\bar{m},\bar{M}^*,\bar{N}) \in \{(0,0,1),(1,0,*)\} \end{cases},$$

$$\partial\vec{V}_1(x,y) = \begin{cases} M^*+m+1, & \text{if } (\bar{m},\bar{M}^*,\bar{N}) \in \{(0,1,*)\} \\ M^*+m, & \text{if } (\bar{m},\bar{M}^*,\bar{N}) \in \{(0,0,*),(1,1,*)\} \\ M^*+m-1, & \text{if } (\bar{m},\bar{M}^*,\bar{N}) \in \{(1,0,*)\} \end{cases},$$

$$\partial\vec{W}_1(x,y) = \begin{cases} M^*+m+2, & \text{if } (\bar{m},\bar{M}^*,\bar{N}) \in \{(0,0,0),(1,1,*)\} \\ M^*+m+1, & \text{if } (\bar{m},\bar{M}^*,\bar{N}) \in \{(0,1,*),(1,0,*)\} \\ M^*+m, & \text{if } (\bar{m},\bar{M}^*,\bar{N}) \in \{(0,0,1)\} \end{cases},$$

where "$*$" stands for 0 and 1, for example, $(0,1,*)$ means $(0,1,0)$ and $(0,1,1)$. Let

$$\vec{T}_2(x,y) = \frac{\vec{U}_2(x,y)}{V_2(x,y)} = \vec{s}_{c_1-1}(y) + \frac{x - x_{c_1-1}}{\vec{T}_1(x,y)}, \tag{2.16}$$

then there exists a polynomial $W_2(x,y)$ such that

$$|\vec{U}_2(x,y)|^2 = V_2(x,y)W_2(x,y)$$

and

$$\partial\vec{U}_2(x,y) = \begin{cases} M^*+2m+2, & \text{if } (\bar{m},\bar{M}^*,\bar{N}) \in \{(0,0,0),(*,1,*)\} \\ M^*+2m+1, & \text{if } (\bar{m},\bar{M}^*,\bar{N}) \in \{(0,0,1),(1,0,*)\} \end{cases},$$

$$\partial\vec{V}_2(x,y) = \begin{cases} M^*+2m+2, & \text{if } (\bar{m},\bar{M}^*,\bar{N}) \in \{(0,0,0)\} \\ M^*+2m+1, & \text{if } (\bar{m},\bar{M}^*,\bar{N}) \in \{(*,1,*)\} \\ M^*+2m, & \text{if } (\bar{m},\bar{M}^*,\bar{N}) \in \{(0,0,1),(1,0,*)\} \end{cases},$$

$$\partial\vec{W}_2(x,y) = \begin{cases} M^*+2m+3, & \text{if } (\bar{m},\bar{M}^*,\bar{N}) \in \{(*,1,*)\} \\ M^*+2m+2, & \text{if } (\bar{m},\bar{M}^*,\bar{N}) \in \{(*,0,*)\} \end{cases}.$$

Let

$$\vec{T}_3(x,y) = \frac{\vec{U}_3(x,y)}{V_3(x,y)} = \vec{s}_{c_1-2}(y) + \frac{x-x_{c_1-2}}{\vec{T}_2(x,y)}, \tag{2.17}$$

and

$$\vec{T}_4(x,y) = \frac{\vec{U}_4(x,y)}{V_4(x,y)} = \vec{s}_{c_1-3}(y) + \frac{x-x_{c_1-3}}{\vec{T}_3(x,y)}, \tag{2.18}$$

similarly one gets

$$\partial\vec{U}_3(x,y) = \begin{cases} M^*+3m+3, & \text{if } (\bar{m},\bar{M}^*,\bar{N}) \in \{(0,0,0),(*,1,*)\} \\ M^*+3m+2, & \text{if } (\bar{m},\bar{M}^*,\bar{N}) \in \{(0,0,1),(1,0,*)\} \end{cases},$$

$$\partial\vec{V}_3(x,y) = \begin{cases} M^*+3m+3, & \text{if } (\bar{m},\bar{M}^*,\bar{N}) \in \{(0,1,*)\} \\ M^*+3m+2, & \text{if } (\bar{m},\bar{M}^*,\bar{N}) \in \{(0,0,*),(1,1,*)\} \\ M^*+3m+1, & \text{if } (\bar{m},\bar{M}^*,\bar{N}) \in \{(1,0,*)\} \end{cases},$$

$$\partial\vec{U}_4(x,y) = \begin{cases} M^*+4m+4, & \text{if } (\bar{m},\bar{M}^*,\bar{N}) \in \{(0,0,0),(*,1,*)\} \\ M^*+4m+3, & \text{if } (\bar{m},\bar{M}^*,\bar{N}) \in \{(0,0,1),(1,0,*)\} \end{cases},$$

$$\partial\vec{V}_4(x,y) = \begin{cases} M^*+4m+4, & \text{if } (\bar{m},\bar{M}^*,\bar{N}) \in \{(0,0,0)\} \\ M^*+4m+3, & \text{if } (\bar{m},\bar{M}^*,\bar{N}) \in \{(*,1,*)\} \\ M^*+4m+2, & \text{if } (\bar{m},\bar{M}^*,\bar{N}) \in \{(0,0,1),(1,0,*)\} \end{cases}.$$

One finally sees

$$\vec{R}(x,y) = \vec{T}_{c_1+1}(x,y) = \frac{\vec{U}_{c_1+1}(x,y)}{V_{c_1+1}(x,y)} = \vec{s}_0(y) + \frac{x-x_0}{\vec{T}_{c_1}(x,y)} \tag{2.19}$$

and derives by induction that

$$\partial\vec{U}_{c_1+1}(x,y) = \begin{cases} M^*+(c_1+1)(m+1), & \text{if } (\bar{m},\bar{M}^*,\bar{N}) \in \{(0,0,0),(*,1,*)\} \\ M^*+(c_1+1)(m+1)-1, & \text{if } (\bar{m},\bar{M}^*,\bar{N}) \in \{(0,0,1),(1,0,*)\} \end{cases},$$

$$\partial\vec{V}_{c_1+1}(x,y)=\begin{cases} M^*+(c_1+1)(m+1), & \text{if } (\bar{m},\bar{M}^*,\bar{N})\in\{(0,1,*)\} \\ M^*+(c_1+1)(m+1)-1, & \text{if } (\bar{m},\bar{M}^*,\bar{N})\in\{(0,0,*),(1,1,*)\} \\ M^*+(c_1+1)(m+1)-2, & \text{if } (\bar{m},\bar{M}^*,\bar{N})\in\{(1,0,*)\} \end{cases},$$

if c_1 is even, and

$$\partial\vec{V}_{c_1+1}(x,y)=\begin{cases} M^*+(c_1+1)(m+1), & \text{if } (\bar{m},\bar{M}^*,\bar{N})\in\{(0,0,0)\} \\ M^*+(c_1+1)(m+1)-1, & \text{if } (\bar{m},\bar{M}^*,\bar{N})\in\{(*,1,*)\} \\ M^*+(c_1+1)(m+1)-2, & \text{if}(\bar{m},\bar{M}^*,\bar{N})\in\{(0,0,1),(1,0,*)\} \end{cases}.$$

if c_1 is odd. Let

$$B=M^*+(c_1+1)(m+1). \tag{2.20}$$

Noticing that

$$\begin{aligned} M^* &= M+(c_2-c_1-1)(N+1)-1 \\ &= (m+1)(n+1-c_2)+(c_2-c_1-1)(m+2+r_1-r_2)-1, \end{aligned}$$

we have

$$\begin{aligned} B =&(m+1)(n+2+c_1-c_2)+(c_2-c_1-1)(m+2+r_1-r_2)-1 \\ =&(m+1)(n+1)+(m+1)(1+c_1-c_2) \\ &-(1+c_1-c_2)(m+1)-(1+c_1-c_2)(1+r_1-r_2)-1 \\ =&(m+1)(n+1)-(c_2-c_1-1)(r_2-r_1-1)-1 \end{aligned}$$

which is only one less than the number of interpolation points contained in Π. Therefore

$$\vec{R}(x,y)\in\begin{cases} \vec{H}_{B,B}, & \text{if } (\bar{m},\bar{M}^*,\bar{N})\in\{(0,0,0)\} \\ \vec{H}_{B,B-1}, & \text{if } (\bar{m},\bar{M}^*,\bar{N})\in\{(*,1,*)\} \\ \vec{H}_{B-1,B-2}, & \text{if } (\bar{m},\bar{M}^*,\bar{N})\in\{(0,0,1),(1,0,*)\} \end{cases}. \tag{2.21}$$

if c_1 is odd, and

$$\vec{R}(x,y)\in\begin{cases} \vec{H}_{B,B}, & \text{if } (\bar{m},\bar{M}^*,\bar{N})\in\{(0,1,*)\} \\ \vec{H}_{B,B-1}, & \text{if } (\bar{m},\bar{M}^*,\bar{N})\in\{(1,1,*),(0,0,0)\} \\ \vec{H}_{B-1,B-1}, & \text{if } (\bar{m},\bar{M}^*,\bar{N})\in\{(0,0,1)\} \\ \vec{H}_{B-1,B-2}, & \text{if } (\bar{m},\bar{M}^*,\bar{N})\in\{(1,0,*)\} \end{cases} \tag{2.22}$$

if c_1 is even. In order to find out the direct relation between the characterisation of $\vec{R}(x,y)$ and the given size parameters m,n,c_1,c_2,r_1,r_2, we define

$$\begin{aligned} &\varphi_1(\bar{m},\bar{n},\bar{c}_1,\bar{c}_2,\overline{r_2-r_1})=\bar{m}, \\ &\varphi_2(\bar{m},\bar{n},\bar{c}_1,\bar{c}_2,\overline{r_2-r_1})=(\bar{m}+1)(\bar{n}+1-\bar{c}_2)+(\bar{c}_2-\bar{c}_1-1)(\bar{m}+2-\overline{r_2-r_1})-1, \\ &\varphi_3(\bar{m},\bar{n},\bar{c}_1,\bar{c}_2,\overline{r_2-r_1})=\bar{m}+1-\overline{r_2-r_1}, \end{aligned} \tag{2.23}$$

where

$$\bar{m}=mod_2 m,\ \bar{n}=mod_2 n,\ \bar{c}_1=mod_2 c_1,\ \bar{c}_2=mod_2 c_2,\ \overline{r_2-r_1}=mod_2(r_2-r_1),$$

and we call $(\bar{m}, \bar{n}, \bar{c}_1, \bar{c}_2, \overline{r_2 - r_1})$ the parameter class equivalent to the set $\{(m, n, c_1, c_2, r_2 - r_1)\}$. Let

$$\bar{\varphi}_i = mod_2 \varphi_i, \ i = 1, 2, 3 \tag{2.24}$$

and establish the following mapping

$$\psi : (\bar{m}, \bar{n}, \bar{c}_1, \bar{c}_2, \overline{r_2 - r_1}) \longmapsto (\bar{\varphi}_1, \bar{\varphi}_2, \bar{\varphi}_3). \tag{2.25}$$

As a matter of fact, the mapping ψ bridges between $(\bar{m}, \bar{n}, \bar{c}_1, \bar{c}_2, \overline{r_2 - r_1})$ and $(\bar{m}, \bar{M}^*, \bar{N})$, and the following tables can be obtained by careful computation and classification

Table 2.1 $\bar{c}_1 = 0$

$(1,1,0,1,1)$	$\longmapsto$	$(1,1,1)$	$(1,1,0,0,0)$	$\longmapsto$	$(1,0,0)$
$(1,0,0,1,1)$	$\longmapsto$	$(1,1,1)$	$(1,0,0,0,0)$	$\longmapsto$	$(1,0,0)$
$(1,1,0,0,1)$	$\longmapsto$	$(1,1,1)$	$(0,0,0,1,1)$	$\longmapsto$	$(0,1,0)$
$(1,0,0,0,1)$	$\longmapsto$	$(1,1,1)$	$(0,0,0,0,1)$	$\longmapsto$	$(0,1,0)$
$(1,1,0,1,0)$	$\longmapsto$	$(1,1,0)$	$(0,1,0,1,0)$	$\longmapsto$	$(0,0,1)$
$(1,0,0,1,0)$	$\longmapsto$	$(1,1,0)$	$(0,0,0,0,0)$	$\longmapsto$	$(0,0,1)$
$(0,0,0,1,0)$	$\longmapsto$	$(0,1,1)$	$(0,1,0,1,1)$	$\longmapsto$	$(0,0,0)$
$(0,1,0,0,0)$	$\longmapsto$	$(0,1,1)$	$(0,1,0,0,1)$	$\longmapsto$	$(0,0,0)$

Table 2.2 $\bar{c}_1 = 1$

$(1,1,1,1,1)$	$\longmapsto$	$(1,1,1)$	$(1,1,1,1,0)$	$\longmapsto$	$(1,0,0)$
$(1,1,1,0,1)$	$\longmapsto$	$(1,1,1)$	$(1,0,1,1,0)$	$\longmapsto$	$(1,0,0)$
$(1,0,1,1,1)$	$\longmapsto$	$(1,1,1)$	$(0,1,1,1,1)$	$\longmapsto$	$(0,1,0)$
$(1,0,1,0,1)$	$\longmapsto$	$(1,1,1)$	$(0,1,1,0,1)$	$\longmapsto$	$(0,1,0)$
$(1,1,1,0,0)$	$\longmapsto$	$(1,1,0)$	$(0,1,1,1,0)$	$\longmapsto$	$(0,0,1)$
$(1,0,1,0,0)$	$\longmapsto$	$(1,1,0)$	$(0,0,1,0,0)$	$\longmapsto$	$(0,0,1)$
$(0,1,1,0,0)$	$\longmapsto$	$(0,1,1)$	$(0,0,1,1,1)$	$\longmapsto$	$(0,0,0)$
$(0,0,1,1,0)$	$\longmapsto$	$(0,1,1)$	$(0,0,1,0,1)$	$\longmapsto$	$(0,0,0)$

Hence we have

Theorem 2.1 (Characterisation Theorem.) *Suppose* $\vec{R}(x,y)$ *is given by* (2.1)–(2.3), *then*

$$\vec{R}(x,y) \in \begin{cases} \vec{H}_{B,B}, & \text{for } (\bar{m}, \bar{n}, \bar{c}_1, \bar{c}_2, \overline{r_2 - r_1}) \in \{(0,0,*,*,1), \\ & \qquad (0,0,0,1,0), (0,1,0,0,0)\} \\ \vec{H}_{B,B-1}, & \text{for } (\bar{m}, \bar{n}, \bar{c}_1, \bar{c}_2, \overline{r_2 - r_1}) \in \\ & \qquad \left\{ \begin{array}{lll} (0,1,*,*,1), & (1,*,*,*,1), & (1,*,1,0,0) \\ (1,*,0,1,0), & (0,1,1,0,0), & (0,0,1,1,0) \end{array} \right. \\ \vec{H}_{B-1,B-1}, & \text{for } (\bar{m}, \bar{n}, \bar{c}_1, \bar{c}_2, \overline{r_2 - r_1}) \in \{(0,1,0,1,0), (0,0,0,0,0)\} \\ \vec{H}_{B-1,B-2}, & \text{for } (\bar{m}, \bar{n}, \bar{c}_1, \bar{c}_2, \overline{r_2 - r_1}) \in \left\{ \begin{array}{ll} (1,*,1,1,0), & (1,*,0,0,0) \\ (0,1,1,1,0), & (0,0,1,0,0) \end{array} \right\}. \end{cases}$$

From the theorem it can easily be seen that the characterisation of $\vec{R}(x,y)$ is independent of r_1 and r_2 as long as $r_2 - r_1$ is fixed, but it depends on the parameters c_1 and

c_2 to a large extent even if $c_2 - c_1$ is fixed. In other words, if one keeps the geometric shape of the block B unchanged by fixing $r_2 - r_1$ and $c_2 - c_1$, then Theorem 2.1 illustrates that moving the block B up or down within the range of $\Pi^{n,m}$ does not affect the characterisation of $\vec{R}(x, y)$, but a horizontal movement of the block B may result in a change of the characterisation of $\vec{R}(x, y)$. However if the movement is assumed to be carried out once a row vertically or once a column horizontally, then a horizontal movement by even times does not destroy the characterization.

Example 2.1. Let $m = n = 10, c_1 = 3, c_2 = 7, r_1 = 2, r_2 = 8$, then $B = 105$ and $\vec{R}(x, y) \in \vec{H}_{105,104}$, but $m = n = 10, c_1 = 4, c_2 = 8, r_1 = 2, r_2 = 8$ results in $B = 105$ and $\vec{R}(x, y) \in \vec{H}_{104,104}$. However choosing $m = n = 10, c_1 = 1, c_2 = 5$ (or $c_1 = 5, c_2 = 9$), $r_1 = 2, r_2 = 8$ leads back to $B = 105$ and $\vec{R}(x, y) \in \vec{H}_{105,104}$.

Example 2.2. Let $m = 8, n = 6, c_1 = 2, c_2 = 5, r_1 = 1, r_2 = 5$, then $B = 56$ and $\vec{R}(x, y) \in \vec{H}_{56,56}$, but $m = 8, n = 6, c_1 = 1, c_2 = 4, r_1 = 1, r_2 = 5$ results in $B = 56$ and $\vec{R}(x, y) \in \vec{H}_{55,54}$.

Furthermore from all 32 equivalent parameter classes regarding $(\bar{m}, \bar{n}, \bar{c}_1, \bar{c}_2, \overline{r_2 - r_1})$ one may pick out those which are prone to changing the characterisation of interpolant $\vec{R}(x, y)$ once some certain block movement yields, they are

$$\begin{array}{llll}
(0,0,0,1,0), & (0,1,0,0,0) & \text{for type} & (B/B) \\
(0,1,1,0,0), & (0,0,1,1,0) & \text{for type} & (B/B-1) \\
(0,1,0,1,0), & (0,0,0,0,0) & \text{for type} & (B-1/B-1) \\
(0,1,1,1,0), & (0,0,1,0,0) & \text{for type} & (B-1/B-2).
\end{array} \tag{2.26}$$

It is not difficult to observe that if the parameters of the interpolation set of points Π is associated with one of the above classes, then a horizontal movement of the block by a column causes the exchanges of the characterisation of $\vec{R}(x, y)$ between type $(B/B-1)$ and type $(B-1/B-1)$ (as illustrated by Example 2.1) or between type (B/B) and type $(B-1/B-2)$ (as illustrated by Example 2.2). A more careful observation reveals that only the classes with both $\bar{m} = 0$ and $\overline{r_2 - r_1} = 0$ appear in equation (2.26), which implies that the characterisation of interpolant $\vec{R}(x, y)$ over Π has some property of topological invariance, provided that at least one of the parameters m and $r_2 - r_1$ is odd. Based on this ground, each of all other 24 equivalent paremeter classes of the forms $(1, *, *, *, *)$ and $(0, *, *, *, 1)$ is said to be topologically invariable class with respect to block movements.

We point out that the characterisation theorem applies to the case where interpolation is considered over rectangular grids without block. In fact, it may be regarded as the special block structure with $c_1 = n, c_2 = n+1$, and $r_2 - r_1 = 0$. Then there exist only four possible equivalent classes regarding the parameters $m, n, c_1, c_2, r_2 - r_1$, i.e.,

$$(1,0,0,1,0),\ (0,1,1,0,0),\ (1,1,1,0,0),\ (0,0,0,1,0).$$

We see $\vec{R}(x, y) \in \vec{H}_{B,B-1}$ for the first three classes and $\vec{R}(x, y) \in \vec{H}_{B,B}$ for the last class, where $B = (m+1)(n+1) - 1$, therefore $\vec{R}(x, y) \in \vec{H}_{(m+1)(n+1)-1, 2[((m+1)(n+1)-1)/2]}$, as asserted in [7] and [9].

3. Uniqueness of BVRIHs

For the mesh shown in Fig.2.1, we can also construct another type of BVRIH

$$\vec{DR}(x,y) = \vec{t}_0(x) + \frac{y-y_0}{\vec{t}_1(x)} + \cdots + \frac{y-y_{m-1}}{\vec{t}_m(x)}, \tag{3.1}$$

where for $i = 0, 1, \cdots, r_1, r_2, \cdots, m$

$$\vec{t}_i(x) = \vec{d}_{i,0} + \frac{x-x_0}{\vec{d}_{i,1}} + \cdots + \frac{x-x_{n-1}}{\vec{d}_{i,n}} \tag{3.2}$$

and for $i = r_1+1, \cdots, r_2-1$

$$\vec{t}_i(x) = \vec{d}_{i,0} + \frac{x-x_0}{\vec{d}_{i,1}} + \cdots + \frac{x-x_{c_1-1}}{\vec{d}_{i,c_1}} + \frac{x-x_{c_1}}{\vec{d}_{i,c_2}} + \frac{x-x_{c_2}}{\vec{d}_{i,c_2+1}} + \cdots + \frac{x-x_{n-1}}{\vec{d}_{i,n}}. \tag{3.3}$$

As a direct consequence of Theorem 2.1 we immediately have

Theorem 3.1. *Suppose $\vec{DR}(x,y)$ is given by (3.1)–(3.3), then*

$$\vec{DR}(x,y) \in \begin{cases} \vec{H}_{B,B}, & \text{for } (\bar{n},\bar{m},\bar{r}_1,\bar{r}_2,\overline{c_2-c_1}) \in \{(0,0,*,*,1), \\ & \qquad (0,0,0,1,0),(0,1,0,0,0)\} \\ \vec{H}_{B,B-1}, & \text{for } (\bar{n},\bar{m},\bar{r}_1,\bar{r}_2,\overline{c_2-c_1}) \in \\ & \left\{ \begin{array}{lll} (0,1,*,*,1), & (1,*,*,*,1), & (1,*,1,0,0) \\ (1,*,0,1,0), & (0,1,1,0,0), & (0,0,1,1,0) \end{array} \right\}, \\ \vec{H}_{B-1,B-1}, & \text{for } (\bar{n},\bar{m},\bar{r}_1,\bar{r}_2,\overline{c_2-c_1}) \in \{(0,1,0,1,0),(0,0,0,0,0)\} \\ \vec{H}_{B-1,B-2} & \text{for } (\bar{n},\bar{m},\bar{r}_1,\bar{r}_2,\overline{c_2-c_1}) \in \left\{ \begin{array}{ll} (1,*,1,1,0), & (1,*,0,0,0) \\ (0,1,1,1,0), & (0,0,1,0,0) \end{array} \right\} \end{cases}$$

where $\bar{r}_1 = mod_2 r_1, \bar{r}_2 = mod_2 r_2$ and $\overline{c_2-c_1} = mod_2(c_2-c_1)$.

Although both $\vec{R}(x,y)$ and $\vec{DR}(x,y)$ interpolate $\vec{V}$ over Π, they may belong to different rational types owing to the geometric structure of the block. To find out the common characterisation of $\vec{R}(x,y)$ and $\vec{DR}(x,y)$ we extend the equivalent classes $(\bar{m},\bar{n},\bar{c}_1,\bar{c}_2,\overline{r_2-r_1})$ to $(\bar{m},\bar{n},\bar{c}_1,\bar{c}_2,\bar{r}_1,\bar{r}_2)$. From Theorem 2.1 and Theorem 3.1 it is not difficult to prove

Theorem 3.2. *Suppose $\vec{R}(x,y)$ and $\vec{DR}(x,y)$ are given by* (2.1)–(2.3) *and* (3.1)–(3.3) *respectively and they interpolate $\vec{V}$ over Π, then both $\vec{R}(x,y)$ and $\vec{DR}(x,y)$ are normally of the same type (B/B) if*

$$(\bar{m},\bar{n},\bar{c}_1,\bar{c}_2,\bar{r}_1,\bar{r}_2) \in \left\{ \begin{array}{lll} (0,0,0,1,*,*), & (0,0,1,0,0,1), & (0,0,1,0,1,0) \\ (0,0,0,0,0,1), & (0,0,1,1,0,1) & \end{array} \right\};$$

$\vec{R}(x,y)$ and $\vec{DR}(x,y)$ are normally of the same type $(B/B-1)$ if

$$(\bar{m},\bar{n},\bar{c}_1,\bar{c}_2,\bar{r}_1,\bar{r}_2) \in$$

$$\left\{\begin{array}{llll}(1,0,0,1,*,*), & (1,0,1,0,*,*), & (*,1,1,0,*,*), & (*,1,0,0,1,0) \\ (*,1,1,1,1,0), & (*,1,0,0,0,1), & (*,1,1,1,0,1), & (1,1,0,1,*,*) \\ (1,0,0,0,1,0), & (1,0,1,1,1,0), & (0,0,1,1,1,1), & (0,1,0,1,0,1) \\ (0,1,0,1,1,0) & & & \end{array}\right\};$$

$\vec{R}(x,y)$ and $D\vec{R}(x,y)$ are normally of the same type $(B-1/B-1)$ if

$$(\bar{m},\bar{n},\bar{c}_1,\bar{c}_2,\bar{r}_1,\bar{r}_2)\in\{(0,0,0,0,0,0)\}$$

and $\vec{R}(x,y)$ and $D\vec{R}(x,y)$ are normally of the same type $(B-1/B-2)$ if

$$(\bar{m},\bar{n},\bar{c}_1,\bar{c}_2,\bar{r}_1,\bar{r}_2)\in\left\{\begin{array}{lll}(*,1,1,1,0,0), & (*,1,1,1,1,1), & (1,*,0,0,1,1) \\ (1,1,0,0,0,0), & (1,0,1,1,1,1) & \end{array}\right\},$$

where $B=(m+1)(n+1)-(c_2-c_1-1)(r_2-r_1-1)-1$.

Theorem 3.3. *Suppose $\vec{R}(x,y)$ and $D\vec{R}(x,y)$ are given by* (2.1)–(2.3) *and* (3.1)–(3.3) *respectively and they interpolate $\vec{V}$ over Π, then $\vec{R}(x,y)\in\vec{H}_{B,B}$ while $D\vec{R}(x,y)\in\vec{H}_{B-1,B-2}$ if*

$$(\bar{m},\bar{n},\bar{c}_1,\bar{c}_2,\bar{r}_1,\bar{r}_2)\in\{(0,0,0,0,1,0),(0,0,1,1,1,0),(0,1,0,0,0,0),(0,1,0,0,1,1)\};$$

$\vec{R}(x,y)\in\vec{H}_{B,B-1}$ while $D\vec{R}(x,y)\in\vec{H}_{B-1,B-1}$ if

$$(\bar{m},\bar{n},\bar{c}_1,\bar{c}_2,\bar{r}_1,\bar{r}_2)\in\{(1,0,0,0,0,1),(0,0,1,1,0,0),(1,0,1,1,0,1)\};$$

$\vec{R}(x,y)\in\vec{H}_{B-1,B-1}$ while $D\vec{R}(x,y)\in\vec{H}_{B,B-1}$ if

$$(\bar{m},\bar{n},\bar{c}_1,\bar{c}_2,\bar{r}_1,\bar{r}_2)\in\{(0,1,0,1,0,0),(0,0,0,0,1,1),(0,1,0,1,1,1)\}$$

and $\vec{R}(x,y)\in\vec{H}_{B-1,B-2}$ while $D\vec{R}(x,y)\in\vec{H}_{B,B}$ if

$$(\bar{m},\bar{n},\bar{c}_1,\bar{c}_2,\bar{r}_1,\bar{r}_2)\in\{(0,0,1,0,0,0),(0,0,1,0,1,1),(1,0,0,0,0,0),(1,0,1,1,0,0)\},$$

where $B=(m+1)(n+1)-(c_2-c_1-1)(r_2-r_1-1)-1$.

Theorem 3.2 and Theorem 3.3 show that among all 64 equivalent classes with regard to $(\bar{m},\bar{n},\bar{c}_1,\bar{c}_2,\bar{r}_1,\bar{r}_2)$ there are 50 allowing $\vec{R}(x,y)$ and $D\vec{R}(x,y)$ to possess the same rational type (i.e., 8 classes for (B/B) type, 33 classes for $(B/B-1)$ type, one class for $(B-1/B-1)$ type and 8 classes for $(B-1/B-2)$ type) and the left 14 classes are divided into four groups switching the characterisation of $\vec{R}(x,y)$ and $D\vec{R}(x,y)$ either between (B/B) type and $(B-1/B-2)$ type or between $(B/B-1)$ type and $(B-1/B-1)$ type.

Even if $\vec{R}(x,y)$ and $D\vec{R}(x,y)$ are of the same rational type, we can by no means come to the conclusion $\vec{R}(x,y)\equiv D\vec{R}(x,y)$. However there holds the following

Theorem 3.4. *suppose $\vec{R}(x,y)$ and $D\vec{R}(x,y)$ are defined in* (2.1)–(2.3) *and* (3.1)–(3.3) *respectively and they interpolate $\vec{V}$ over Π. If the block appears neither on the left boundary nor on the upper boundary, then $\vec{R}(x,y)$ and $D\vec{R}(x,y)$ satisfy*
(a) $\vec{R}(x_0,y)\equiv D\vec{R}(x_0,y)$
(b) $\vec{R}(x,y_0)\equiv D\vec{R}(x,y_0)$.

Proof. From (2.1)–(2.3) and (3.1)–(3.3) we have

$$\vec{R}(x_0,y) \equiv \vec{s}_0(y) = \vec{b}_{0,0} + \frac{y-y_0}{\vec{b}_{0,1}} + \cdots + \frac{y-y_{m-1}}{\vec{b}_{0,m}}$$

and

$$D\vec{R}(x_0,y) = \vec{t}_0(x_0) + \frac{y-y_0}{\vec{t}_1(x_0)} + \cdots + \frac{y-y_{m-1}}{\vec{t}_m(x_0)},$$

therefore from $\vec{R}(x_0,y_j) = D\vec{R}(x_0,y_j)$, $j=0,1,\cdots,m$, it follows $\vec{R}(x_0,y) \equiv D\vec{R}(x_0,y)$. One can similarly prove (b).

Theorem 3.5. *Suppose $\vec{R}(x,y)$ and $D\vec{R}(x,y)$ are given by* (2.1)–(2.3) *and* (3.1)–(3.3) *respectively and they both interpolate $\vec{V}$ over the mesh Π, then $\vec{R}(x,y)$ is unique in the sense that it is independent of the ordering of the elements of every column in Π while $D\vec{R}(x,y)$ is unique in the sense that it is independent of the ordering of the elements of every row in Π.*

Proof. Denote by $\vec{U}(x,y)$ another BVRIH which differs from $\vec{R}(x,y)$ only in branches. We might as well suppose that the block does not lie on any boundary and hence may write

$$\vec{R}(x,y) = \vec{s}_0(y) + \frac{x-x_0}{\vec{s}_1(y)} + \cdots + \frac{x-x_{n-1}}{\vec{s}_n(y)},$$

$$\vec{U}(x,y) = \vec{v}_0(y) + \frac{x-x_0}{\vec{v}_1(y)} + \cdots + \frac{x-x_{n-1}}{\vec{v}_n(y)},$$

where

$$\vec{s}_i(y) = \vec{b}_{i,0} + \frac{y-y_0}{\vec{b}_{i,1}} + \cdots + \frac{y-y_{m-1}}{\vec{b}_{i,m}},$$

$$\vec{v}_i(y) = \vec{e}_{i,i_0} + \frac{y-y_0}{\vec{e}_{i,i_1}} + \cdots + \frac{y-y_{m-1}}{\vec{e}_{i,i_m}}$$

with $i_0,i_1,\cdots,i_m$ being a reordering of $0,1,\cdots,m$ for $i=0,1,\cdots,c_1,c_2,c_2+1,\cdots,n$ and

$$\vec{s}_i(y) = \vec{b}_{i,0} + \frac{y-y_0}{\vec{b}_{i,1}} + \cdots + \frac{y-y_{r_1-1}}{\vec{b}_{i,r_1}} + \frac{y-y_{r_1}}{\vec{b}_{i,r_2}} + \frac{y-y_{r_2}}{\vec{b}_{i,r_2+1}} + \cdots + \frac{y-y_{m-1}}{\vec{b}_{i,m}},$$

$$\vec{v}_i(y) = \vec{e}_{i,i_0} + \frac{y-y_0}{\vec{e}_{i,i_1}} + \cdots + \frac{y-y_{r_1-1}}{\vec{e}_{i,i_{r_1}}} + \frac{y-y_{r_1}}{\vec{e}_{i,i_{r_2}}} + \frac{y-y_{r_2}}{\vec{e}_{i,i_{r_2+1}}} + \cdots + \frac{y-y_{m-1}}{\vec{e}_{i,i_m}}$$

with $i_0,i_1,\cdots,i_{r_1},i_{r_2},\cdots,i_m$ being a reordering of $0,1,\cdots,r_1,r_2,\cdots,m$ for $i=c_1+1,\cdots,c_2-1$. From

$$\vec{R}(x_0,y_j) = D\vec{R}(x_0,y_j),\ j=0,1,\cdots,m$$

it follows

$$\vec{s}_0(y) \equiv \vec{v}_0(y).$$

Assume

$$\vec{s}_i(y) \equiv \vec{v}_i(y),\ i=0,1,\cdots,k-1\,(k \le n),$$

then we derive

$$\vec{s}_k(y) \equiv \vec{v}_k(y)$$

from

$$\vec{s}_k(y) = -\frac{x_k - x_{k-1}}{\vec{s}_{k-1}(y)} + \cdots + \frac{x_k - x_1}{\vec{s}_1(y)} + \frac{x_k - x_0}{\vec{s}_0(y) - \vec{R}(x_k, y)}$$
$$\vec{v}_k(y) = -\frac{x_k - x_{k-1}}{\vec{v}_{k-1}(y)} + \cdots + \frac{x_k - x_1}{\vec{v}_1(y)} + \frac{x_k - x_0}{\vec{v}_0(y) - \vec{U}(x_k, y)}$$

and

$$\vec{R}(x_k, y_j) = \vec{U}(x_k, y_j)$$

for $j = 0, 1, \cdots, m$ if $k \le c_1$ or $k \ge c_2$ and for $j = 0, 1, \cdots, r_1, r_2, \cdots, m$ if $c_1 + 1 \le k \le c_2 - 1$. The uniqueness of $\vec{R}(x, y)$ is thus proved by induction. Similar procedure can be used to prove the uniqueness of $\vec{DR}(x, y)$.

Finally we point out that if the size parameters of Π satisfy $m = n, c_1 = r_1, c_2 = r_2$, $x_i = y_i$ for $i = 0, 1, \cdots, n$ and $\vec{v}_{i,j} = \vec{v}_{j,i}$ for $(x_i, y_j) \in \Pi$, then $\vec{R}(x, y) \equiv \vec{DR}(y, x)$. On the basis of this paper it seems interesting to pursue the further study of BVRIHs with other holes instead of rectangular one, and other types of branched continued fractions, such as those treated by Cuyt and Verdonk in [1,3], may be extended to the vector valued case with hole structure taken into consideration.

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