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NONCONFORMING FINITE ELEMENT APPROXIMATIONS TO THE UNILATERAL PROBLEM*1)

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Abstract

The nonconforming finite element (two Crouzeix-Raviart linear elements and Wilson element) approximations to the unilateral problem are considered. The error bounds for these elements are obtained in the appropriate assumptions of regularity of solution of the problem.

Key words: Unilateral problem, Nonconforming finite element

1. Introduction

There have been numerous work in the analysis of finite element methods for the unilateral problem (c.f.[4] and the references therein). It should be mentioned that in F. Scarpini et. al.[6], I.Hlavacek et. al.[5] and F. Brezzi et. al.[1], the conforming linear element approximation to the unilateral problem have been considered, and the various error bounds have been obtained in the different assumption of regularity of solution of the problem.

In this paper, we consider three nonconforming finite element (i.e. two Crouzrix-Raviart linear elements and Wilson element) approximations to the unilateral problem, and the error bounds for these elements are obtained in the appropriate assumptions of regularity of solution of the problem.

The unilateral problem is the following

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0, \\ u \geq 0,\ \partial u/\partial \nu \geq 0,\ u\partial u/\partial \nu = 0, & \text{on } \Gamma_1, \end{cases} \tag{1.1}$$

where Ω is a convex domain in R^2 with piecewise smooth boundary $\partial\Omega$, $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$ and $\partial u/\partial \nu$ is the outer normal derivative of u on Γ_1. It is well known that the problem (1.1) is equivalent to the following variational inequality:

$$\begin{cases} \text{to find } u \in K, & \text{such that} \\ a(u, v-u) \geq \ \langle f, v-u\rangle & \forall v \in K, \end{cases} \tag{1.2}$$

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where

$$K = \{v \in H^1(\Omega)|\ v = 0 \text{ on } \Gamma_0, v \geq 0 \text{ on } \ \Gamma_1\}, \tag{1.3}$$

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v dx, \quad \langle f, v\rangle = \int_\Omega f \cdot v dx. \tag{1.4}$$

the solution of the problem (1.2) will be approximated by the finite element method with a regular subdivision. For each $h > 0$, let $\mathcal{T}_h$ be a regular subdivision of Ω. For the sake of simplisity, let Ω be a convex polygon, then $\Omega = \bigcup_{\tau\in\mathcal{T}_h} \tau$. Let V_h be a finite element space of approximating the space $H^1_{\Gamma_0}(\Omega) = \{v \in H^1(\Omega)|v = 0 \text{ on } \Gamma_0\}$, with norm $\|\cdot\|$:

$$\|v\|_h = \Big(\sum_{\tau\in\mathcal{T}_h} |v|^2_{1,\tau}\Big)^{\frac{1}{2}} \quad \forall\ v \in V_h, \tag{1.5}$$

and K_h be a convex closed subset of V_h, as an approximation of K. Then the approximate problem of the unilateral problem (1.2) is the following:

$$\begin{cases} \text{to find} \quad u_h \in K_h, & \text{such that} \\ a_h(u_h, v_h - u_h) \geq \ \langle f, v_h - u_h\rangle & v_h \in K_h, \end{cases} \tag{1.6}$$

where

$$a_h(u_h, v_h) = \sum_{\tau\in\mathcal{T}_h} \int_\tau \nabla u_h \nabla v_h dx. \tag{1.7}$$

We now show abstract error estimate

Theorem 1.1. *Assume that u and u_h are the solutions of the problems* (1.2) *and* (1.6) *respectively, then*

$$\|u - u_h\|^2_h \leq C \inf_{v_h\in K_h} \{\|u - v_h\|^2_h + a_h(u, v_h - u_h) - \langle f, v_h - u_h\rangle\}. \tag{1.8}$$

Proof. Using the triangle inequality

$$\|u - u_h\|_h \leq \|u - v_h\|_h + \|v_h - u_h\|_h, \quad \forall\ v_h \in K_h.$$

And noting that u_h is the solution of the problem (1.6),

$$\begin{aligned} \|v_h - u_h\|^2_h &= a_h(v_h - u_h, v_h - u_h) \\ &= a_h(v_h - u, v_h - u_h) + a_h(u - u_h, v_h - u_h) \\ &\leq \|v_h - u\|_h \cdot \|v_h - u_h\|_h + a_h(u, v_h - u_h) - \langle f, v_h - u_h\rangle. \end{aligned}$$

Summarizing the previous two inequalities, the theorem is proved.

2. Crouzeix-Raviart Linear Element Approximation(I)

For the Crouzeix-Raviart linear element approximation to the unilateral problem (1.2), the subdivision $\mathcal{T}_h$ is a triangulation, $\tau \in \mathcal{T}_h$ triangle element,

$V_h = \{v_h \in L^2(\Omega) : v_h|_\tau \in P_1(\tau),\ v_h$ is continuous at the midpoints of edges of element

$$\tau \in \mathcal{T}_h, v_h(a_{ij}) = 0 \ \forall \text{ midpoints } a_{ij} \text{ of edges on } \Gamma_0\}. \tag{2.1}$$

$$\begin{aligned} K_h^1 = &\{v_h \in V_h : v_h(a_{12}) \geq |v_h(a_{23}) - v_h(a_{13})| \ \forall \text{ edge } \overline{a_1 a_2} \subset \Gamma_1, \\ &a_i, i = 1,2,3, \quad \text{the vertices of element } \tau\} \end{aligned} \tag{2.2}$$

(c.f.Fig.2.1)

Then the following lemmas can be proved easily:

Lemma 2.1. *K_h^1 is a convex subset of V_h.*

Let $\Pi_h : H^2(\Omega) \to V_h$ the interpolation operator defined as follows: for any given $v \in H^2(\Omega)$,

$$\Pi_h v|_\tau = \Pi_\tau v = v(a_{23})\mu_1(x) + v(a_{13})\mu_2(x) + v(a_{12})\mu_3(x), \tag{2.3}$$

$$\begin{aligned} \mu_1(x) = &\lambda_2(x) + \lambda_3(x) - \lambda_1(x), \mu_2(x) = \lambda_3(x) + \lambda_1(x) - \lambda_2(x), \\ &\mu_3(x) = \lambda_1(x) + \lambda_2(x) - \lambda_3(x), \end{aligned} \tag{2.4}$$

where $\lambda_i(x), i = 1,2,3$, are the barycentric coordinates. (c.f.Fig.2.2)

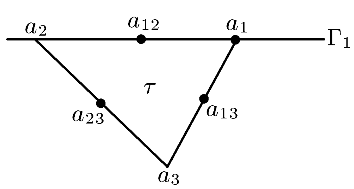

Fig.2.1

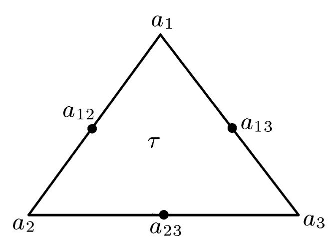

Fig.2.2

Lemma. 2.2.

$$\begin{cases} \Pi_\tau v(a_1) = -v(a_{23}) + v(a_{13}) + v(a_{12}), \\ \Pi_\tau v(a_2) = v(a_{23}) - v(a_{13}) + v(a_{12}), \\ \Pi_\tau v(a_3) = v(a_{23}) + v(a_{13}) - v(a_{12}). \end{cases} \tag{2.5}$$

We now introduce another interpolation operator $\widetilde{\Pi_h} : H^2(\Omega) \to V_h$ defined as follows: let

$$\mathcal{T}_h^0 = \{\tau : \partial\tau \cap \Gamma_1 = \emptyset\}, \ \mathcal{T}_h^1 = \{\tau : \partial\tau \cap \Gamma_1 \neq \emptyset\}, \ \mathcal{T}_h = \mathcal{T}_h^0 \cup \mathcal{T}_h^1, \tag{2.6}$$

then for any given $v \in H^2(\Omega)$,

$$\begin{cases} \widetilde{\Pi_\tau} v = \widetilde{\Pi_h} v|_\tau = \Pi_\tau v \quad \forall \ \tau \in \mathcal{T}_h^0; \\ \widetilde{\Pi_\tau} v = \widetilde{\Pi_h} v|_\tau = \Pi_\tau v \text{ for } v(a_{12}) \geq |v(a_{23}) - v(a_{13})|, \ \forall \tau \in \mathcal{T}_h^1; \\ \widetilde{\Pi_\tau} v = \widetilde{\Pi_h} v|_\tau = v(a_{23})\mu_1 + v(a_{13})\mu_2 + |v(a_{23}) - v(a_{13})|\mu_3, \\ \qquad \text{for } v(a_{12}) \leq |v(a_{23}) - v_(a_{13})|, \ \forall \tau \in \mathcal{T}_h^1. \end{cases} \tag{2.7}$$

Lemma 2.3. $\forall v \in K$

$$\widetilde{\Pi_h} v \in K_h^1. \tag{2.8}$$

Lemma 2.4. $\forall v_h \in K_h^1$,

$$v_h|_{\Gamma_1} \geq 0. \tag{2.9}$$

The proof of Lemma 2.4 can be completed by the Lemma 2.2 and the definition of K_h^1 (2.2).

Lemma 2.5. (c.f.[7])

$$\int_{\partial\tau} |w|^2 ds \leq C\{h^{-1}\|w\|_{0,\tau}^2 + h|w|_{1,\tau}^2\} \quad \forall w \in H^1(\tau). \tag{2.10}$$

We now establish the error estimate of the Crouzeix-Raviart linear element approximation to the unilateral problem.

Theorem 2.1. *Assume that u and u_h are the solutions of the problems* (1.2) *and* (1.6) *with K_h^1* (2.2) *respectively, and that $u \in H^2(\Omega)$ and $u \in W^{1,\infty}(\Omega_{\Gamma_1})$, where Ω_{Γ_1} is any given neighbourhood of Γ_1. Then the following error estimate holds*

$$\|u - u_h\|_h \leq C(h|u|_{2,\Omega} + h^{\frac{1}{2}}|u|_{1,\infty,\Omega_{\Gamma_1}}). \tag{2.11}$$

Proof. (i) We first estimate

$$\begin{aligned} E_h(u, v_h - u_h) =& a_h(u, v_h - u_h) - \langle f, v_h - u_h\rangle \\ =& \sum_{\tau\in\mathcal{T}_h} \int_\tau \nabla u \cdot \nabla(v_h - u_h)dx - \int_\Omega f(v_h - u_h)dx \\ =& \sum_{\tau\in\mathcal{T}_h} \int_{\partial\tau} \frac{\partial u}{\partial\nu}(v_h - u_h)ds = \sum_{\tau\in\mathcal{T}_h} \sum_{\gamma\subset\partial\tau\cap\Gamma_1} \int_\gamma \frac{\partial u}{\partial\nu}(v_h - u_h)ds \\ &+ \sum_{\tau\in\mathcal{T}_h} \sum_{\gamma\in\partial\tau,\gamma\not\subset\Gamma_1} \int_\gamma \frac{\partial u}{\partial\nu}(v_h - u_h)ds. \end{aligned} \tag{2.12}$$

It is well known that (c.f.[7], [3])

$$\sum_{\tau\in\mathcal{T}_h} \sum_{\gamma\in\partial\tau,\gamma\not\subset\Gamma_1} \int_\gamma \frac{\partial u}{\partial\nu}(v_h - u_h)ds \leq Ch|u|_{2,\Omega}\|u_h - v_h\|_h. \tag{2.13}$$

Noting that $\frac{\partial u}{\partial\nu}\cdot u = 0$ on Γ_1 and $u_h \geq 0$ on Γ_1 (Lemma 2.4), it can be seen that $\forall\gamma \subset \Gamma_1$

$$\begin{aligned} \int_\gamma \frac{\partial u}{\partial\nu}(v_h - u_h)ds &= \int_\gamma \frac{\partial u}{\partial\nu}(v_h - u)ds + \int_\gamma \frac{\partial u}{\partial\nu}(u - u_h)ds \\ &= \int_\gamma \frac{\partial u}{\partial\nu}(v_h - u)ds - \int_\gamma \frac{\partial u}{\partial\nu}u_h ds \leq \int_\gamma \frac{\partial u}{\partial\nu}(v_h - u)ds. \end{aligned} \tag{2.14}$$

Summarizing (2.12)–(2.14) and Theorem 1.1, we have

$$\|u - u_h\|_h^2 \leq C_1 \inf_{v_h\in K_h}\{\|u - v_h\|_h^2 + \sum_{\tau\in\mathcal{T}_h} \sum_{\gamma\subset\partial\tau\cap\Gamma_1} \int_\gamma \frac{\partial u}{\partial\nu}(v_h - u)ds\} + C_2h^2|u|_{2,\Omega}^2. \tag{2.15}$$

(ii) Let $v_h = \widetilde{\Pi}_h u$ in (2.15), we first estimate

$$I_2 = \sum_{\tau\in\mathcal{T}_h} \sum_{\gamma\subset\partial\tau\cap\Gamma_1} \int_\gamma \frac{\partial u}{\partial\nu}(\widetilde{\Pi}_h u - u)ds. \tag{2.16}$$

For $u(a_{12}) \geq |u(a_{23}) - u(a_{13})|$, $\widetilde{\Pi}_\tau u = \Pi_\tau u$, $\gamma \subset \partial\tau$, $\tau \in \mathcal{T}_h^1$, by the lemma 2.5, we have

$$\begin{aligned}\Big|\int_\gamma \frac{\partial u}{\partial \nu}(\widetilde{\Pi_h} u - u)ds\Big| &\leq \Big\|\frac{\partial u}{\partial \nu}\Big\|_{0,\gamma}\|\Pi_h u - u\|_{0,\gamma}\\ &\leq C\Big\|\frac{\partial u}{\partial \nu}\Big\|_{0,\gamma}\{h^{-1}\|\Pi_h u - u\|_{0,\tau}^2 + h|\Pi_h u - u|_{1,\tau}^2\}^{\frac{1}{2}}\\ &\leq Ch^{\frac{3}{2}}\Big\|\frac{\partial u}{\partial \nu}\Big\|_{0,\gamma}|u|_{2,\tau}. \end{aligned} \tag{2.17}$$

For $u(a_{12}) < |u(a_{23}) - u(a_{13})|$, without lost of generality, assume that $u(a_{23}) > u(a_{13})$, then

$$\begin{aligned}\widetilde{\Pi_h} u|_\gamma &= [\Pi_h u + (\widetilde{\Pi_h} u - \Pi_h u)]|_\gamma = \Pi_h u|_\gamma + [(u(a_{23}) - u(a_{13})) - u(a_{12})]\mu_3|_\gamma\\ &= \Pi_h u|_\gamma + (u(a_{23}) - u(a_{13}) - u(a_{12}))(\lambda_1 + \lambda_2)|_\gamma = \Pi_h u|_\gamma - \Pi_\tau u(a_1).\end{aligned}$$

So

$$\begin{aligned}\Big|\int_\gamma \frac{\partial u}{\partial \nu}(\widetilde{\Pi_h} u - u)ds\Big| &\leq \Big|\int_\gamma \frac{\partial u}{\partial \nu}(\Pi_h u - u)ds\Big| + \Big|\int_\gamma \frac{\partial u}{\partial \nu}(\widetilde{\Pi_h} u - \Pi_h u)ds\Big|\\ &\leq Ch^{\frac{3}{2}}\Big\|\frac{\partial u}{\partial \nu}\|_{0,\gamma}|u|_{2,\tau} + h^{\frac{1}{2}}\Big\|\frac{\partial u}{\partial \nu}\|_{0,\gamma}|\Pi_\tau u(a_1)|. \end{aligned} \tag{2.18}$$

Since $0 < u(a_{12}) < u(a_{23}) - u(a_{13})$, by Lemma 2.2,

$$\begin{aligned}\Pi_\tau u(a_1) &= u(a_{12}) - (u(a_{23}) - u(a_{13})) < 0,\\ \Pi_\tau u(a_2) &= u(a_{12}) + (u(a_{23}) - u(a_{13})) > 0,\end{aligned}$$

then there exists a point $a \in \overline{a_1 a_2} = \gamma$, such that

$$\Pi_\tau u(a) = 0.$$

So that

$$\begin{aligned}|\Pi_\tau u(a_1)| &= |\Pi_\tau u(a_1) - \Pi_\tau u(a)| \leq h\Big|\frac{d\Pi_\tau u|_\gamma}{ds}\Big| = Ch\Big|\frac{\Pi_\tau u(a_1) - \Pi_\tau u(a_2)}{|a_1 a_2|}\Big|\\ &= Ch\Big|\frac{u(a_{23}) - u(a_{13})}{|a_{13}a_{23}|}\Big| \leq Ch \max_{x\in\overline{a_{13}a_{23}}}|\nabla u(x)|. \end{aligned} \tag{2.19}$$

From (2.18) and (2.19), we have, for $u(a_{12}) < |u(a_{23}) - u(a_{13})|$,

$$\Big|\int_\gamma \frac{\partial u}{\partial \nu}(\widetilde{\Pi_h} u - u)ds\Big| \leq Ch^{\frac{3}{2}}\Big\|\frac{\partial u}{\partial \nu}\Big\|_{0,\gamma}(|u|_{2,\tau} + \max_{\overline{a_{13}a_{23}}}|\nabla u|)$$

Therefore

$$I_2 \leq C_1 h^{\frac{3}{2}}\Big\|\frac{\partial u}{\partial \nu}\Big\|_{0,\Gamma_1}|u|_{2,\Omega} + C_2 h\Big\|\frac{\partial u}{\partial \nu}\Big\|_{0,\Gamma_1}\max_{\Omega_{\Gamma_1}}|\nabla u|. \tag{2.20}$$

(iii) Next we estimate

$$I_1 = \|u - \widetilde{\Pi_h} u\|_h^2 \leq 2(\|u - \Pi_h u\|_h^2 + \|\Pi_h u - \widetilde{\Pi_h} u\|_h^2). \tag{2.21}$$

It is well known that (c.f. [3])

$$\|u-\Pi_h u\|_h^2 \le Ch^2|u|_{2,\Omega}^2. \tag{2.22}$$

For $\tau \in \mathcal{T}_h^0$ and $\tau \in \mathcal{T}_h^1$ with $u(a_{12}) \ge |u(a_{23})-u(a_{13})|$,

$$\widetilde{\Pi_h} u|_\tau = \Pi_h u|_\tau. \tag{2.23}$$

For $\tau \in \mathcal{T}_h^1$ with $u(a_{12}) < |u(a_{23})-u(a_{13})|$ (assume that $u(a_{23}) > u(a_{13})$) :

$$\Pi_\tau u - \widetilde{\Pi}_\tau u = (u(a_{12}) - u(a_{23}) + u(a_{13}))\mu_3,$$

and

$$\begin{aligned}|\Pi_\tau u - \widetilde{\Pi}_\tau u|_{1,\tau}^2 &= |u(a_{12}) - u(a_{23}) + u(a_{13})|^2 \int_\tau |\nabla \mu_3|^2 dx \\ &\le C|\Pi_\tau u(a_1)|^2 \le Ch^2 \max_{\overline{a_{13}a_{23}}} |\nabla u|^2, \end{aligned} \tag{2.24}$$

since (2.19). From (2.21)–(2.24), we have

$$I_1 \le C_1 h^2 |u|_{2,\Omega}^2 + \sum_{\tau\in\mathcal{T}_h^1} |\Pi_\tau u - \widetilde{\Pi}_\tau u|_{1,\tau}^2 \le C_1 h^2 |u|_{2,\Omega}^2 + C_2 h \max_{\Omega_{\Gamma_1}} |\nabla u|^2. \tag{2.25}$$

Summarizing (i), (ii) and (iii), the proof is completed.

3. Crouzeix-Raviart Linear Element Approximation (II)

In the previous section, the solution of the unilateral problem (1.2) has been approximated by the Crouzeix-Raviart linear element solution u_h which is restricted in the convex set K_h^1 and the element v_h is nonnegative on Γ_1 (Lemma 2.4). In this section, we propose another Crouzeix-Raviart linear element solution of the unilateral problem (1.2), which belongs to a convex set K_h^2 (see below) and the restriction $v_h|_{\Gamma_1} \ge 0$ is relaxed. Let (c.f.Fig.2.1)

$$K_h^2 = \{v_h \in V_h : v_h(a_{12}) \ge 0 \ \forall \text{ edge } \overline{a_1 a_2} \subset \Gamma_1\}, \tag{3.1}$$

then the approximation problem of the unilateral problem (1.2) is (1.6) with $K_h = K_h^2$.

We have the following result

Theorem 3.1. *Assume that u and u_h are the solutions of the problems* (1.2) *and* (1.6) *with $K_h = K_h^2$* (3.1) *respectively, and that $u \in H^2(\Omega)$, $f \in L^2(\Omega)$. Then the following error estimate holds*

$$\|u-u_h\|_h \le Ch^{\frac{1}{2}}(|u|_{2,\Omega} + \|f\|_{0,\Omega}). \tag{3.2}$$

Proof. (i) In the same way as the first paragraph in the Proof of the Theorem 2.1, except that $u_h \ge 0$ on Γ_1, it can be seen that

$$\|u-u_h\|_h^2 \le C_1 \inf_{v_h\in K_h^2}\{\|u-v_h\|_h^2 + \sum_{\gamma\subset\Gamma_1}\int_\gamma \frac{\partial u}{\partial \nu}(v_h - u)ds\}$$

$$+ C_2 h^2 |u|_{2,\Omega}^2 - \sum_{\gamma \subset \Gamma_1} \int_\gamma \frac{\partial u}{\partial \nu} u_h ds. \tag{3.3}$$

(ii) Let $v_h = \Pi_h u$ in (3.3), with the operator Π_h defined in (2.3), then from (2.22) and (2.18), we have

$$\|u - \Pi_h u\|_h^2 \le Ch^2 |u|_{2,\Omega}^2, \tag{3.4}$$

and

$$\Big| \sum_{\gamma \subset \Gamma_1} \int_\gamma \frac{\partial u}{\partial \nu} (\Pi_h u - u) ds \Big| \le Ch^{\frac{3}{2}} \sum_{\gamma \subset \partial\tau \cap \Gamma_1} \Big\| \frac{\partial u}{\partial \nu} \Big\|_{0,\gamma} |u|_{2,\tau} \le Ch^{\frac{3}{2}} \Big\| \frac{\partial u}{\partial \nu} \Big\|_{0,\Gamma_1} |u|_{2,\Omega}. \tag{3.5}$$

(iii) We now estimate the last term on the right hand side of (3.3)

$$I = - \sum_{\gamma \subset \Gamma_1} \int_\gamma \frac{\partial u}{\partial \nu} u_h ds. \tag{3.6}$$

Let P_0^γ be the L^2-projection operator $L^2(\gamma) \to R$ as follows:

$$P_0^\gamma(v) = \frac{1}{|\gamma|} \int_\gamma v ds, \quad R_0^\gamma(v) = v - P_0^\gamma(v), \quad |\gamma| = \int_\gamma 1 ds, \tag{3.7}$$

and P_0^τ be the L^2-projection operator $L^2(\tau) \to R$ as follows:

$$P_0^\tau(v) = \frac{1}{|\tau|} \int_\tau v dx, \quad R_0^\tau(v) = v - P_0^\tau(v), \quad |\tau| = \int_\tau 1 dx. \tag{3.8}$$

Taking into account that $\frac{\partial u}{\partial \nu} \ge 0$ on Γ_1, it can be seen that

$$- \int_\gamma P_0^\gamma (\frac{\partial u}{\partial \nu}) u_h ds = -P_0^\gamma (\frac{\partial u}{\partial \nu}) \int_\gamma u_h ds = -P_0^\gamma (\frac{\partial u}{\partial \nu}) |\gamma| u_h(a_{12}) \le 0$$

from which we have

$$\begin{aligned} - \int_\gamma \frac{\partial u}{\partial \nu} u_h ds &= - \int_\gamma R_0^\gamma \Big(\frac{\partial u}{\partial \nu} \Big) u_h ds - \int_\gamma P_0^\gamma \Big(\frac{\partial u}{\partial \nu} \Big) u_h ds \\ &\le - \int_\gamma R_0^\gamma \Big(\frac{\partial u}{\partial \nu} \Big) u_h ds = - \int_\gamma R_0^\gamma \Big(\frac{\partial u}{\partial \nu} \Big) R_0^\gamma(u_h) ds - \int_\gamma R_0^\gamma \Big(\frac{\partial u}{\partial \nu} \Big) P_0^\gamma(u_h) ds \\ &= - \int_\gamma R_0^\gamma \Big(\frac{\partial u}{\partial \nu} \Big) R_0^\gamma(u_h) ds \le \Big\{ \int_\gamma R_0^\gamma \Big(\frac{\partial u}{\partial \nu} \Big)^2 ds \Big\}^{\frac{1}{2}} \Big\{ \int_\gamma R_0^\gamma (u_h)^2 ds \Big\}^{\frac{1}{2}}. \end{aligned} \tag{3.9}$$

We now estimate, with use of the projective property and Lemma 2.5,

$$\begin{aligned} \int_\gamma R_0^\gamma \Big(\frac{\partial u}{\partial \nu} \Big)^2 ds &\le \int_\gamma R_0^\gamma \Big(\frac{\partial u}{\partial x} \Big)^2 ds + \int_\gamma R_0^\gamma \Big(\frac{\partial u}{\partial y} \Big)^2 ds \\ &\le \int_\gamma R_0^\tau \Big(\frac{\partial u}{\partial x} \Big)^2 ds + \int_\gamma R_0^\tau \Big(\frac{\partial u}{\partial y} \Big)^2 ds \le Ch |u|_{2,\tau}^2. \end{aligned} \tag{3.10}$$

As for estimating the second factor on the right hand side of (3.9), we have

$$\int_\gamma R_0^\gamma(u_h)^2 ds \le \int_\gamma R_0^\tau(u_h)^2 ds \le Ch|u_h|_{1,\tau}^2. \tag{3.11}$$

We now take $v_h = 0$ in the discrete problem (1.6) with $K_h = K_h^2$, then

$$|u_h|_{1,h}^2 = a_h(u_h, u_h) \le \langle f, u_h\rangle \le \|f\|_{0,\Omega}\|u_h\|_{0,\Omega}, \tag{3.12}$$

from which it can be seen that

$$|u_h|_{1,h} \le C\|f\|_{0,\Omega}, \tag{3.13}$$

since a Poincare inequality in nonconforming finite element spaces[8,9]:

$$\|v_h\|_{0,\Omega} \le C|v_h|_{1,h} \;\forall\; v_h \in V_h. \tag{3.14}$$

Combining (3.9)–(3.11) and (3.13), we have

$$I \le Ch \sum_{\tau\in\mathcal{T}_h} |u|_{2,\tau}|u_h|_{1,\tau} \le Ch|u|_{2,\Omega}\|f\|_{0,\Omega} \tag{3.15}$$

Summarizing (i)–(iii), the theorem is proved.

4. Wilson Element Approximation

For the Wilson element approximation, let Ω be a rectangle, $\mathcal{T}_h$ be a rectangular subdivision, $\tau \in \mathcal{T}_h$ rectangular element, and $\Gamma_1 \subset \partial\Omega$ be parallel to x_1-axis,

$$\begin{aligned} V_h = \{v_h \in L^2(\Omega) : v_h|_\tau \in P_2(\tau), v_h \text{ is continuous at the vertices of element } \tau\ \forall \tau \in \mathcal{T}_h,\\ \text{and } v_h(a) = 0\ \forall \text{ vertices } a \in \Gamma_0\}, \end{aligned} \tag{4.1}$$

$$K_h = \{v_h \in V_h : v_h(a) \ge 0\ \forall\ \text{ vertices } a \in \Gamma_1\}. \tag{4.2}$$

Let the interpolation operator $\Pi_h : H^2(\Omega) \to V_h$ be defined as follows (c.f.Fig.4.1): for any given $v \in H^2(\Omega)$,

$$\Pi_h v|_\tau = \Pi_\tau v = \sum_{i=1}^{4} v(a_i)p_i(x) + \sum_{j=1}^{2} \phi_j(v)q_j(x) \;\; \forall \;\; \tau \in \mathcal{T}_h \tag{4.3}$$

where

$$\begin{cases} p_1(x) = \frac14\Big(1 + \frac{x_1-c_1}{h_1}\Big)\Big(1 + \frac{x_2-c_2}{h_2}), \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ p_4(x) = \frac14\Big(1 + \frac{x_1-c_1}{h_1}\Big)\Big(1 - \frac{x_2-c_2}{h_2}\Big); \end{cases} \tag{4.4}$$

$$\begin{cases} q_j(x) = \frac18\Big[\Big(\frac{x_j-c_j}{h_j}\Big)^2 - 1\Big], \\ \phi_j(v) = \frac{h_j^2}{h_1h_2}\int_\tau \partial_{jj}v\,dx,\ j = 1, 2; \end{cases} \tag{4.5}$$

$$c = \frac{1}{4}\sum_{i=1}^{4} a_i. \tag{4.6}$$

Then it can be seen easily that $\Pi_h v \in K_h \ \forall v \in K$.

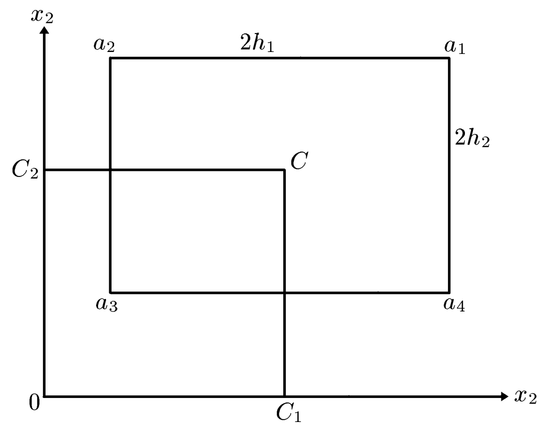

Fig.4.1

We have the following error estimate

Theorem 4.1. *Assume that* u *and* u_h *are the solutions of the problem* (1.2) *and the problem* (1.6) *with* K_h (4.2) *respectively, and that* $u \in H^2(\Omega)$, $u|_{\Gamma_1} \in H^{2-\epsilon}(\Gamma_1)$, *where* $0 < \epsilon \le \frac{1}{2}$, $u|_{\Gamma_1}$ *means a function on* Γ_1. *Then the following error estimate holds*

$$\|u - u_h\|_h \le Ch^{1-\frac{\epsilon}{2}}(|u|_{2,\Omega} + |u|_{1,\Gamma_1} + |u|_{\Gamma_1}|_{2-\epsilon,\Gamma_1}). \tag{4.7}$$

Proof. (i) Let Q_1 be the peicewise bilinear operator on V_h, and $R_1(w_h) = w_h - Q_1(w_h)\ \forall w_h \in V_h$, then

$$\begin{aligned} E_h(u, v_h - u_h) &= a_h(u, v_h - u_h) - \langle f, v_h - u_h\rangle = \sum_{\tau\in\mathcal{T}_h}\int_{\partial\tau}\frac{\partial u}{\partial\nu}(v_h - u_h)ds \\ &= \sum_{\tau\in\mathcal{T}_h}\int_{\partial\tau}\frac{\partial u}{\partial\nu}Q_1(v_h - u_h)ds + \sum_{\tau\in\mathcal{T}_h}\int_{\partial\tau}\frac{\partial u}{\partial\nu}R_1(v_h - u_h)ds. \end{aligned} \tag{4.8}$$

By the standard error estimate of Wilson element[7],

$$\Big|\sum_{\tau\in\mathcal{T}_h}\int_{\partial\tau}\frac{\partial u}{\partial\nu}R_1(v_h - u_h)ds\Big| \le Ch|u|_{2,\Omega}\|v_h - u_h\|_h. \tag{4.9}$$

Since $v_h, u_h \in V_h$, then $Q_1(v_h - u_h) \in C^0(\Omega)$ and $Q_1(v_h - u_h)|_{\Gamma_0} = 0$, thus

$$\begin{aligned} &\sum_{\tau\in\mathcal{T}_h}\int_{\partial\tau}\frac{\partial u}{\partial\nu}Q_1(v_h - u_h)ds = \sum_{\gamma\subset\Gamma_1}\int_{\gamma}\frac{\partial u}{\partial\nu}Q_1(v_h - u_h)ds \\ &\quad = \sum_{\gamma\subset\Gamma_1}\int_{\gamma}\frac{\partial u}{\partial\nu}(Q_1(v_h) - u)ds - \sum_{\gamma\subset\Gamma_1}\int_{\gamma}\frac{\partial u}{\partial\nu}Q_1(u_h)ds \le \sum_{\gamma\subset\Gamma_1}\int_{\gamma}\frac{\partial u}{\partial\nu}(Q_1(v_h) - u)ds, \end{aligned} \tag{4.10}$$

here we have used the following relations: $u \cdot \frac{\partial u}{\partial \nu} = 0$, $\frac{\partial u}{\partial \nu} \geq 0$ and $Q_1(u_h) \geq 0$ on Γ_1.

Summing (4.8)–(4.10), and by Theorem 1.1, we have

$$\|u - u_h\|_h^2 \leq C_1 \inf_{v_h \in K_h} \Big\{ \|u - v_h\|_h^2 + \sum_{\gamma \subset \Gamma_1} \int_\gamma \frac{\partial u}{\partial \nu}(Q_1(v_h) - u) ds \Big\} + C_2 h^2 |u|_{2,\Omega}^2 \quad (4.11)$$

(ii) Let $v_h = \Pi_h u$ in (4.11), then, with use of the interpolation error estimate[2],

$$\|u - \Pi_h u\|_h^2 \leq C h^2 |u|_{2,\Omega}^2, \quad (4.12)$$

and

$$Q_1(\Pi_h u) = Q_1(u). \quad (4.13)$$

The estimate of the second term on the right hand side of (4.11) is as follows: $\forall \gamma \subset \Gamma_1$,

$$\begin{aligned} \int_\gamma \frac{\partial u}{\partial \nu}(Q_1(\Pi_h u) - u) ds &= \int_\gamma \frac{\partial u}{\partial \nu}(Q_1(u) - u) ds \\ &\leq \Big\| \frac{\partial u}{\partial \nu} \Big\|_{0,\gamma} \|Q_1(u) - u\|_{0,\gamma} \leq C h^{2-\epsilon} \Big\| \frac{\partial u}{\partial \nu} \Big\|_{0,\gamma} |u|_\gamma|_{2-\epsilon,\gamma}, \end{aligned} \quad (4.14)$$

here we have used the bilinear interpolation error estimate[2]. Thus

$$\Big| \sum_{\gamma \subset \Gamma_1} \int_\gamma \frac{\partial u}{\partial \nu}(Q_1(\Pi_h u) - u) ds \Big| \leq C h^{2-\epsilon} \| \frac{\partial u}{\partial \nu} \|_{0,\gamma} |u|_{\Gamma_1}|_{2-\epsilon,\Gamma_1}. \quad (4.15)$$

From (4.11), (4.12) and (4.15), the proof is completed.

Remark. Theorem 4.1 means that the error bound of Wilson element, as the same as the conforming bilinear element, approximation to the unilateral problem. And it is well known that the error bounds are the same for Wilson element and the conforming bilinear element approximations to the second order elliptic problem.

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